THE TRIGONOMETRIC FUNCTIONS

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In this note we *define* the sine and cosine by the following power series. For any $x \in \mathbb{R}$, we define

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

This is possible since the radius of convergence of the two power series above are infinite. One may thus differentiate term by term, and obtain that

$$\frac{d}{dx} \sin x = \cos x, \quad \text{and} \quad \frac{d}{dx} \cos x = -\sin x.$$

Also, from this definition, it is clear that $\sin 0 = 0$, $\cos 0 = 1$, and $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$ for all $x \in \mathbb{R}$.

Below we deduce all familiar properties of the sine and cosine from this definition.

The key is the following corollary of the mean value theorem:

Corollary 1. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$. If*

$$f'(x) = 0$$

for all $x \in \mathbb{R}$, then f is constant on $\mathbb{R}$; in particular, $f(x) = f(0)$ for all $x \in \mathbb{R}$.

Hence we have:

Proposition 2.

$$\sin^2 x + \cos^2 x = 1 \quad \textit{for all } x \in \mathbb{R}. \tag{1}$$

Proof. Let

$$f(x) = \sin^2 x + \cos^2 x.$$

Then

$$f(0) = \sin^2 0 + \cos^2 0 = 1.$$

Also, f is differentiable on $\mathbb{R}$, and

$$f'(x) = 2 \sin x \cos x + 2 \cos x(-\sin x) = 0$$

for all $x \in \mathbb{R}$. Hence Corollary 1 implies that $f(x) = f(0) = 1$ for all $x \in \mathbb{R}$, which is the desired conclusion. □

Next we will prove the compound angle formula:

Proposition 3. *For all $x, y \in \mathbb{R}$, we have*

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \tag{2}$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y \tag{3}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{4}$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y. \tag{5}$$

In fact, it is easy to see that (3) follows from (2) by replacing y by $-y$ (and using that $\sin(-y) = -\sin y$, $\cos(-y) = \cos y$). Similarly, (5) follows from (4) by replacing y by $-y$. Hence it suffices to prove (2) and (4).

To do so, one way is to proceed via the following result about differential equation:

Lemma 4. *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a twice differentiable function, with*

$$f''(x) + f(x) = 0 \quad \text{for all } x \in \mathbb{R},$$

and

$$f(0) = f'(0) = 0.$$

Then $f(x) = 0$ for all $x \in \mathbb{R}$.

It says the zero function is the *only* function that satisfies the conditions of the Lemma.

Proof of Lemma 4. We introduce two auxiliary functions $F\colon \mathbb{R} \to \mathbb{R}$, $G\colon \mathbb{R} \to \mathbb{R}$, such that

$$F(x) = f(x)\cos x - f'(x)\sin x, \quad G(x) = f(x)\sin x + f'(x)\cos x$$

for all $x \in \mathbb{R}$. Then F and G are both differentiable, and

$$F'(x) = -f(x)\sin x + f'(x)(\cos x - \cos x) - f''(x)\sin x = 0$$

$$G'(x) = f(x)\cos x + f'(x)(\sin x - \sin x) + f''(x)\cos x = 0$$

for all $x \in \mathbb{R}$ (the last equalities uses the assumption $f'' + f = 0$). Hence both F and G are constants. Also, by our assumptions on f and f', we have

$$F(0) = f(0)\cos 0 - f'(0)\sin 0 = f(0) = 0$$

and

$$G(0) = f(0)\sin 0 + f'(0)\cos 0 = f'(0) = 0.$$

Hence

$$F(x) = 0 = G(x) \quad \text{for all } x \in \mathbb{R},$$

i.e.

$$\begin{cases} f(x)\cos x - f'(x)\sin x = 0 \\ f(x)\sin x + f'(x)\cos x = 0 \end{cases} \quad \text{for all } x \in \mathbb{R}.$$

We now solve for $f(x)$ by eliminating $f'(x)$: we multiply the first equation by $\cos x$, and the second equation by $\sin x$, and take the sum of the resulting equations. Then

$$(\cos^2 x + \sin^2 x) f(x) = 0$$

for all $x \in \mathbb{R}$, which by (1) implies $f(x) = 0$ for all $x \in \mathbb{R}$, as desired. □

We can now prove (2) and (4).

