MATH 1010A/K 2017-18

University Mathematics

Tutorial Notes VIII

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Taylor's Theorem

Let f be a function which is $n+1$-times differentiable on some interval I with some $c \in I$.

Let $x \in I$, then there exist some ξ between x and c, such that

$$f(x) = \underbrace{f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3!}(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n}_{n\text{-th Taylor's Polynomial of } f \text{ centered at } c} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}}_{\text{Reminder term}}.$$

Question

(Q1a) Let $f(x) = \dfrac{1}{\sqrt{1-x}}$ and $p(x)$ be the Taylor Polynomial of degree 4 centered at $x=0$.

- **(i)** Find $p(x)$.
- **(ii)** Show for any $|x| \leq \frac{1}{4}$, we have

$$\left|f(x) - p(x)\right| \leq \frac{7}{3456\sqrt{3}}.$$

(Q1b) Find Taylor Polynomial of $g(x) = \sin^{-1} x$ of degree 9 centered at $x=0$.

(Q2) Show that for all $x > 0$,

$$1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1+x} \leq 1 + \frac{x}{2}.$$

(Q3) By considering appropriate Taylor series expansions, evaluate the limits below:

- **(a)** $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$
- **(b)** $\displaystyle\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x}$
- **(c)** $\displaystyle\lim_{x\to 0} \frac{\sin^3 x}{x\left(1 - \cos x\right)}$
- **(d)** $\displaystyle\lim_{x\to 0} \frac{\ln\left(1 + x^2\right)}{x \sin x}$

(Q4) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying

$$\begin{cases} f'(x) = f(x) + 2e^{-x} \\ f(0) = 1 \end{cases}.$$

- **(a)** Use $f^{(n-1)(x)}$ and e^{-x} to represent $f^{(n)}(x)$.
- **(b)** Find $f^{(n)}(0)$.
- **(c)** Write down the Taylor's Series of f centered at $x=0$.

Answer

(A1a) Note that

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{1-x}}, & f(0) &= 1,\\
f'(x) &= \frac{1}{2\left(1-x\right)^{\frac{3}{2}}}, & f'(0) &= \frac{1}{2},\\
f''(x) &= \frac{3}{4\left(1-x\right)^{\frac{5}{2}}}, & f''(0) &= \frac{3}{4},\\
f'''(x) &= \frac{15}{8\left(1-x\right)^{\frac{7}{2}}}, & f'''(0) &= \frac{15}{8},\\
f^{(4)}(x) &= \frac{105}{16\left(1-x\right)^{\frac{9}{2}}}, & f^{(4)}(0) &= \frac{105}{16},\\
f^{(5)}(x) &= \frac{945}{32\left(1-x\right)^{\frac{11}{2}}}.
\end{aligned}
$$

Then $p(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4$.

If $|x| \leq \frac{1}{4}$, then by Taylor's Theorem, there exist some ξ between x and 0, such that

$$f(x) - p(x) = \frac{f^{(5)}(\xi)}{5!}x^5.$$

Note that $\xi \in \left(-\frac{1}{4}, \frac{1}{4}\right)$, we have

$$\left|f^{(5)}(\xi)\right| = \frac{945}{32\left|1-x\right|^{\frac{11}{2}}} \leq \frac{3^3 \cdot 5 \cdot 7}{2^5\left(\frac{3}{4}\right)^{5+\frac{1}{2}}} = \frac{2^6 \cdot 5 \cdot 7}{3^2\sqrt{3}}.$$

Hence, we have

$$\left|f(x) - p(x)\right| = \frac{\left|f^{(5)}(x)\right|}{5!}|x|^5 \leq \frac{2^6 \cdot 5 \cdot 7}{2^3 \cdot 3^3 \cdot 5\sqrt{3}} \cdot \frac{1}{2^{10}} = \frac{7}{2^7 \cdot 3^3\sqrt{3}} = \frac{7}{3456\sqrt{3}}.$$

