Differentiation of an infinite series

In deriving the formula for $\frac{d}{dx}\exp(x)$, we formally differentiated the series defining $\exp(x)$ term by term: we said something like

$$\frac{d}{dx}\exp(x) = \frac{d}{dx}\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\overset{(*)}{=} \sum_{k=0}^{\infty}\frac{d}{dx}\left(\frac{x^k}{k!}\right)$$

$$= \sum_{k=0}^{\infty}\frac{kx^{k-1}}{k!}$$

$$= \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$= \exp(x).$$

However, 2 questions must be addressed in step $(*)$:

1. We know that the sum of finitely many differentiable functions is differentiable. But does this apply to infinite sums? In particular, is the infinite sum $\sum_{k=0}^{\infty}\frac{x^k}{k!}$ really a differentiable function of x?

2. Even if the infinite sum of functions is differentiable, is it true that we can differentiate term by term?

In 1872, Karl Weierstrass constructed the following very counter-intuitive example:

Let

$$f(x) = \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k \cos(9^k \pi x) \qquad \forall x \in \mathbb{R}.$$

It can be shown that this series converges $\forall x \in \mathbb{R}$ (so that $f: \mathbb{R} \to \mathbb{R}$ is well-defined), that the f such defined is continuous at every $x \in \mathbb{R}$, and yet this f fails to be differentiable at every $x \in \mathbb{R}$. The series defining f is a sum of very nice differentiable functions (in fact each term $\left(\frac{2}{3}\right)^k \cos(9^k \pi x)$ is even infinitely differentiable), so this example shows that even infinite sums of infinitely differentiable functions can fail to be differentiable at every point. This definitely shows that Q1 on the previous page is a delicate one, and it is only with luck that we could really differentiate the series defining $\exp(x)$ term by term and get the correct answer.

The graph of f (or rather, an approximation of f by summing the first 300 terms) is shown on the next page.

Basically, f fails to be differentiable, because $\cos(9^k \pi x)$ is a very rapidly oscillating function in x, as $k \to \infty$.

Each time we add a term like $\left(\frac{2}{3}\right)^k \cos(9^k \pi x)$, we are adding a very rapid oscillation (whose amplitude is very small due to the presence of the factor $\left(\frac{2}{3}\right)^k$.).

The result is that we add a lot of small wiggles, making the resulting function more and more wiggly.

This makes the infinite series defining f a nowhere differentiable function. (The f we gave above is an example of what people sometimes call a lacunary Fourier series, and you'll learn more about it when you study Fourier analysis.)

In[1]:= `Plot[Sum[(2 / 3)^k * Cos[9^k * Pi * x], {k, 1, 300}], {x, 0, 1}]`

Out[1]=