Proof of (2). Fix $y \in \mathbb{R}$. Let $f(x) = \sin(x+y) - \sin x \cos y - \cos x \sin y$. We want to show that $f(x) = 0$ for all $x \in \mathbb{R}$. To do so, we appeal to our lemma. First,

$$f''(x) = -\sin(x+y) + \sin x \cos y + \cos x \sin y = -f(x)$$

for all $x \in \mathbb{R}$, i.e. $f''(x) + f(x) = 0$ for all $x \in \mathbb{R}$. Next,

$$f(0) = \sin(0+y) - \sin 0 \cos y - \cos 0 \sin y = 0.$$

Also,

$$f'(x) = \cos(x+y) - \cos x \cos y + \sin x \sin y$$

for all $x \in \mathbb{R}$. Hence

$$f'(0) = \cos(0+y) - \cos 0 \cos y + \sin 0 \sin y = 0.$$

By the lemma, it follows that $f(x) = 0$ for all $x \in \mathbb{R}$, as desired. □

The proof of (4) is similar, and left as an exercise.

From Proposition 3 we have the usual double angle formula:

Proposition 5. *For all $x \in \mathbb{R}$, we have*

$$\sin(2x) = 2\sin x \cos x \tag{6}$$

$$\cos(2x) = \cos^2 x - \sin^2 x \tag{7}$$

$$\cos(2x) = 2\cos^2 x - 1 \tag{8}$$

$$\cos(2x) = 1 - 2\sin^2 x \tag{9}$$

Proof. (6) and (7) just follows from (2) and (4) by setting $y = x$. (8) and (9) follow from (7) and an application of (1). □

Hence we have the half-angle formula:

Proposition 6. *For all $x \in \mathbb{R}$, we have*

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \tag{10}$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \tag{11}$$

Proof. Just rearrange (8) and (9). □

Sometimes the following triple-angle formula are also useful:

Proposition 7. *For all $x \in \mathbb{R}$, we have*

$$\sin(3x) = 3\sin x - 4\sin^3 x \tag{12}$$

$$\cos(3x) = 4\cos^3 x - 3\cos x \tag{13}$$

Proof. For all $x \in \mathbb{R}$, we have (by the compound and double angle formula)

$$\begin{aligned}
\sin(3x) &= \sin(x + 2x) \\
&= \sin x \cos 2x + \cos x \sin 2x \\
&= \sin x(1 - 2\sin^2 x) + 2\sin x \cos^2 x \\
&= \sin x(1 - 2\sin^2 x) + 2\sin x(1 - \sin^2 x) \\
&= 3\sin x - 4\sin^3 x,
\end{aligned}$$

and

$$\begin{aligned}
\cos(3x) &= \cos(x + 2x) \\
&= \cos x \cos 2x - \sin x \sin 2x \\
&= \cos x(2\cos^2 x - 1) - 2\sin^2 x \cos x \\
&= \cos x(2\cos^2 x - 1) - 2(1 - \cos^2 x)\cos x \\
&= 4\cos^3 x - 3\cos x.
\end{aligned}$$

□

Also from Proposition 3, we have the following product-to-sum formula (allowing one to convert the product of two trigonometric functions into a sum):

Proposition 8. *For all $x, y \in \mathbb{R}$, we have*

$$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y)) \tag{14}$$

$$\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y)) \tag{15}$$

$$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y)) \tag{16}$$

Proof. (14) follows by averaging (2) and (3). (15) follows by averaging (4) and (5). (16) follows by subtracting (4) from (5), and dividing by 2. □

Next, we look at why sine and cosine are periodic. We know $\cos 0 = 1$, and we claim $\cos 2 < 0$: in fact $\cos 2$ is defined by an alternating series, and thus

$$\begin{aligned}
\cos 2 &= 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \dots \\
&\leq 1 - \frac{2^2}{2!} + \frac{2^4}{4!} \\
&< 0.
\end{aligned}$$

Since $\cos x$ is continuous on $\mathbb{R}$, there exists a smallest positive number a such that $\cos a = 0$. (Such a is unique.) We now make the following *definition*:

Definition 1. We define the real number π so that

$$\pi := 2a,$$

where a is as above.

