**Math 1010C Term 1 2014**
**Supplementary exercises 1**

The following exercises are optional, and for your own enjoyment only.

1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *even* if $f(x) = f(-x)$ for all $x \in \mathbb{R}$, and *odd* if $f(x) = -f(-x)$ for all $x \in \mathbb{R}$.
   (a) Suppose $p\colon \mathbb{R} \to \mathbb{R}$ is the polynomial function
   $$p(x) = \sum_{n=0}^{d} a_n x^n.$$
   Show that $p$ is even if and only if $a_n = 0$ for all odd integers $n$.
   (b) Let $p$ be as in part (a). Find a necessary and sufficient condition on the coefficients of $p$, such that $p$ is odd.
   (c) Is there a function $g\colon \mathbb{R} \to \mathbb{R}$ that is neither even nor odd?
   (d) Is there a function $h\colon \mathbb{R} \to \mathbb{R}$ that is both even and odd?
   (e) Show that every function $f\colon \mathbb{R} \to \mathbb{R}$ can be written as the sum of an odd function and an even function.
   (f) For those who know derivatives already: show that the derivative of an odd function is even, and the derivative of an even function is odd.
   (g) For those who know some linear algebra: Does the set of all even functions from $\mathbb{R}$ to $\mathbb{R}$ form a vector space over $\mathbb{R}$? What about the set of all odd functions?

2. The following generalizes the concept of odd and even functions defined above. Suppose $X$ is a set, and $\theta\colon X \to X$ is an *involution*, in the sense that $\theta \circ \theta$ is the identity function on $X$ (i.e. $\theta(\theta(x)) = x$ for all $x \in X$).
   (a) Show that $\theta\colon X \to X$ is a bijection.
   (b) A function $f\colon X \to \mathbb{R}$ is said to be even with respect to $\theta$ if $f(\theta(x)) = f(x)$ for all $x \in X$. A function $f\colon X \to \mathbb{R}$ is said to be odd with respect to $\theta$ if $f(\theta(x)) = -f(x)$ for all $x \in X$.
      (i) Find all functions $F\colon X \to \mathbb{R}$ that is both even with respect to $\theta$, and odd with respect to $\theta$.
      (ii) Show that every function $f\colon X \to \mathbb{R}$ can be written as the sum $g + h$, where $g\colon X \to \mathbb{R}$ is odd with respect to $\theta$, and $h\colon X \to \mathbb{R}$ is even with respect to $\theta$.
   (c) How is all this relevant to Question 1?
   (d) For those who know complex numbers already: Did it matter that we considered functions that took values in $\mathbb{R}$? What if we considered complex-valued functions?

3. (a) Is there a bijection from $\mathbb{N}$ to $\mathbb{Z}$? If yes, construct one.
   (b) Is there a bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$? (The latter is the set of ordered pairs $(m, n)$, where $m$ and $n$ are both positive integers.) If yes, construct one. (Hint: Draw a picture to visualize $\mathbb{N} \times \mathbb{N}$.)
   (c) Is there a bijection from $\mathbb{N}$ to $\mathbb{Q}$? If yes, construct one. (Hint: Use part (b).)

(d) (Challenge) A sequence of positive integers is an ordered list $(a_1, a_2, a_3, \dots)$, where each $a_i$ is a positive integer. Let's denote the set of all sequences of positive integers by $Y$. Is there a bijection from $\mathbb{N}$ to $Y$?