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Advances in Mathematics 195 (2005) 24–101
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers

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Received 21 January 2004; accepted 28 June 2004

Communicated by R.D. Mauldin
Available online 11 September 2004

Abstract

In this paper, we give a systematical study of the local structures and fractal indices of the limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers. For a given Pisot number in the interval $(1, 2)$, we construct a finite family of non-negative matrices (maybe non-square), such that the corresponding fractal indices can be re-expressed as some limits in terms of products of these non-negative matrices. We are especially interested in the case that the associated Pisot number is a simple Pisot number, i.e., the unique positive root of the polynomial $x^k - x^{k-1} - \dots - x - 1$ $(k=2, 3, \dots)$. In this case, the corresponding products of matrices can be decomposed into the products of scalars, based on which the precise formulas of fractal indices, as well as the multifractal formalism, are obtained.

MSC: primary 28A78; secondary 28D20; 58F11; 54C30; 11R06

Keywords: Limited Rademacher functions; Bernoulli convolutions; Pisot numbers; Hausdorff dimension; Information dimension; L^q-spectrum; Multifractals; Product of matrices

1. Introduction

In this paper, we provide a systematical study of the local structures and different fractal indices of the limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers. We also verify the validity of the multifractal formalism for

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[1]Partially supported by the Special Funds for Major State Basic Research Projects in China.

doi:10.1016/j.aim.2004.06.011

the corresponding Bernoulli convolutions associated with a special class of Pisot numbers. Recall that a real algebraic integer is called a *Pisot number* if all its conjugates are less than 1 in modulus. There are infinitely many Pisot numbers in the interval $(1, 2)$: for example, for each $k = 2, 3, \ldots$, the unique positive root of the polynomial

$$p_k(x) = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1$$

is a Pisot number. We shall call these the *simple* Pisot numbers. The readers may see the books [3,55] for the detailed properties of Pisot numbers.

Recall that for $\frac{1}{2} < \rho < 1$, the *limited Rademacher function f* with parameter ρ is defined by

$$f(x) = (1-\rho)\sum_{n=0}^{\infty} \rho^n R(2^n x), \qquad x \in [0, 1], \tag{1.1}$$

where R denotes the classic Rademacher function: $R(x)$ is defined on the line $\mathbb{R}$ with period 1, taking values 0 and 1 on the intervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$, respectively. The distribution of f induces a probability measure μ on $[0, 1]$. That is,

$$\mu(E) = \mathcal{L}\{x \in [0, 1] : \ f(x) \in E\}, \qquad \forall \text{ Borel set } E \subset [0, 1],$$

where $\mathcal{L}$ denotes the one-dimensional Lebesgue measure. The measure μ is called *the Bernoulli convolution* with parameter ρ, since it is the infinite convolution of $\frac{1}{2}(\delta_0 + \delta_{(1-\rho)\rho^n})$. An important property of μ is its self-similarity (see, e.g., [31, Theorem 4.3]):

$$\mu = \tfrac{1}{2}\,\mu \circ S_0^{-1} + \tfrac{1}{2}\,\mu \circ S_1^{-1}, \tag{1.2}$$

where $S_0(x) = \rho x$ and $S_1(x) = \rho x + 1 - \rho$.

The limited Rademacher functions and Bernoulli convolutions have been studied for a long time, revealing many connections with harmonic analysis, number theory, fractal geometry and dynamical systems, see [1,30,48,53,57]. It is well known [28] that for each parameter $\rho \in (1/2, 1)$, μ is either absolutely continuous or totally singular. Erdös [6] proved that μ is totally singular if ρ is the reciprocal of a Pisot number. In the opposite direction, Solomyak [56] proved that μ is absolutely continuous with $\frac{d\mu}{dx} \in L^2$ for almost all $\rho \in (1/2, 1)$, extending an early result of Erdös [7]. Please see [49] for a simpler proof. Mauldin and Simon [39] showed that μ is in fact equivalent to the Lebesgue measure for almost all $\rho \in (1/2, 1)$. Later Peres and Schlag [47] strengthed Solomyak's result by showing the Hausdorff dimension of the exceptional ρ's in $[a, 1]$ is strictly smaller than 1 for each $a > \frac{1}{2}$. Recently, Feng and Wang [20] found a sequence of ρ's such that ρ^{-1} is not Pisot number and ρ lies in the exceptional set (i.e. the corresponding μ, if it is absolutely continuous, has no L^2 density).