Note that if we adopt this definition, then we will have to *prove* that the circumference of a circle of radius r is $2\pi r$; that we will do when we learn about integration, which allows one to make sense of the length of a curve.

Adopting the above definition of π, we show the following:

Proposition 9.

$$\cos x > 0 \quad \textit{for all } x \in [0, \frac{\pi}{2}), \tag{17}$$

and

$$\sin x > 0 \quad \textit{for all } x \in (0, \frac{\pi}{2}]. \tag{18}$$

Also,

$$\cos \frac{\pi}{2} = 0, \tag{19}$$

and

$$\sin \frac{\pi}{2} = 1 \tag{20}$$

Proof. (19) follows immediately from our definition of π.

Now note that $\frac{\pi}{2}$ is the first positive zero of cos, and that $\cos 0 = 1 > 0$. Hence by continuity of cos, we conclude that (17) holds. In particular, $\frac{d}{dx} \sin x = \cos x > 0$ for $x \in [0, \frac{\pi}{2})$. Hence sin is strictly increasing on $[0, \frac{\pi}{2}]$. Since $\sin 0 = 0$, it follows that (18) holds; in particular, $\sin \frac{\pi}{2} > 0$.

We can now determine the value of $\sin \frac{\pi}{2}$. In fact, from (1), we get

$$\sin^2 \frac{\pi}{2} + \cos^2 \frac{\pi}{2} = 1,$$

so from (19), we get $\sin \frac{\pi}{2} = \pm 1$. But we already knew that $\sin \frac{\pi}{2} > 0$, so $\sin \frac{\pi}{2}$ must be 1 (and not -1). This proves (20). □

We can now prove the periodicity of sine and cosine:

Proposition 10. *For all $x \in \mathbb{R}$, we have*

$$\sin(x + \frac{\pi}{2}) = \cos x \tag{21}$$

$$\cos(x + \frac{\pi}{2}) = -\sin x \tag{22}$$

$$\sin(x + \pi) = -\sin x \tag{23}$$

$$\cos(x + \pi) = -\cos x \tag{24}$$

$$\sin(x + 2\pi) = \sin x \tag{25}$$

$$\cos(x + 2\pi) = \cos x \tag{26}$$

Proof. To prove (21), let $f(x) = \sin(x + \frac{\pi}{2}) - \cos x$. We want to invoke Lemma 4. To do so, let's check that

$$f''(x) + f(x) = [-\sin(x + \frac{\pi}{2}) + \cos x] + [\sin(x + \frac{\pi}{2}) - \cos x] = 0$$

for all $x \in \mathbb{R}$. Also,

$$f(0) = \sin \frac{\pi}{2} - \cos 0 = 1 - 1 = 0,$$

$$f'(0) = \cos \frac{\pi}{2} + \sin 0 = 0 + 0 = 0.$$

So one can apply Lemma 4, and conclude that $f(x) = 0$ for all $x \in \mathbb{R}$. This proves (21).

The proof of (22) is similar, and left to the reader.

