L'Hôpital

Suppose $f: (a,b) \to \mathbb{R}$, $g: (a,b) \to \mathbb{R}$ is differentiable at every $x \in (a,b) \setminus \{c\}$, with $c \in (a,b)$

Suppose further $g'(x) \neq 0$ $\forall x \in (a,c)$, and

either ① $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$

or ② $\lim_{x \to c} |g(x)| = +\infty$

If $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to c} \frac{f(x)}{g(x)}$ exists, & $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

(Note: there is a variant that works when $c = \pm\infty$
there is also a variant that works when $\lim_{x \to c} \frac{f'(x)}{g'(x)} = +\infty$ or $-\infty$)

eg. $\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{1 - \cos x}$

Step 1: Compute limit of denominator & numerator

$\lim_{x \to 0} (e^{2x} - 1 - 2x) = 0$

$\lim_{x \to 0} (1 - \cos x) = 0$

$\rightsquigarrow \frac{0}{0}$, try to use L'Hôpital's rule

Step 2: Check $\frac{d}{dx}$(denominator) $\neq 0$ if x is close to, but not equal to, 0.

$\frac{d}{dx}(1 - \cos x) = \sin x$, which is non-zero if $x \in (-\pi, \pi) \setminus \{0\}$.

Step 3: Check whether $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ exists.

$\frac{f'(x)}{g'(x)} = \frac{2e^{2x} - 2}{\sin x}$ Does this have a limit as $x \to 0$?

Repeat Step 1: $\lim_{x \to 0} (2e^{2x} - 2) = 2e^{2 \cdot 0} - 2 = 0$

$\lim_{x \to 0} \sin x = \sin 0 = 0$

$\rightsquigarrow \frac{0}{0}$ again, try L'Hôpital's rule again!

Step 2: Check $\frac{d}{dx} \sin x = \cos x \neq 0$ near 0

Step 3: Check whether $\lim_{x \to 0} \frac{\frac{d}{dx}(2e^{2x} - 2)}{\frac{d}{dx}(\sin x)}$ exists

$$\frac{\frac{d}{dx}(2e^{2x} - 2)}{\frac{d}{dx} \sin x} = \frac{4e^{2x}}{\cos x} \to \frac{4e^{2 \cdot 0}}{\cos 0} = 4 \text{ as } x \to 0$$

$$\therefore \lim_{x\to 0} \frac{4e^{2x}}{\cos x} \text{ exists \& equals } 4$$

$$\therefore \lim_{x\to 0} \frac{2e^{2x}-2}{\sin x} \text{ exists \& equals } 4$$

$$\therefore \lim_{x\to 0} \frac{e^{2x}-1-2x}{1-\cos x} \text{ exists \& equals } 4$$

Solution

$$\lim_{x\to 0} \frac{e^{2x}-1-2x}{1-\cos x}$$

$$= \lim_{x\to 0} \frac{2e^{2x}-2}{\sin x} \qquad \left(\text{L'Hôpital's rule, since } \lim_{x\to 0}(e^{2x}-1-2x)=0,\ \lim_{x\to 0}(1-\cos x)=0\right)$$

$$= \lim_{x\to 0} \frac{4e^{2x}}{\cos x} \qquad \left(\text{L'Hôpital's rule, since } \lim_{x\to 0}(2e^{2x}-2)=0,\ \lim_{x\to 0} x = 0\right)$$

$$= \frac{4e^{2\cdot 0}}{\cos 0}$$

$$= 4.$$

eg.

$$\lim_{x\to 0} \frac{3\sin x - \sin 3x}{x - \sin x}$$

$$= \lim_{x\to 0} \frac{3\cos x - 3\cos 3x}{1-\cos x} \qquad \left(\text{L'Hôpital's rule, } \lim_{x\to 0}(3\sin x - \sin 3x)=0,\ \lim_{x\to 0}(x-\sin x)=0\right)$$

$$= \lim_{x\to 0} \frac{-3\sin x + 9\sin 3x}{\sin x} \qquad \left(\text{L'Hôpital's rule } \lim_{x\to 0}(3\cos x - 3\cos 3x)=0,\ \lim_{x\to 0}(1-\cos x)=0\right)$$

$$= \lim_{x\to 0} \frac{-3\cos x + 27\cos 3x}{\cos x} \qquad \left(\text{L'Hôpital's rule: } \lim_{x\to 0}(-3\sin x + 9\sin 3x)=0,\ \lim_{x\to 0}\sin x = 0\right)$$

$$= \frac{-3+27}{1} = 24.$$

eg. $\lim_{x\to\infty} \frac{x^2}{1+x^2} = \lim_{x\to\infty} \frac{2x}{2x}$ (L'Hôpital's rule, $\lim_{x\to\infty} |1+x^2| = +\infty$)