In this paper, we always assume that the parameter $\rho \in (\frac{1}{2}, 1)$ is a *Pisot reciprocal*. That is, ρ^{-1} is a Pisot number. Under this assumption, we would like to analyse the complexity and the degree of singularity of the corresponding limited Rademacher function and Bernoulli convolution. More precisely, we study the following fractal indices for a fixed Pisot reciprocal ρ:

- the Hausdorff dimension of the graph of the limited Rademacher function;
- the Hausdorff dimension of the level sets of the limited Rademacher function;
- the L^q-spectrum, Hausdorff dimension and the range of local dimensions of the Bernoulli convolution.

Furthermore, we will give a complete multifractal analysis of μ when ρ is a simple Pisot number.

Our basic approach is the following: using an algebraic property of Pisot numbers found by Garcia [22], for each given Pisot reciprocal ρ in $(1/2, 1)$ we construct a finite family of non-negative matrices (maybe non-square) and re-express a major part of the above fractal indices as limits in terms of products of these matrices. Particularly interesting is the case where ρ is the reciprocal of a simple Pisot number. In this case we find that the corresponding product of matrices is degenerate and can be decomposed as the product of a sequence of scalars (this fact implies that μ is locally a self-similar measure with countably many generators which satisfy the separation condition). Using this key property, we obtain the precise formulas of all the above fractal indices, and verify the validity of the multifractal formalism of μ.

At first we give some necessary definitions and notations. We use $\dim_{\mathrm{H}}$, $\dim_{\mathrm{B}}$ to denote the Hausdorff dimension and the box-counting dimension, respectively (see [8,38] for the definitions). For a real function g defined on $[0, 1]$, the *graph* of the function g, denoted as $\Gamma(g)$ or simply Γ, is defined by

$$\Gamma = \{(x, g(x)) \in \mathbb{R}^2 : \ x \in [0, 1]\}.$$

For $t \in \mathbb{R}$, the *t-level set* of g, denoted as $L_t(g)$ or simply L_t, is defined by

$$L_t = \{x \in [0, 1] : \ g(x) = t\}.$$

For a given finite Borel measure ν on the line, the *upper local dimension of* ν *at* $x \in \mathrm{supp}(\nu)$ is defined by

$$\overline{d}(\nu, x) = \limsup_{r \to 0+} \frac{\log \nu([x - r, x + r])}{\log r},$$

and the *lower local dimension* $\underline{d}(\nu, x)$ *at* x is defined similarly by taking the lower limit. When $\overline{d}(\nu, x) = \underline{d}(\nu, x)$, the common value is called the *local dimension* of ν at x and is denoted by $d(\nu, x)$. The *range* of local dimensions of ν, denoted by $\mathcal{R}(\nu)$, is defined by

$$\mathcal{R}(\nu) = \{y \in \mathbb{R} : \ d(\nu, x) = y \text{ for some } x \in \mathrm{supp}(\nu)\}.$$

Recall the *Hausdorff dimension* of ν is defined by

$$\dim_{\mathrm{H}}(\nu) = \inf \{\dim_{\mathrm{H}}(E) : \ E \subset \mathbb{R} \text{ is a Borel set and } \nu(E) = 1\},$$

and the L^q*-spectrum* ($q \in \mathbb{R}$) of ν is defined by

$$\tau(q) = \tau(\nu, q) = \liminf_{\delta \to 0+} \frac{\log \left(\sup \sum_i \ \nu([x_i - \delta, x_i + \delta])^q\right)}{\log \delta},$$

where the supremum is taken over all the families $\{[x_i - \delta, x_i + \delta]\}_i$ of disjoint intervals with $x_i \in \mathrm{supp}(\nu)$. The readers may see the books [8,38,50,60] for more information about the above definitions.

