SOME CONSEQUENCES OF THE MEAN-VALUE THEOREM

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Below we establish some important theoretical consequences of the mean-value theorem. First recall the mean value theorem:

Theorem 1 (Mean value theorem)**.** *Suppose $f\colon [a,b] \to \mathbb{R}$ is a function defined on a closed interval $[a,b]$ where $a, b \in \mathbb{R}$. If f is continuous on the closed interval $[a,b]$, and f is differentiable on the open interval (a,b), then there exists $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a).$$

This has some important corollaries.

1. Monotone functions

We introduce the following definitions:

Definition 1. Suppose $f\colon I \to \mathbb{R}$ is defined on some interval I.

(i) f is said to be *constant* on I, if and only if $f(x) = f(y)$ for any $x, y \in I$.
(ii) f is said to be *increasing* on I, if and only if for any $x, y \in I$ with $x < y$, we have $f(x) \le f(y)$.
(iii) f is said to be *strictly increasing* on I, if and only if for any $x, y \in I$ with $x < y$, we have $f(x) < f(y)$.
(iv) f is said to be *decreasing* on I, if and only if for any $x, y \in I$ with $x < y$, we have $f(x) \ge f(y)$.
(v) f is said to be *strictly decreasing* on I, if and only if for any $x, y \in I$ with $x < y$, we have $f(x) > f(y)$.

(Can you draw some examples of such functions?)

Corollary 2. *Suppose $f\colon [a,b] \to \mathbb{R}$ is a function defined on a closed interval $[a,b]$ where $a, b \in \mathbb{R}$. Suppose also that f is continuous on the closed interval $[a,b]$, and that f is differentiable on the open interval (a,b).*

(i) If $f'(t) = 0$ for all $t \in (a,b)$, then f is constant on $[a,b]$.
(ii) If $f'(t) \ge 0$ for all $t \in (a,b)$, then f is increasing on $[a,b]$.
(iii) If $f'(t) > 0$ for all $t \in (a,b)$, then f is strictly increasing on $[a,b]$.
(iv) If $f'(t) \le 0$ for all $t \in (a,b)$, then f is decreasing on $[a,b]$.
(v) If $f'(t) < 0$ for all $t \in (a,b)$, then f is strictly decreasing on $[a,b]$.

Proof. Suppose f is continuous on $[a,b]$, and differentiable on (a,b). Fix two points $x, y \in [a,b]$, with say $x < y$. Then f is continuous on the closed interval $[x,y]$, and differentiable on the open interval (x,y), so the mean value theorem applies, and there exists some $c \in (x,y) \subset (a,b)$ such that

$$f(y) - f(x) = f'(c)(y-x).$$

(i) If $f'(t) = 0$ for all $t \in (a,b)$, then in particular $f'(c) = 0$. So

$$f(y) - f(x) = f'(c)(y-x) = 0,$$

i.e. $f(x) = f(y)$. Since x, y are arbitrary in $[a,b]$, this shows f is a constant on $[a,b]$.

(ii) If $f'(t) \ge 0$ for all $t \in (a,b)$, then in particular $f'(c) \ge 0$. So using also $y - x > 0$, we see that

$$f(y) - f(x) = f'(c)(y-x) \ge 0,$$

i.e. $f(x) \le f(y)$. Since x, y are arbitrary in $[a,b]$, this shows f is increasing on $[a,b]$.

(iii) If $f'(t) > 0$ for all $t \in (a, b)$, then in particular $f'(c) > 0$. So using also $y - x > 0$, we see that
$$f(y) - f(x) = f'(c)(y - x) > 0,$$
i.e. $f(x) < f(y)$. Since x, y are arbitrary in $[a, b]$, this shows f is strictly increasing on $[a, b]$.

(iv) Similar to (ii) (or apply (ii) to $-f$)

(v) Similar to (iii) (or apply (iii) to $-f$)

□

Note: The converse of (i), (ii) and (iv) are true, but the converse of (iii), (v) are not! Verify yourself.

2. First and second derivative tests

The above can help us determine whether a critical point is a local maxima and minima. We recall the following definitions:

Definition 2. Suppose $f\colon (a, b) \to \mathbb{R}$ is defined on an open interval (a, b).

(i) If $c \in (a, b)$ is a point, such that $f(x) \geq f(c)$ in a neighborhood of c (i.e. there exists a (possibly tiny) open interval (α, β), such that $c \in (\alpha, \beta) \subset (a, b)$, and $f(x) \geq f(c)$ for all $x \in (\alpha, \beta)$), then c is called a local minimum of the function f.

