MATH 1010A/K 2017-18

University Mathematics

Tutorial Notes IV

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Question

(Q1) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$.

Show f is differentiable at $x = 0$ and hence find $f'(x)$ for any $x \in \mathbb{R}$. Is f' continuous at $x = 0$?

(Q2) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x^2 \tan^{-1} \dfrac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$.

Show f is differentiable at $x = 0$ and hence find $f'(x)$ for any $x \in \mathbb{R}$. Is f' continuous at $x = 0$?

(Q3) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} \sin x & , x < \pi \\ ax + b & , x \geq \pi \end{cases}$.

Find $a, b \in \mathbb{R}$ such that f is differentiable on $\mathbb{R}$.

(Q4) For all $m = 0, 1, 2, 3, ..,$ show the Chebyshev Polynomial $T_m : \mathbb{R} \to \mathbb{R}$ by

$$T_m(x) = \frac{1}{2^{m-1}} \cos\left(m \cos^{-1} x\right)$$

satisfy $\left(1 - x^2\right) T_m''(x) - xT_m'(x) + m^2 T_m(x) = 0$.

(Q5) Find $\dfrac{dy}{dx}$ if

(a) $xe^{xy} = 1$,

(b) $\cos\left(\dfrac{y}{x}\right) = \ln(x + y)$,

(c) $y = x^{\ln x}$.

(Q6) Prove that for any $x > 0$, we have

$$\frac{x}{1+x} < \ln\left(1 + x\right) < x.$$

And hence, show for any $x > 0$, we have

$$\frac{1}{1+x} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}.$$

(Q7) Let $f(x)$ be a function defined on $[0, \infty)$ such that

(i) $f(0) = 0$,

(ii) f is continuous on $[0, \infty)$

(iii) f is differentiable on $(0, \infty)$ and f' is monotonic increasing on $(0, \infty)$.

Prove that

$$f(a + b) \geq f(a) + f(b)$$

for any $0 \leq a \leq b \leq a + b$.

Answer

(A1) By Sandwich Theorem in the special tutorial notes, we have $\lim_{h\to 0} h\sin\frac{1}{h} = 0$. Then

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h}-0}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0.$$

Hence, f differentiable at 0 with $f'(0)=0$. Using product rule, we have

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & , x\neq 0 \\ 0 & , x = 0 \end{cases}.$$

Since $\lim_{x\to 0} 2x\sin\frac{1}{x}$ exists ($=0$) and $\lim_{x\to 0}\cos\frac{1}{x}$ does NOT exist (Why?),

we must have $\lim_{x\to 0} f'(x)$ does NOT exist,

otherwise, we have $\lim_{x\to 0}\cos\frac{1}{x} = \lim_{x\to 0}\left(2x\sin\frac{1}{x} - f'(x)\right)$ exists in $\mathbb{R}$. (Which is a contradiction)

Therefore, f' is NOT continuous at 0.

(A2) Note that $-\frac{\pi}{2}\leq \tan^{-1}\frac{1}{x}\leq\frac{\pi}{2}$ for any $x\neq 0$.

By Sandwich Theorem (try to write the proof down!), we have $\lim_{h\to 0} h\tan^{-1}\frac{1}{h} = 0$. Then

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\tan^{-1}\frac{1}{h}-0}{h} = \lim_{h\to 0} h\tan^{-1}\frac{1}{h} = 0.$$

Hence, f differentiable at 0 with $f'(0)=0$.

Note if $y=\tan^{-1}x$, then

$$\begin{aligned} \tan y &= x \\ \left(\sec^2 y\right)\frac{dy}{dx} &= 1 \\ \frac{d}{dx}\tan^{-1}x = \frac{dy}{dx} &= \cos^2 y = \cos^2\left(\tan^{-1}x\right) = \frac{1}{1+x^2}. \end{aligned}$$

Note that the last step is obtained by drawing a triangle. Using product rule, we have

$$f'(x) = \begin{cases} 2x\tan^{-1}\frac{1}{x} + \dfrac{x^2}{1+x^2} & , x\neq 0 \\ 0 & , x = 0 \end{cases}.$$

Note $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x\tan^{-1}\frac{1}{x} + \frac{x^2}{1+x^2}\right) = 0+0 = 0 = f'(0)$,

so f' continuous at 0.