We will state our matrix product results for general Pisot reciprocals in Section 3. In the following we only present our results for the cases $\rho = \lambda_k$ ($k = 2, 3, \ldots$), where λ_k is the largest real root of $x^k + x^{k-1} + \cdots + x - 1$. Define two 2×2 matrices M_0, M_1 by

$$M_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}. \tag{1.3}$$

For $n \geqslant 1$, denote by $\mathcal{A}_n$ the set of all indices $j_1 \ldots j_n$ over $\{0, 1\}$. Denote

$$M_J = M_{j_1} M_{j_2} \ldots M_{j_n}$$

for $J = j_1 \ldots j_n$. For our convenience, we write $\emptyset$ for the empty word and define

$$M_{\emptyset} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Set $\mathcal{A}_0 = \{\emptyset\}$. For any 2×2 non-negative matrix B, denote its norm by $\|B\| = (1, 1)B(1, 1)^T$.

Our main results for the cases $\rho = \lambda_k$ ($k \geqslant 2$) are the following theorems:

Theorem 1.1. *For $k = 2, 3, \ldots$, let Γ be the graph of the limited Rademacher function f with parameter $\rho = \lambda_k$. Then*

$$\dim_{\mathrm{H}} \Gamma = \frac{\log x_k}{\log \rho},$$

where x_k is the unique root in $(0, \lambda_{k-1})$ (defining $\lambda_1 = 1$) of the equation

$$\frac{1 - 2x^{k-1} + x^k}{1 - 2x + x^k} \cdot \sum_{n=0}^{\infty} \left(\sum_{J \in \mathcal{A}_n} \|M_J\|^{\alpha_k} \right) x^{kn+k+1} = 1$$

with $\alpha_k = -\frac{\log \rho}{\log 2}$.

Theorem 1.2. *For $k = 2, 3, \ldots$, the Hausdorff dimension and box-counting dimension of t-level set of the limit Rademacher function f with parameter $\rho = \lambda_k$ are equal to*

$$d_k := \frac{\rho^k\left(1 - 2\rho^k\right)^2}{\left(2 - (k+1)\rho^k\right)\log 2} \sum_{n=0}^{\infty} \left(\rho^{kn} \sum_{J \in \mathcal{A}_n} \log \|M_J\| \right)$$

for $\mathcal{L}$ almost all $t \in [0, 1]$.

Theorem 1.3. (i) *For any $q \in \mathbb{R}$, the L^q-spectrum $\tau(q)$ of the Bernoulli convolution with parameter $\rho = \lambda_2$ is equal to*

$$-\frac{q \log 2}{\log \rho} - \frac{\log x(2, q)}{\log \rho},$$

where $x(2, q)$ is defined by

$$x(2, q) = \sup \left\{ x \geqslant 0 : \sum_{n=0}^{\infty} \left(\sum_{J \in \mathcal{A}_n} \|M_J\|^q \right) x^{2n+3} \leqslant 1 \right\}.$$

There exists a unique real number $q_0 < -2$ satisfying

$$\sum_{n=0}^{\infty} \left(\sum_{J \in \mathcal{A}_n} \|M_J\|^{q_0} \right) = 1.$$

For $q > q_0$, $x(2, q)$ is the unique positive root of

$$\sum_{n=0}^{\infty} \left(\sum_{J \in \mathcal{A}_n} \|M_J\|^q \right) x^{2n+3} = 1,$$

and it is an infinitely differentiable function of q on $(q_0, +\infty)$. For $q \leqslant q_0$, $x(2, q) = 1$. Moreover $x(2, q)$ is not differentiable at $q = q_0$.

(ii) *For any integer $k \geqslant 3$ and any real number q, the L^q-spectrum $\tau(q)$ of the Bernoulli convolution μ with parameter $\rho = \lambda_k$ is equal to*