(A1b) Note $g(x) = \sin^{-1} x$, then $g(0) = 0$ and $g'(x) = \frac{1}{\sqrt{1-x^2}}$, by (a),

the Taylor Polynomial of degree 4 of $g'(x) = f\left(x^2\right)$ centered at $x = 0$ is

$$p\left(x^2\right) = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8.$$

Then we know $g''(0) = g^{(4)}(0) = g^{(6)}(0) = g^{(8)}(0) = 0$ and

$g'(0) = 1, g^{(3)}(0) = 1, g^{(5)}(0) = 9, g^{(7)}(0) = 225, g^{(9)}(0) = 11025.$

Hence, the required polynomial is

$$x + \frac{1}{6}x^3 + \frac{1}{15}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9.$$

(A2) Let $f(x) = \sqrt{1+x}$, then f is infinitely differentiable, $f(0) = 1$, and

$$\begin{aligned} f'(x) &= \frac{1}{2(1+x)^{\frac{3}{2}}}, & f'(0) &= \frac{1}{2}, \\ f''(x) &= -\frac{3}{4(1+x)^{\frac{5}{2}}}, & f''(0) &= -\frac{3}{4}, \\ f'''(x) &= \frac{15}{8(1+x)^{\frac{7}{2}}}. \end{aligned}$$

Hence, the Taylor's Polynomial of degree 2 of f centered at 0 is

$$p_2(x) = 1 + \frac{x}{2} - \frac{x^2}{8}.$$

Hence, the Taylor's Polynomial of degree 1 of f centered at 0 is

$$p_1(x) = 1 + \frac{x}{2}.$$

Let $x > 0$. By Taylor's Theorem, there exist some $\xi_1, \xi_2 \in (0, x)$, such that

$$\begin{aligned} f(x) - p_1(x) &= f''(\xi_1)x^2 & f(x) - p_2(x) &= f'''(\xi_2)x^3 \\ &= -\frac{3x^2}{4(1+\xi)^{\frac{5}{2}}} & &= \frac{15x^3}{8(1+\xi_2)^{\frac{7}{2}}} \\ &\le 0 & &\ge 0 \end{aligned}$$

That is, $p_2(x) \le f(x) \le p_1(x)$, i.e. $1 + \frac{x}{2} - \frac{x^2}{8} \le \sqrt{1+x} \le 1 + \frac{x}{2}$.

(A3) *Remark: Try to compute the Taylor's Polynomial and I will skip it.*

(a) Note that $\frac{1}{x} - \frac{1}{e^x - 1} = \frac{e^x - x - 1}{x\left(e^x - 1\right)}$, by your exercise,

the Taylor's Polynomial of degree 2 of $e^x - x - 1$ centered at 0 is $\frac{x^2}{2}$,

and the Taylor's Polynomial of degree 2 of $x\left(e^x - 1\right)$ centered at 0 is x^2.

Let $x \in \mathbb{R}$. By Taylor's Theorem, there exist some ξ, η between 0 and x, such that

$$e^x - x - 1 = \frac{x^2}{2} + \frac{e^\xi}{6}x^3, \qquad x\left(e^x - 1\right) = x^2 + \frac{e^\eta(\eta+3)}{6}x^3.$$

Then we have

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0}\frac{e^x - x - 1}{x\left(e^x - 1\right)} = \lim_{x\to 0}\frac{\frac{x^2}{2} + \frac{e^\xi}{6}x^3}{x^2 + \frac{e^\eta(\eta+3)}{6}x^3} = \lim_{x\to 0}\frac{\frac{1}{2} + \frac{e^\xi}{6}x}{1 + \frac{e^\eta(\eta+3)}{6}x} = \frac{1}{2}.$$

(b) The Taylor's Polynomial of degree 3 of $2\sin x - \sin 2x$ centered at 0 is x^3,

and the Taylor's Polynomial of degree 3 of $x - \sin x$ centered at 0 is $\frac{x^3}{6}$.