(A3) When $x<\pi$, $f(x)=\sin x$ which is a differentiable function.

When $x>\pi$, $f(x)=ax+b$ which is a differentiable function.

It suffice to find $a,b\in\mathbb{R}$ such that f is differentiable at $x=\pi$.

Suppose such differentiable f exists, f must be continuous at $x=0$, that is

$$\begin{aligned} \lim_{x\to\pi^-} f(x) &= f(\pi) = \lim_{x\to\pi^+} f(x) \\ 0 = \lim_{x\to\pi^-}\sin x &= a\pi + b = \lim_{x\to\pi^+}(ax+b) = a\pi+b. \end{aligned}$$

Hence, we have $b = -a\pi$, that is $f(\pi) = a\pi + b = 0$.

Since f is differentiable at $x = 0$, $\lim_{h\to 0} \frac{f(\pi+h) - f(\pi)}{h}$ exists, that is

$$\lim_{h\to 0^-} \frac{f(\pi+h) - f(\pi)}{h} = \lim_{h\to 0^+} \frac{f(\pi+h) - f(\pi)}{h}$$
$$\lim_{h\to 0^-} \frac{\sin h}{h} = \lim_{h\to 0^-} \frac{\sin(\pi+h)}{h} = \lim_{h\to 0^+} \frac{a(\pi+h)+b}{h} = \lim_{h\to 0^+} \frac{ah}{h}.$$

Hence, $a = 1$ and $b = -\pi$.

(A4) Note if $y = \cos^{-1} x$, then

$$\begin{aligned}
\cos y &= x \\
-\sin y \frac{dy}{dx} &= 1 \\
\frac{d}{dx}\cos^{-1} x = \frac{dy}{dx} &= -\csc y = -\csc\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}
\end{aligned}$$

Note that the last step is obtained by drawing a triangle. Then

$$\begin{aligned}
T_m(x) &= \frac{1}{2^{m-1}} \cos\left(m\cos^{-1} x\right), \\
T_m'(x) &= \frac{-1}{2^{m-1}} \sin\left(m\cos^{-1} x\right)\left(\frac{-m}{\sqrt{1-x^2}}\right) \\
&= \frac{m}{2^{m-1}} \frac{\sin\left(m\cos^{-1} x\right)}{\sqrt{1-x^2}}, \\
T_m''(x) &= \frac{m}{2^{m-1}} \frac{\sqrt{1-x^2}\cos\left(m\cos^{-1} x\right)\left(\frac{-m}{\sqrt{1-x^2}}\right) - \frac{-2x}{2\sqrt{1-x^2}}\sin\left(m\cos^{-1} x\right)}{1-x^2} \\
&= \frac{m}{2^{m-1}} \frac{x\frac{\sin\left(m\cos^{-1} x\right)}{\sqrt{1-x^2}}}{1-x^2} - \frac{1}{2^{m-1}} \frac{m^2\cos\left(m\cos^{-1} x\right)}{1-x^2}.
\end{aligned}$$

Hence, $\left(1-x^2\right)T_m''(x) - xT_m'(x) + m^2 T_m(x) = 0$.

(A5) Using product rule and chain rule,

(a) we have

$$\begin{aligned}
xe^{xy} &= 1 \\
x\frac{d}{dx}e^{xy} + e^{xy}\frac{d}{dx}x &= \frac{d}{dx}1 \\
xe^{xy}\left(x\frac{d}{dx}y + y\frac{d}{dx}x\right) + e^{xy} &= 0 \\
xe^{xy}\left(x\frac{dy}{dx} + y\right) + e^{xy} &= 0 \\
\frac{dy}{dx} &= -\frac{1+xy}{x^2}.
\end{aligned}$$