$$-\frac{q \log 2}{\log \rho} - \frac{\log x(k, q)}{\log \rho},$$

Table 1
Numerical estimations

k	$\dim_{\mathrm{H}}\,\Gamma(f_{\lambda_k})$	d_{λ_k}	$\dim_{\mathrm{H}}\,\mu_{\lambda_k}$
2	1.304 ± 0.001	0.302 ± 0.001	0.9957 ± 10^{-4}
3	1.11875217 ± 10^{-8}	0.1025001503 ± 10^{-10}	$0.98040931953 \pm 10^{-11}$
4	1.052565407 ± 10^{-9}	$0.041560454940769\pm 10^{-14}$	$0.9869264743338\pm 10^{-12}$
5	1.024596045 ± 10^{-9}	$0.01842625239655\pm 10^{-14}$	$0.9925853002741\pm 10^{-12}$
6	1.011844824 ± 10^{-9}	$0.00859023108854\pm 10^{-14}$	$0.9960325915849\pm 10^{-12}$
7	1.005796386 ± 10^{-9}	$0.00412363866083\pm 10^{-14}$	$0.9979374455070\pm 10^{-12}$
8	1.002862729 ± 10^{-9}	$0.00201383805752\pm 10^{-14}$	$0.9989449154498\pm 10^{-12}$
9	1.001421378 ± 10^{-9}	$0.00099344117302\pm 10^{-14}$	$0.9994653680555\pm 10^{-12}$
10	1.000707890 ± 10^{-9}	$0.00049294459129\pm 10^{-14}$	$0.9997306068783\pm 10^{-12}$

where $x(k,q) \in (0, \lambda_{k-1})$ *satisfies*

$$\frac{1-2x^{k-1}+x^k}{1-2x+x^k}\cdot\sum_{n=0}^{\infty}\left(\sum_{J\in\mathcal{A}_n}\|M_J\|^q\right)x^{kn+k+1}=1.$$

Moreover $x(k,q)$ *is an infinitely differentiable function of* q *on the whole line.*

Theorem 1.4. *For* $k=2,3,\ldots$, *the Hausdorff dimension of the Bernoulli convolution* μ *with parameter* $\rho=\lambda_k$ *satisfies*

$$\dim_{\mathrm{H}}\,\mu = -\frac{\log 2}{\log\rho}+\left(\frac{2^k-3}{2^k-1}\right)^2\cdot\frac{\sum_{n=0}^{\infty}2^{-kn-k-1}\sum_{J\in\mathcal{A}_n}\|M_J\|\log\|M_J\|}{\log\rho}.$$

Theorem 1.5. *For* $k=2,3,\ldots$, *let* $\mathcal{R}(\mu)$ *be the range of local dimensions of the Bernoulli convolution* μ *with parameter* $\rho=\lambda_k$. *Then*

$$\mathcal{R}(\mu)=\begin{cases}\left[-\frac{\log 2}{\log\rho}-\frac{1}{2},-\frac{\log 2}{\log\rho}\right] & \text{if } k=2,\\[2ex] \left[-\frac{k\log 2}{(k+1)\log\rho},\quad -\frac{\log 2}{\log\rho}\right] & \text{if } k\geqslant 3.\end{cases}$$

In Table 1, we give some numerical estimations of $\dim_{\mathrm{H}}\,\Gamma$, d_k and $\dim_{\mathrm{H}}\,\mu$ in the above theorems for $2\leqslant k\leqslant 10$.

Theorem 1.6. *For* $k=2,3,\ldots$, *let* μ *be the Bernoulli convolution* μ *with parameter* $\rho=\lambda_k$. *For each* $\alpha\geqslant 0$, *define*

$$K(\alpha)=\left\{x\in[0,1]:\ \lim_{\delta\to 0}\frac{\log\mu([x-\delta,x+\delta])}{\log\delta}=\alpha\right\}.$$

Then

(i) *If* $k = 2$, *then for any* $q \in \mathbb{R}\backslash\{q_0\}$,

$$\dim_{\mathrm{H}} K(\alpha(q)) = \alpha(q)q - \tau(q), \tag{1.4}$$

where q_0 *is the real number defined as in* (i) *of Theorem* 1.3, *and* $\alpha(q) = \tau'(q)$.

(ii) *If* $k \geqslant 3$, *then* (1.4) *holds for any* $q \in \mathbb{R}$.

Besides the above results, in Section 4.10 we give some results on biased Bernoulli convolutions with parameter $\rho = \lambda_2$ (including an explicit formula of the Hausdorff dimension).

