Math 1010C Term 1 2015
Supplementary exercises 10

1. (a) Let $f(x) = \displaystyle\int_{\frac{1}{x}}^{x} \sin\sqrt{t}\,dt$ for $x > 0$. Find $f'(1)$.

   (b) Let $g(x) = \displaystyle\int_{\frac{1}{x}}^{x} \sin\sqrt{xt}\,dt$ for $x > 0$. Find $g'(1)$.

2. (a) Evaluate $\displaystyle\frac{d}{du}\int_0^u (\sqrt{2})^{t^2}\,dt$.

   (b) Define $F(x) = \displaystyle\int_{\tan x}^{\sec x} (\sqrt{2})^{t^2}\,dt$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$
   Solve $F'(x) = 0$.

3. Let $f(x) = \displaystyle\int_1^x \sin(\cos t)\,dt$, where $x \in [0, \frac{\pi}{2})$.
   (a) Show that f is strictly increasing on $[0, \frac{\pi}{2})$.

   (b) If g is the inverse function of f, find $g'(0)$.

4. (a) Let f be a non-negative continuous function on $[a, b]$. Define

$$F(x) = \int_0^x f(t)\,dt \quad \text{for } x \in [a, b].$$

   Show that F is an increasing function on $[a, b]$.
   Hence deduce that if $\displaystyle\int_a^b f(t)\,dt = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

   (b) Let g be a continuous function on [a,b]. If $\displaystyle\int_a^b g(x)u(x)\,dx = 0$ for any continuous function u on $[a, b]$, show that $g(x) = 0$ for all $x \in [a, b]$.

   (c) Let h be a continuous function on $[a, b]$. Define $A = \displaystyle\frac{1}{b-a}\int_a^b h(t)\,dt$.

      (i) If $v(x) = h(x) - A$ for all $x \in [a, b]$, show that $\displaystyle\int_a^b v(x)\,dx = 0$.

      (ii) If $\displaystyle\int_a^b h(x)w(x)\,dx = 0$ for any continuous function w on $[a, b]$ satisfying $\displaystyle\int_a^b w(x)\,dx = 0$, show that $h(x) = A$ for all $x \in [a, b]$

5. Let f be a real-valued function continuous on $[0,1]$ and differentiable in $(0,1)$. Suppose f satisfies

$$\begin{array}{ll} A. & f(0)=0 \\ B. & f(1)=\frac{1}{3} \\ C. & 0<f'(t)<1 \quad \text{for} \quad t\in(0,1). \end{array}$$

Define $F(x)=2\displaystyle\int_0^x f(t)\,dt-\Big[f(x)\Big]^2$ for $x\in[0,1]$.

(a) Show that $F'(x)>0$ for $x\in(0,1)$.

(b) Show that $\displaystyle\int_0^1 f(t)\,dt>\frac{1}{18}$.

6. Let

$$f(t)=\begin{cases} \dfrac{e^t-1}{t} & \text{if } t\neq 0, \\ 1 & \text{if } t=0 \end{cases}.$$

Find $\displaystyle\lim_{x\to 0}\frac{\int_0^x f(t)\,dt-x}{x^2}$.

7. (a) Let $u:\mathbb{R}\to\mathbb{R}$ be a twice differentiable function satisfying the following conditions:

$$\begin{array}{ll} (1) & u''(x)=-u(x) \quad \text{for all } x\in\mathbb{R} \\ (2) & u(0)=0 \\ (3) & u'(0)=1 \end{array}$$

Define $v(x)=u(x)-\sin x$ for all $x\in\mathbb{R}$.

By differentiating $w(x)=\Big(v(x)\Big)^2+\Big(v'(x)\Big)^2$, prove that $u(x)=\sin x$ for all $x\in\mathbb{R}$

(b) Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ be continuous functions such that $f(x)=e^{-x}\displaystyle\int_0^x e^t g(t)\,dt$ and $g(x)=e^{-x}-e^{-x}\displaystyle\int_0^x e^t f(t)\,dt$ for all $x\in\mathbb{R}$.

(i) Prove that $f(x)+f'(x)=g(x)$ for all $x\in\mathbb{R}$.

(ii) (A) Prove that $f''(x)+2f'(x)+2f(x)=0$ for all $x\in\mathbb{R}$.

(B) Let $h(x)=e^x f(x)$ for all $x\in\mathbb{R}$
Prove that $h''(x)=-h(x)$. Using (a), find $f(x)$.

(iii) Find $g(x)$.