(b) we have

$$
\begin{aligned}
\cos\left(\frac{y}{x}\right) &= \ln(x+y)\\
-\sin\left(\frac{y}{x}\right)\frac{d}{dx}\left(\frac{y}{x}\right) &= \frac{1}{x+y}\frac{d}{dx}(x+y)\\
-\sin\left(\frac{y}{x}\right)\frac{x\frac{d}{dx}y - y\frac{d}{dx}x}{x^2} &= \frac{1}{x+y}\left(1+\frac{dy}{dx}\right)\\
-\sin\left(\frac{y}{x}\right)\frac{x\frac{dy}{dx} - y}{x^2} &= \frac{1}{x+y}\left(1+\frac{dy}{dx}\right)\\
y(x+y)\sin\left(\frac{y}{x}\right) - x(x+y)\sin\left(\frac{y}{x}\right)\frac{dy}{dx} &= x^2 + x^2\frac{dy}{dx}\\
\frac{dy}{dx} &= \frac{y(x+y)\sin\left(\frac{y}{x}\right) - x^2}{x(x+y)\sin\left(\frac{y}{x}\right) + x^2}.
\end{aligned}
$$

(c) we have

$$
\begin{aligned}
y &= x^{\ln x}\\
\ln y &= \ln\left(x^{\ln x}\right) = \left(\ln x\right)^2\\
\frac{1}{y}\frac{dy}{dx} &= 2\left(\ln x\right)\left(\frac{1}{x}\right)\\
\frac{dy}{dx} &= \frac{2y\ln x}{x} = 2x^{(\ln x)-1}\ln x
\end{aligned}
$$

(A6) Define $f : (0,\infty) \to \mathbb{R}$ by $f(x) = \ln x$ for any $x > 0$.

Fixed any $x > 0$, note f is continuous on $\left[1, x+1\right]$ and differentiable on $\left(1, x+1\right)$.

By (Lagrange's) Mean Value Theorem, there exists some ξ with $1 < \xi < x+1$, such that

$$\frac{\ln\left(x+1\right)}{x} = \frac{f(x+1) - f(1)}{x+1-1} = f'(\xi) = \frac{1}{\xi}.$$

Note $0 < 1 < \xi < x+1$, so

$$
\begin{aligned}
1 > &\frac{1}{\xi} > \frac{1}{x+1}\\
1 > &\frac{\ln\left(x+1\right)}{x} > \frac{1}{x+1}\\
x > &\ln\left(x+1\right) > \frac{x}{x+1} \quad \text{Since } x > 0.
\end{aligned}
$$

Therefore, $\frac{x}{1+x} < \ln\left(1+x\right) < x$ for any $x > 0$.

Fixed any $x > 0$, note $y = \frac{1}{x} > 0$, hence

$$
\begin{aligned}
&\frac{y}{1+y} < \ln\left(1+y\right) < y\\
\text{that is} \quad &\frac{1}{1+x} = \frac{\frac{1}{x}}{1+\frac{1}{x}} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}.
\end{aligned}
$$

Therefore, $\frac{1}{1+x} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$ for any $x > 0$.

(A7) Note that the case that $a = 0$ is trival since $f(0) = 0$.

Now fixed any a, b with $0 < a \leq b < a + b$,

since f is continuous on $[0, a]$ and f is differentiable on $(0, a)$,

by (Lagrange's) Mean Value Theorem, there exists some η with $0 < \eta < a$, such that

$$(*) \quad \frac{f(a)}{a} = \frac{f(a) - f(0)}{a - 0} = f'(\eta) \underset{\text{increasing}}{\overset{f' \text{ is}}{\leq}} f'(a)$$

Since f is continuous on $[b, a + b]$ and f is differentiable on $(b, a + b)$,

by (Lagrange's) Mean Value Theorem, there exists some ξ with $b < \xi < a + b$, such that

$$\frac{f(a + b) - f(b)}{a} = \frac{f(a + b) - f(b)}{a + b - b} = f'(\xi) \underset{\text{increasing}}{\overset{f' \text{ is}}{\geq}} f'(a) \overset{(*)}{\geq} \frac{f(a)}{a}.$$

Since $a > 0$, we have $f(a + b) \geq f(a) + f(b)$.

Therefore, $f(a + b) \geq f(a) + f(b)$ for any $0 \leq a \leq b \leq a + b$.