Let us give some backgrounds and remarks about our results. The limited Rademacher function f looks very closed to the Weierstrass function W, which is defined by

$$W(x) = \sum_{n=1}^{\infty} \lambda^n \sin(2^n x)$$

with parameter $\lambda \in (\frac{1}{2}, 1)$. A famous question, which still remains open, is whether or not the Hausdorff dimension and box-counting dimension of the graph of W coincide (it is known that both the box-counting dimension of the graph of W and that of f equal $2 + \frac{\log \lambda}{\log 2}$. See e.g. [8]). We refer the reader to Mauldin and Williams [41] and the references therein for more details. It is natural to consider the same question for the graph of f. Przytycki and Urbański [53] proved that if λ is a Pisot reciprocal, the Hausdorff dimension of the graph of f is strictly smaller than the box dimension. Przytycki and Urbański obtained results for $\dim_{\mathrm{H}} \Gamma$ in this case, but their results are not explicit, and cannot be used to estimate the Hausdorff dimension of Γ delicately. To our best knowledge, Theorem 1.1 is the first explicit result about the Hausdorff dimension of $\dim_{\mathrm{H}} \Gamma$ in the Pisot reciprocal case.

For the level sets of f with parameter λ, Hu and Lau [26] proved that if the corresponding μ is absolutely continuous, then the Hausdorff dimension of the t-level set of f is equal to $1 + \frac{\log \lambda}{\log 2}$ for $\mathcal{L}$ almost all $t \in [0, 1]$. Recall that μ is absolutely continuous for almost all $\lambda \in (\frac{1}{2}, 1)$ and it is totally singular if λ is a Pisot reciprocal. One may see that Theorem 1.2 describes the different behavior of level sets in the Pisot reciprocal cases.

Theorem 1.3 concerns the L^q-spectra of the Bernoulli convolutions and their differentiability. We need to point out that the L^q-spectrum of a measure is one of the basic ingredients in the study of multifractal phenomena. It is well known that if ν is the self-similar measure associated with an iterated function system (IFS) $\{\phi_j\}_{j=1}^m$ satisfying the so-called *open set condition* [27], then $\tau(q)$ can be calculated by an explicit formula and it is analytic on $\mathbb{R}$ [5,44]. However if the IFS does not satisfy the open set condition, it is much harder to obtain a formula for $\tau(q)$. In their fundamental work [32], Lau and Ngai considered the IFS satisfying the weak separation condition, and proved that each associated self-similar measure partially satisfies the multifractal formalism. Their result strongly relies on the differentiability property of $\tau(q)$. The weak separation condition is strictly weaker than the open set condition, and it is satisfied by many interesting cases, such as the Bernoulli convolutions associated with Pisot numbers. In a later paper [33], Lau and Ngai considered the Bernoulli convolution with parameter $\rho = \lambda_2$. They gave an explicit formula of $\tau(q)$ for $q > 0$ and proved that it is infinitely differentiable on $(0, \infty)$. They also raised a question how to determine the formula of $\tau(q)$ for $q < 0$ when $\rho = \lambda_2$, and more generally how to determine $\tau(q)$ and check its differentiability for some other Pisot reciprocal parameters. Theorem 1.3 answers their question considerably. It is very surprising for the case $\rho = \lambda_2$, $\tau(q)$ is not differentiable at one point $q_0 < 0$. This leads to the phase transition of the corresponding Bernoulli convolution [18]. Some similar phenomena (non-differentiability of $\tau(q)$) were found in the study of another self-similar measure (i.e., the $\underline{3}$-fold convolution of the standard Cantor measure) [16,35].

Theorem 1.4 gives the formulas of $\dim_{\mathrm{H}} \mu$ of μ with parameters $\rho = \lambda_k$, $k \geqslant 2$. The formula for $\rho = \lambda_2$ is already known, which was obtained by several authors [2,36,43,58] through different approaches. In all cases their methods depend on the specific algebraic properties of λ_2 and cannot be used with other parameters. We mention that Lalley [29] has expressed $\dim_{\mathrm{H}} \mu$ as the top Lyapunov exponent of a sequence of random matrices for any Pisot reciprocal parameter. Nevertheless, the involved Lyapunov exponent is hard to calculate, and Lalley only gave the numerical estimation in the case $\rho = \lambda_2$. Our result for $\rho = \lambda_k$ $(k \geqslant 3)$ verifies a claim of Alexander and Zagier [2] that "it seems likely that" one can give a formula for $\dim_{\mathrm{H}} \mu$ when $\rho = \lambda_k$ $(k \geqslant 3)$.