$= 1$

eg. $\lim_{x\to 0} \frac{x^2}{1+x^2}$ <u>can't</u> use L'Hôpital's rule: $\lim_{x\to 0} x^2 = 0$

$\lim_{x\to 0} (1+x^2) = 1$

In fact, by limit rules earlier, $\lim_{x\to 0} \frac{x^2}{1+x^2} = \frac{0}{1} = 0$.

eg: $\lim_{x\to\infty} \frac{\sinh x}{\cosh x} = \lim_{x\to\infty} \frac{\cosh x}{\sinh x}$ (L'Hôpital's rule, $\lim_{x\to\infty} \sinh x = \infty$)

$= \lim_{x\to\infty} \frac{\sinh x}{\cosh x}$ (L'Hôpital's rule, $\lim_{x\to\infty} \cosh x = \infty$)

$= ?$

Trick: $\lim_{x\to\infty} \frac{\sinh x}{\cosh x} = \lim_{x\to\infty} \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} = \lim_{x\to\infty} \frac{\frac{1}{2}(e^x - e^{-x}) \cdot 2e^{-x}}{\frac{1}{2}(e^x + e^{-x}) \cdot 2e^{-x}}$

$= \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1-0}{1+0} = 1.$

eg. $\lim_{x\to 0^+} x \ln x$ ($0 \cdot \infty$, not a quotient, but can rewrite as a quotient!)

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$

$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$ (L'Hôpital's rule, $\lim_{x\to 0^+} \ln x = -\infty$, $\lim_{x\to 0^+} \frac{1}{x} = +\infty$)

$= \lim_{x\to 0^+} (-x)$ (simplify before trying another L'Hôpital's!)

$= 0$

eg. $\lim_{x\to 0^+} x^{2x}$ (0^0, need to rewrite using exp)

$= \lim_{x\to 0^+} e^{2x\ln x}$

$= e^{\lim_{x\to 0^+} 2x\ln x}$ (since exp is continuous)

$= e^{2\cdot 0}$ (previous eg)

$= 1$

eg. $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$ ($\infty - \infty$; need to combine terms & simplify first)

$= \lim_{x\to 1} \frac{x\ln x - x + 1}{(x-1)\ln x}$ then apply L'Hôpital's ...

eg. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ (1^∞; need to rewrite using exponential)

$= \lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})}$

$= e^{\lim_{x\to\infty} x\ln(1+\frac{1}{x})}$ Now use L'Hôpital's to evaluate

$\lim_{x\to\infty} x\ln(1+\frac{1}{x})$: this limit exists & equals 1

$= e^1$

$= e$.

Towards the proof of L'Hôpital's rule

Cauchy mean-value thm:

Suppose $f: [x,y] \to \mathbb{R}$, $g: [x,y] \to \mathbb{R}$ are both continuous on same closed & bounded interval $[x,y]$

Suppose also that f & g are both differentiable on the open interval (x,y),

& that $g'(t) \neq 0$ $\forall t \in (x,y)$.

Then $\exists Z \in (x,y)$ s.t.

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(Z)}{g'(Z)}.$$

(when $g(x)=x$ this is just the usual mean-value theorem)

Proof of L'Hôpital (using Cauchy's mean-value thm)

Since $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, let L be this limit

(The following argument will work if L is finite. If $L \in \{+\infty, -\infty\}$, a modification of the following will work, but we will not give the details).

Then for any given $\varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $0 < |Z - c| < \delta$, we have

$$\left| \frac{f'(z)}{g'(z)} - L \right| < \varepsilon \quad , \quad \text{ie. } L - \varepsilon < \frac{f'(z)}{g'(z)} < L + \varepsilon \qquad (*)$$

Now pick $x, y \in (c-\delta, c)$ with $x \neq y$.