Finally, (23) follows by first applying (21), and then (22):

$$\sin(x + \pi) = \cos(x + \frac{\pi}{2}) = -\sin x,$$

Similarly one can prove (24). Also, (25) follows from applying (23) twice:

$$\sin(x + 2\pi) = -\sin(x + \pi) = \sin x.$$

Similarly one can prove (26). □

We can now sketch the graph of sin and cos. First, from the periodicity established in the earlier proposition, it suffices to sketch sin and cos on $[0, \frac{\pi}{2}]$. But then this is easy: for instance, to sketch sin, knowing that

$$\sin 0 = 0, \quad \sin \frac{\pi}{2} = 1, \quad \sin x \text{ is continuous for } x \in [0, \frac{\pi}{2}],$$

$$\frac{d}{dx} \sin x = \cos x > 0 \quad \text{for } x \in (0, \frac{\pi}{2}),$$

$$\frac{d^2}{dx^2} \sin x = -\sin x < 0 \quad \text{for } x \in (0, \frac{\pi}{2}),$$

allows us to sketch fairly well the graph of sin on $[0, \frac{\pi}{2}]$; it is strictly increasing and concave there. Similarly one can sketch the graph of cos on $[0, \frac{\pi}{2}]$, hence on the whole $\mathbb{R}$ by periodicity.

Another useful formula is

Proposition 11. *For any $x \in \mathbb{R}$,*

$$\sin(\frac{\pi}{2} - x) = \cos x \tag{27}$$

$$\cos(\frac{\pi}{2} - x) = \sin x \tag{28}$$

Proof. To prove (27), just use (21) with x replaced by $-x$, and use that cos is even:

$$\sin(\frac{\pi}{2} - x) = \cos(-x) = \cos x.$$

Similarly one can prove (28). □

If one wants to calculate the values of sin and cos at some special angles, we can also do so:

Proposition 12.

$$\sin \pi = 0 \tag{29}$$

$$\cos \pi = -1 \tag{30}$$

$$\sin \frac{3\pi}{2} = -1 \tag{31}$$

$$\cos \frac{3\pi}{2} = 0 \tag{32}$$

$$\sin 2\pi = 0 \tag{33}$$

$$\cos 2\pi = 1 \tag{34}$$

Proof. Just use (23) and (24) by setting $x = 0$, $\frac{\pi}{2}$ or π. □

Proposition 13.

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \tag{35}$$

$$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \tag{36}$$

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \tag{37}$$

$$\cos \frac{\pi}{3} = \frac{1}{2} \tag{38}$$

$$\sin \frac{\pi}{6} = \frac{1}{2} \tag{39}$$

$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \tag{40}$$

Proof. (35) follows from (11), (19) and (18):

$$\sin^2 \frac{\pi}{4} = \frac{1}{2}(1 - \cos \frac{\pi}{2}) = \frac{1}{2}$$

so from (18) we get $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Similarly one can prove (36).

To prove (37), note that from (12) and (29),

$$3 \sin \frac{\pi}{3} - 4 \sin^3 \frac{\pi}{3} = \sin \pi = 0.$$

Hence $\sin \frac{\pi}{3}$ is a root of $3x - 4x^3 = 0$. Now we solve this cubic equation: if $3x - 4x^3 = 0$, then $x = 0$, or $3 - 4x^2 = 0$, so $x = 0, \pm \frac{\sqrt{3}}{2}$. But $\sin \frac{\pi}{3} > 0$ by (18). Hence $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, as desired.

(38) then follows from (1) and (17):

$$\cos^2 \frac{\pi}{6} = 1 - \sin^2 \frac{\pi}{6} = 1 - \frac{3}{4} = \frac{1}{4}$$

so $\cos \frac{\pi}{3} > 0$ implies $\cos \frac{\pi}{3} = \frac{1}{2}$.

Also, (39) follows from (27) and (38):

$$\sin \frac{\pi}{6} = \sin(\frac{\pi}{2} - \frac{\pi}{3}) = \cos \frac{\pi}{3} = \frac{1}{2}.$$

Similarly one can deduce (40). □

In what follows, we deduce some properties of tangent and cotangent: recall

$$\tan x = \frac{\sin x}{\cos x}$$

when $\cos x \neq 0$, and

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

when $\sin x \neq 0$. Also

$$\sec x = \frac{1}{\cos x}$$

when $\cos x \neq 0$, and

$$\csc x = \frac{1}{\sin x}$$

when $\sin x \neq 0$.