For a given measure ν, the range $\mathcal{R}(\nu)$ of local dimensions of ν is important in considering the local structure and multifractal property of ν. However, it is very hard to determine $\mathcal{R}(\nu)$ when ν is a self-similar measure with overlaps. Hu first determined $\mathcal{R}(\mu)$ in the case $\rho = \lambda_2$ by using a combinatorial method [24]. He also claimed (see Theorems A, B of [24]) without proof that for $\rho = \lambda_k$ $(k \geqslant 3)$,

$$\mathcal{R}(\mu) = \left[-\frac{\log \lambda_2}{k \log \rho},\ -\frac{\log 2}{\log \rho} \right].$$

However, the above formula is not true. In Theorem 1.5 we present the correct one.

Theorem 1.6 verifies the validity of the multifractal formalism of μ parameters $\rho = \lambda_k$, $k \geqslant 2$. We say the multifractal formalism of μ holds at $\alpha \in \mathcal{R}(\mu)$ if

$$\dim_{\mathrm{H}} K(\alpha) = \inf_{t \in \mathbb{R}} \{\alpha t - \tau(t)\}.$$

Before our result some partial multifractal results for μ with $\rho = \lambda_2$ were obtained. In [32] Lau and Ngai showed that (1.4) is true for $q > 0$, and Porzio [52], based on the previous work [37] joint with Ledrappier, extended the valid range to $q > -\frac{1}{2}$. We remark that Theorem 1.6 has not yet set up the validity of the multifractal formalism of μ with $\rho = \lambda_2$ for those $\alpha \in (\tau'(q_0+), \tau'(q_0-)$. However this has been done recently by Feng and Olivier [18] by viewing μ as a weak Gibbs measure associated to some dynamical system. For all the Pisot reciprocals besides simple Pisot reciprocals, by extending an idea in this paper and using a result on the product of non-negative matrices in [15], Feng [14] recently proved that $\tau(q)$ is always differentiable for $q > 0$ and (1.4) holds for $q > 0$.

An essential property of simple Pisot reciprocals is the following: For a given simple Pisot reciprocal ρ, let Ω be the class of characteristic vectors and $\xi : \Omega \to \Omega^*$ the transition map. There is a $\gamma \in \Omega$ with $v(\gamma) = 1$ such that for any $\beta \in \Omega$, there exists $n \in \mathbb{N}$ (depending on β) so that γ is a letter in the word $\xi^n(\beta)$ (see Section 2 for all involved definitions and notations). This property guarantees that the products of the corresponding transition matrices can be decomposed as the products of scalars. We did find another number (the positive root of $1 - x + 2x^2 - x^3$) that also satisfies this property. However this property is not generic, for example it is not satisfied by the positive root of $x^3 + x^2 - 1$.

This paper is organized as follows. In Section 2, we introduce some basic notations such as net intervals, characteristic vectors and multiplicity vectors; and we give a symbolic expression for each net interval; furthermore we construct a finite family of non-negative matrices (maybe non-square) such that the distribution of μ on each net interval can be expressed as the products of these matrices. In Section 3, we re-express some fractal indices (local dimension and L^q-spectrum of μ, the Hausdorff dimension of $\Gamma(f)$, the box dimension of the level sets of f) as the limits in terms of product of these matrices. In Section 4, we focus on the golden ratio case $\rho = \frac{\sqrt{5}-1}{2}$ and give a series of explicit formulas of fractal indices. We prove in this case μ has locally infinite similarity and satisfies the multifractal formalism. In Section 5, we consider the simple Pisot reciprocals other than $\frac{\sqrt{5}-1}{2}$.