Let $x \in \mathbb{R}$. By Taylor's Theorem, there exist some number C, D,

note that C, D depends on x and bounded near 0, such that

$$2\sin x - \sin 2x = x^3 + Cx^4, \qquad x - \sin x = \frac{x^3}{6} + Dx^4.$$

Then we have

$$\lim_{x\to 0}\frac{2\sin x - \sin 2x}{x - \sin x} = \lim_{x\to 0}\frac{x^3 + Cx^4}{\frac{x^3}{6} + Dx^4} = \lim_{x\to 0}\frac{1 + Cx}{\frac{1}{6} + Dx} = 6$$

(c) The Taylor's Polynomial of degree 1 of $\sin x$ centered at 0 is x and

The Taylor's Polynomial of degree 2 of $1 - \cos x$ centered at 0 is $\frac{x^2}{2}$ and

Let $x \in \mathbb{R}$. By Taylor's Theorem, there exist some number C, D,

note that C, D depends on x and bounded near 0, such that

$$\sin x = x + Cx^2, \qquad 1 - \cos x = \frac{x^2}{2} + Dx^3.$$

Then we have

$$\lim_{x\to 0} \frac{\sin^3 x}{x\left(1 - \cos x\right)} = \lim_{x\to 0} \frac{\left(x + Cx^2\right)^3}{x\left(\frac{x^2}{2} + Dx^3\right)} = \lim_{x\to 0} \frac{(1 + Cx)^3}{\frac{1}{2} + Dx} = 2.$$

(d) The Taylor's Polynomial of degree 2 of $\ln\left(1 + x^2\right)$ centered at 0 is $2x^2$ and

the Taylor's Polynomial of degree 1 of $\sin x$ centered at 0 is x.

Let $x \in \mathbb{R}$. By Taylor's Theorem, there exist some number C, D,

note that C, D depends on x and bounded near 0, such that

$$\ln\left(1 + x^2\right) = 2x^2 + Cx^3, \qquad \sin x = x + Dx^2.$$

Then we have

$$\lim_{x\to 0} \frac{2x^2 + Cx^3}{x\left(x + Dx^2\right)} = \lim_{x\to 0} \frac{2 + Cx}{1 + Dx} = 2.$$

(A4) Note that $\frac{d^n}{d^n x} e^{-x} = \begin{cases} e^{-x}, & \text{if } n \text{ is even} \\ -e^{-x}, & \text{if } n \text{ is odd} \end{cases}$.

Hence, $f^{(n)}(x) = f^{(n-1)}(x) + 2\frac{d^{n-1}}{d^{n-1}x} e^{-x} = \begin{cases} f^{(n-1)}(x) - 2e^{-x}, & \text{if } n \text{ is even} \\ f^{(n-1)}(x) + 2e^{-x}, & \text{if } n \text{ is odd} \end{cases}$.

Let $P(n)$ be the statement that $f^{(n)}(0) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$.

By assumption, $P(0)$ is true.

Assume $P(k)$ is true for some $k \in \mathbb{N}$,

(Case 1) Suppose k is even, that is $f^{(k)}(0) = 1$, hence $f^{(k+1)}(0) = f^{(k)}(0) + 2e^{-0} = 1 + 2 = 3$.

(Case 2) Suppose k is odd, that is $f^{(k)}(0) = 3$, hence $f^{(k+1)}(0) = f^{(k)}(0) - 2e^{-0} = 3 - 2 = 1$.

So $P(k + 1)$ is true.

By First Principal of Mathematical Induction, we know $f^{(n)}(0) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$.

The Taylor's Series of f centered at $x = 0$ is

$$1 + 3x + \frac{1}{2}x^2 + \frac{3}{3!}x^3 + \frac{1}{4!}x^4 + ... + \frac{1}{(2n)!}x^{2n} + \frac{3}{(2n+1)!}x^{2n+1} + ...$$

OR

$$\sum_{n=0}^{\infty} \left(\frac{1}{(2n)!}x^{2n} + \frac{3}{(2n+1)!}x^{2n+1}\right)$$

OR

$$\sum_{n=0}^{\infty} \frac{1 + (-1)^{n+1}}{n!} x^n$$