Proposition 14.

$$1 + \tan^2 x = \sec^2 x \quad \textit{if } \cos x \neq 0 \tag{41}$$

$$1 + \cot^2 x = \csc^2 x \quad \textit{if } \sin x \neq 0 \tag{42}$$

Proof. To prove (41), note that if $\cos x \neq 0$, then

$$1 + \tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

The proof of (42) is similar. □

The following is a compound angle formula for tangent:

Proposition 15. *For all $x, y \in \mathbb{R}$, we have*

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \quad \textit{if } \cos(x + y) \neq 0, \cos x \neq 0 \textit{ and } \cos y \neq 0 \tag{43}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \quad \textit{if } \cos(x - y) \neq 0, \cos x \neq 0 \textit{ and } \cos y \neq 0 \tag{44}$$

Proof. To prove (43), note that for $x, y \in \mathbb{R}$, if $\cos(x - y) \neq 0$, then

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

If further $\cos x \neq 0$ and $\cos y \neq 0$, then one divides both the numerator and denominator by $\cos x \cos y$, and obtain (43).

(44) follows from (43) by replacing y by $-y$, and using that $\tan y = -\tan y$. □

The following are formula that expresses $\sin x$, $\cos x$ and $\tan x$ in terms of $\tan \frac{x}{2}$ only. They are useful formula in computing integrals involving trigonometric functions (via a technique called t-substitution).

Proposition 16. *For all $x \in \mathbb{R}$, we have*

$$\tan x = \frac{2\tan \frac{x}{2}}{1-\tan^2 \frac{x}{2}} \quad \text{if } \cos \frac{x}{2} \neq 0 \text{ and } \cos x \neq 0 \tag{45}$$

$$\sin x = \frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} \quad \text{if } \cos \frac{x}{2} \neq 0 \tag{46}$$

$$\cos x = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} \quad \text{if } \cos \frac{x}{2} \neq 0 \tag{47}$$

Proof. (45) follows from (43) by replacing both x and y there by $\frac{x}{2}$.

To prove (46), note that if $\cos \frac{x}{2} \neq 0$, we can simplify the denominator of the right hand side, and obtain:

$$\frac{2\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{2\tan \frac{x}{2}}{\sec^2 \frac{x}{2}} = 2\tan \frac{x}{2}\cos^2 \frac{x}{2} = 2\sin \frac{x}{2}\cos \frac{x}{2} = \sin x$$

(the last equality following from (6).)

To prove (47), note that if $\cos \frac{x}{2} \neq 0$, we can simplify the denominator of the right hand side, and obtain:

$$\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{1-\tan^2 \frac{x}{2}}{\sec^2 \frac{x}{2}} = (1-\tan^2 \frac{x}{2})\cos^2 \frac{x}{2} = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos x$$

(the last equality following from (7).) □

Finally, here are some half-angle formula for tangent:

Proposition 17. *For all $x \in \mathbb{R}$, we have*

$$\tan \frac{x}{2} = \frac{\sin x}{1+\cos x} \quad \text{if } \cos \frac{x}{2} \neq 0 \tag{48}$$

$$\tan \frac{x}{2} = \frac{1-\cos x}{\sin x} \quad \text{if } \sin \frac{x}{2} \neq 0 \text{ and } \cos \frac{x}{2} \neq 0 \tag{49}$$

Proof. To prove (48), note that if $\cos \frac{x}{2} \neq 0$, then by double angle formula (6) and (8), we have

$$\frac{\sin x}{1+\cos x} = \frac{2\sin \frac{x}{2}\cos \frac{x}{2}}{2\cos^2 \frac{x}{2}} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \tan \frac{x}{2}.$$

To prove (49), note that if $\sin \frac{x}{2} \neq 0$ and $\cos \frac{x}{2} \neq 0$, then by double angle formula (6) and (9), we have

$$\frac{1-\cos x}{\sin x} = \frac{2\sin^2 \frac{x}{2}}{2\sin \frac{x}{2}\cos \frac{x}{2}} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \tan \frac{x}{2}.$$

□