Then by Cauchy mean-value theorem, $\exists z$ between x & y s.t.

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(z)}{g'(z)}$$

Since $z \in (c-\delta, c)$, $(*)$ holds for this z. Hence

$$L-\varepsilon < \frac{f(x)-f(y)}{g(x)-g(y)} < L+\varepsilon \qquad \text{——} \quad (**)$$

Case 1 $\quad \lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$

Note $(**)$ holds for all $x, y \in (c-\delta, c)$ with $x<y$.

Then we let $y \to c$ in $(**)$. We then get

$$L-\varepsilon \le \frac{f(x)}{g(x)} \le L+\varepsilon \quad , \quad \text{ie.} \quad \left|\frac{f(x)}{g(x)} - L\right| \le \varepsilon$$

Since this holds $\forall\, x \in (c-\delta, c)$, we see that

$$\lim_{x\to c^-} \frac{f(x)}{g(x)} = L$$

A similar argument shows that the right hand limit $\lim_{x\to c^+} \frac{f(x)}{g(x)}$ also exists & equals L. Hence $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists & equals L.

This proves L'Hôpital's rule in this case.

Case 2 $\quad \lim_{x\to\infty} |g(x)| = \infty$

Note that $(**)$ is true $\forall\, x, y \in (c-\delta, c)$ with $x>y$.

Hence $\forall\, x, y$ with $c-\delta < y < x < c$, we have

$$L-\varepsilon < \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(y)}{g(x)}}{1 - \dfrac{g(y)}{g(x)}} < L+\varepsilon.$$

Now $\lim_{x\to c} \frac{g(y)}{g(x)} = 0 = \lim_{x\to c} \frac{f(y)}{g(x)}$. Hence for the ε given above,

$\exists\, \delta_1 > 0$ s.t. if $\quad 0 < |x-c| < \delta_1$, then $\quad \left|\frac{f(y)}{g(x)}\right| < \varepsilon$, $\left|\frac{g(y)}{g(x)}\right| < \varepsilon$

& $\quad 1 - \frac{g(y)}{g(x)} > 0$.

Hence for $y \in (c-\delta, c)$ & $x \in (\max\{c-\delta_1, y\}, c)$.

we have

$$(L-\varepsilon)(1-\varepsilon)-\varepsilon < \frac{f(x)}{g(x)} < (L+\varepsilon)(1+\varepsilon)+\varepsilon$$

which implies

$$\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon\,(L+2+\varepsilon)$$

Since this holds for any x that is sufficiently close to, but less than c, we see that

$$\lim_{x\to c^-} \frac{f(x)}{g(x)} = L.$$

Similarly $\lim\limits_{x\to c^+} \frac{f(x)}{g(x)} = L$. Hence $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$ exists & equals L.

This finishes the proof of L'Hôpital's rule in this case

$\blacksquare$

We now need to prove the Cauchy mean-value theorem:

<u>Proof of Cauchy's mean-value theorem</u>

Let $H(t) = [f(x)-f(y)]\,g(t) - [g(x)-g(y)]\,f(t) \quad \forall t \in [x,y]$.

Then H is continuous on $[x,y]$ & differentiable on (x,y)

$\therefore$ By mean-value theorem, $\exists z \in (x,y)$ s.t. $H'(z) = \dfrac{H(y)-H(x)}{y-x}$.

But $H(y)-H(x) = [f(x)-f(y)][g(y)-g(x)] - [g(x)-g(y)][f(y)-f(x)] = 0$

$\therefore \quad H'(z) = 0$

i.e. $\quad [f(x)-f(y)]\,g'(z) - [g(x)-g(y)]\,f'(z) = 0 \quad$ — $(***)$

Now by assumption, $g'(z) \neq 0$.

Also, $g(x)-g(y) \neq 0$, for otherwise by mean-value theorem, $\exists\, s \in (x,y)$ s.t. $g'(s) = 0$, contradicting our hypothesis that $g'(s) \neq 0 \ \forall s \in (x,y)$.

$\therefore$ We can divide by $g'(z)\,[g(x)-g(y)]$ in $(***)$, & get

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(z)}{g'(z)}$$

as desired.

$\boxtimes$

The Cauchy's mean-value theorem has another important application towards Taylor's polynomials.