(ii) If $c \in (a, b)$ is a point, such that $f(x) \leq f(c)$ in a neighborhood of c (i.e. there exists a (possibly tiny) open interval (α, β), such that $c \in (\alpha, \beta) \subset (a, b)$, and $f(x) \leq f(c)$ for all $x \in (\alpha, \beta)$), then c is called a local maximum of the function f.

(Sometimes a local maximum is called a relative maximum, and a local minimum is called a relative minimum. Sometimes we also say a point is a local (or relative) extremum, if it is a local maximum or a local minimum.)

Corollary 3 (First derivative test). *Suppose $f\colon (a, b) \to \mathbb{R}$ is defined on an open interval (a, b), and $c \in (a, b)$. Suppose also that f is differentiable on $(a, b) \setminus \{c\}$, and f is continuous at c.*

(i) If there exists α, $\beta \in (a, b)$, with $\alpha < c < \beta$, such that
$$\begin{cases} f'(t) \leq 0 & \text{for all } t \in (\alpha, c), \\ f'(t) \geq 0 & \text{for all } t \in (c, \beta), \end{cases}$$
then c is a local minimum of f.

(ii) If there exists α, $\beta \in (a, b)$, with $\alpha < c < \beta$, such that
$$\begin{cases} f'(t) \geq 0 & \text{for all } t \in (\alpha, c), \\ f'(t) \leq 0 & \text{for all } t \in (c, \beta), \end{cases}$$
then c is a local maximum of f.

Proof. (i) If f is as in (i), then f is decreasing on $[\alpha, c]$, and increasing on $[c, \beta]$. So $f(x) \geq f(c)$ on (α, β). It follows that c is a local minimum of f.

(ii) If f is as in (ii), then f is increasing on $[\alpha, c]$, and decreasing on $[c, \beta]$. So $f(x) \leq f(c)$ on (α, β). It follows that c is a local maximum of f.

□

Next, we recall the definition of second derivatives: If f is a differentiable function on an open interval (a, b), and if f' is also differentiable on (a, b), then the derivative of f' is denoted f'', and is called the second derivative of f. Such functions are said to be *twice differentiable* on (a, b).

Corollary 4 (Second derivative test). *Suppose $f\colon (a, b) \to \mathbb{R}$ is differentiable on an open interval (a, b), and $c \in (a, b)$. Suppose also that $f'(c) = 0$, and that f' is differentiable at c.*

(i) If $f''(c) > 0$, then c is a local minimum of f.

(ii) If $f''(c) < 0$, then c is a local maximum of f.

Proof. Since $f'(c) = 0$ and $f''(c)$ exists, we have

$$\lim_{x\to c}\frac{f'(x)}{x-c} = \lim_{x\to c}\frac{f'(x)-f'(c)}{x-c}$$

exists and equals $f''(c)$.

(i) If $f''(c) > 0$, then

$$\lim_{x\to c}\frac{f'(x)}{x-c} > 0.$$

It follows that

$$\frac{f'(x)}{x-c} > 0$$

for all x in a deleted neighborhood of c, i.e. there exists $\alpha, \beta \in (a, b)$ with $c \in (\alpha, \beta)$ such that $\frac{f'(x)}{x-c} > 0$ holds for $x \in (\alpha, \beta) \setminus \{c\}$. Since $x - c > 0$ when $x > c$, and $x - c < 0$ when $x < c$, it follows that

$$f'(x) \begin{cases} < 0 & \text{if } x \in (\alpha, c) \\ > 0 & \text{if } x \in (c, \beta) \end{cases}$$

Using the first derivative test, it follows that c is a local minimum of f.

(ii) Similar to (i).

□

3. Convex functions

In the second derivative test, we used only the second derivative at one point. If in addition a function is twice differentiable on an interval, then the sign of the second derivative actually tells us about the *convexity* of a function, which we define below. We note, however, that the notion of convexity is independent of the notion of the second derivative.

Definition 3. Suppose $f : (a, b) \to \mathbb{R}$ is a function defined on some open interval (a, b).

(i) f is said to be *convex* on (a, b), if and only if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

for any $x, y \in (a, b)$ and any $\lambda \in (0, 1)$.

(ii) f is said to be *concave* on (a, b), if and only if

$$f(\lambda x + (1-\lambda)y) \ge \lambda f(x) + (1-\lambda)f(y)$$

for any $x, y \in (a, b)$ and any $\lambda \in (0, 1)$.

