Math 1010C Term 1 2015
Supplementary exercises 5

1. Prove Leibniz's rule for higher order derivatives: if f, g are both n-times differentiable at a point a, then
$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n-k)}(a).$$

2. Suppose f is defined on an open interval I that contains a point a, and that f is differentiable on $I \setminus \{a\}$.
   (a) Show that each of the following statements is false:
      (i) If $\lim\limits_{x \to a} f'(x)$ exists, then f is differentiable at a.
      (ii) If $\lim\limits_{x \to a} f'(x)$ does not exist, then f is not differentiable at a.
   (b) Show, however, that the following statement is true:
   Suppose $\lim\limits_{x \to a} f'(x)$ exists. In addition, suppose f is continuous at a. Then f is differentiable at a, and $f'(a) = \lim\limits_{x \to a} f'(x)$.

3. The following gives a heuristic proof of the first form of L'Hopital's rule. The task here is to make precise the proof (for instance, by using the definition of limits).

   Suppose $f, g \colon (a, b) \to \mathbb{R}$ are differentiable on (a, b), with $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose also that
$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0,$$
   and that
$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} \quad \text{exists and equals } L.$$
   We will prove that
$$\lim_{x \to a^+} \frac{f(x)}{g(x)} \quad \text{also exists and equals } L.$$

   To do so, suppose $x \in (a, b)$. Let y be such that $a < y < x$. Then by Cauchy's mean value theorem, there exists $\xi \in (y, x)$ such that
$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)}.$$
   Note ξ depends on both x and y. Now let $y \to a^+$. The left hand side converges to $f(x)/g(x)$, and the right hand side can be made arbitrarily close to L, as long as x is also sufficiently close to a. This suggests that $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ also exists and equals L.

   Can you make precise the above argument?

4. The following gives a heuristic proof of the second form of L'Hopital's rule. The task here is to make precise the proof (for instance, by using the definition of limits).

   Suppose $f, g\colon (a,b) \to \mathbb{R}$ are differentiable on (a,b), with $g'(x) \neq 0$ for all $x \in (a,b)$. Suppose also that

$$\lim_{x \to a^+} |g(x)| = +\infty,$$

   and that

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} \quad \text{exists and equals } L.$$

   We will prove that

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{also exists and equals } L.$$

   To do so, suppose $x \in (a,b)$. Let y be such that $x < y < b$. Then by Cauchy's mean value theorem, there exists $\xi \in (x,y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)}.$$

   Note ξ depends on both x and y. Now let x be just slightly bigger than a. The left hand side is then approximately $f(x)/g(x)$, and the right hand side can be made arbitrarily close to L, as long as y is also sufficiently close to a. This suggests that $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ also exists and equals L.

   Can you make precise the above argument?