(Can you draw some examples of such functions?)

(Some people call convex functions "convex up", and concave functions "convex down". We will not use these terminologies.)

The above definition is symmetric with respect to x and y. Hence we only need to check it when (say) $x < y$. For fixed $x, y \in (a, b)$ with $x < y$, let $z = \lambda x + (1 - \lambda)y$. When λ varies between $(0, 1)$, z varies between (x, y), and the correspondence between such λ's and z's is one-to-one. Hence the above definitions can be rephrased as follows:

Proposition 5. *Suppose $f : (a, b) \to \mathbb{R}$ is a function defined on some open interval (a, b).*

(i) f is convex on (a, b), if and only if

$$f(z) \le \frac{y-z}{y-x}f(x) + \frac{z-x}{y-x}f(y)$$

for any $x, y, z \in (a, b)$ with $x < z < y$.

(ii) f is concave on (a, b), if and only if

$$f(z) \ge \frac{y-z}{y-x}f(x) + \frac{z-x}{y-x}f(y)$$

for any $x, y, z \in (a, b)$ with $x < z < y$.

By rearranging the above inequalities, one also has the following equivalent characterization of convexity:

Proposition 6. *Suppose $f\colon (a,b) \to \mathbb{R}$ is a function defined on some open interval (a,b).*

(i) f is convex on (a,b), if and only if
$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z}$$
for any $x,y,z \in (a,b)$ with $x<z<y$.

(ii) f is concave on (a,b), if and only if
$$\frac{f(z)-f(x)}{z-x} \geq \frac{f(y)-f(z)}{y-z}$$
for any $x,y,z \in (a,b)$ with $x<z<y$.

Heuristically, this says that f is convex on (a,b), if and only if the slopes of the secant lines are 'increasing' on (a,b). Similarly for concavity.

Not every convex / concave function is differentiable. (What is a simple example here?) But for functions that are differentiable, we can characterize convexity / concavity as follows.

Proposition 7. *Suppose $f\colon (a,b) \to \mathbb{R}$ is a* differentiable *function defined on some open interval (a,b).*

(i) f is convex on (a,b), if and only if f' is increasing on (a,b).

(ii) f is concave on (a,b), if and only if f' is decreasing on (a,b).

Proof. (i) Suppose f is differentiable on (a,b). Then by mean value theorem, for any $x,y,z \in (a,b)$ with $x<z<y$, there exists $c \in (x,z)$, $d \in (z,y)$ such that
$$\frac{f(z)-f(x)}{z-x} = f'(c), \quad \frac{f(y)-f(z)}{y-z} = f'(d).$$
Note that c must then be less than d. Hence if f' is increasing on (a,b), then $f'(c) \leq f'(d)$, and this shows that f is convex on (a,b) by our earlier proposition.

Conversely, suppose f is differentiable and convex on (a,b). Suppose $x,y \in (a,b)$ with $x<y$. Then
$$f'(x) = \lim_{z_1 \to x^+} \frac{f(z_1)-f(x)}{z_1-x}, \quad f'(y) = \lim_{z_2 \to y^-} \frac{f(y)-f(z_2)}{y-z_2}.$$
But for z_1, z_2 with $x<z_1<z_2<y$, we have, by convexity of f, that
$$\frac{f(z_1)-f(x)}{z_1-x} \leq \frac{f(z_2)-f(z_1)}{z_2-z_1} \leq \frac{f(y)-f(z_2)}{y-z_2}.$$
Hence forgetting about the middle term, and letting $z_1 \to x^+$, $z_2 \to y^-$, we see that $f'(x) \leq f'(y)$, and this shows that f' is increasing.

(ii) Similar to (i).

□

If in addition, f is twice differentiable, then f' is increasing on (a,b), if and only if $f'' \geq 0$ on (a,b). This gives us the following corollary:

Corollary 8. *Suppose $f\colon (a,b) \to \mathbb{R}$ is twice differentiable on an open interval (a,b).*

(i) If $f''(t) \geq 0$ for all $t \in (a,b)$, then f is convex on (a,b).

(ii) If $f''(t) \leq 0$ for all $t \in (a,b)$, then f is concave on (a,b).

One can easily show now that e.g. $\exp(x)$ is strictly increasing and convex on $\mathbb{R}$.

It is now easy to sketch the graph of a function using its first (and possibly second) derivatives. This in turn allows us to determine (sometimes) the global maximum or minimum of a function over unbounded intervals, and establish some identities / inequalities.