2. Net intervals, characteristic vectors, multiplicity vectors

In this section we study the properties of so-called *net interval*, *characteristic vector* and *multiplicity vector*. In Section 2.1, we give the definitions of all these notations. In Section 2.2, by using an algebraic property of Pisot numbers, we show that the collection of all possible characteristic vectors, denoted as Ω, is finite. Using the self-similar structure of net intervals, we set up a one-to-one correspondence between nth net intervals and *admissible words* of length $n + 1$ over Ω. We call the corresponding admissible word of an nth net interval the *symbolic expression* of this net interval. Furthermore, we construct some *transition matrices* over Ω, such that the multiplicity vector of a nth net interval can be expressed as a product of these matrices. In Section 2.3, we obtain the distribution of μ on each net interval.

2.1. The definitions

Let ρ be a Pisot reciprocal in the interval $(1/2, 1)$. Define $S_0x = \rho x$ and $S_1x = \rho x + (1-\rho)$. For our convenience, we write $\mathcal{A} = \{0, 1\}$ and let $\mathcal{A}_n$ denote the collection of all indices $j_1 \cdots j_n$ of length n over $\mathcal{A}$. For $\sigma = j_1 \cdots j_n \in \mathcal{A}_n$, write for simplicity $S_\sigma = S_{j_1} \circ \cdots \circ S_{j_n}$. We define two families of sets P_n^0, P_n^1 $(n \geqslant 0)$ in the following way: $P_0^0 = \{0\}$, $P_0^1 = \{1\}$, and $P_n^0 = \{S_\sigma(0) : \sigma \in \mathcal{A}_n\}$, $P_n^1 = \{S_\sigma(1) : \sigma \in \mathcal{A}_n\}$ for $n \geqslant 1$. Define $P_n = P_n^0 \bigcup P_n^1$ for $n \geqslant 0$. Let $h_1, \ldots, h_{s_n}$ be all the elements of P_n ranked in the increasing order. Define

$$\mathcal{F}_n = \left\{[h_j, h_{j+1}] : 1 \leqslant j < s_n\right\}.$$

Each element in $\mathcal{F}_n$ is called a *nth net interval*.

The following facts about net intervals can be checked easily: (i) $\bigcup_{\Delta \in \mathcal{F}_n} \Delta = [0, 1]$ for any $n \geqslant 0$; (ii) For any $\Delta_1, \Delta_2 \in \mathcal{F}_n$ with $\Delta_1 \neq \Delta_2$, $\mathrm{int}(\Delta_1) \cap \mathrm{int}(\Delta_2) = \emptyset$; (iii) For any $\Delta \in \mathcal{F}_n$ $(n \geqslant 1)$, there is a unique element $\widehat{\Delta} \in \mathcal{F}_{n-1}$ such that $\widehat{\Delta} \supset \Delta$.

For each net interval $\Delta = [a, b] \in \mathcal{F}_n$, we define a positive number $\ell_n(\Delta)$, a vector $V_n(\Delta)$ and a positive integer $r_n(\Delta)$ as follows: If $\Delta = [0, 1] \in \mathcal{F}_0$, we define $\ell_0(\Delta) = 1$, $V_0(\Delta) = 0$ and $r_0(\Delta) = 1$; Otherwise for $n \geqslant 1$, we define $\ell_n(\Delta)$ and $V_n(\Delta)$ directly by

$$\ell_n(\Delta) = \rho^{-n}(b - a)$$

and

$$V_n(\Delta) = (a_1, \ldots, a_k), \tag{2.1}$$

where $a_1, \ldots, a_k$ (ranked in the increasing order) are the elements of the following set

$$\left\{\rho^{-n}\left(a - S_\sigma(0)\right) : \sigma \in \mathcal{A}_n,\ a - \rho^n < S_\sigma(0) \leqslant a\right\}.$$

Let $v_n(\Delta)$ denote the dimension of $V_n(\Delta)$, i.e., $v_n(\Delta) = k$. We define $r_n(\Delta)$ in the following way: let $\widehat{\Delta}$ be the unique one interval in $\mathcal{F}_{n-1}$ containing Δ, and $\Delta_1, \ldots, \Delta_m$ (ranked in the increasing order) be all the elements in $\mathcal{F}_n$ satisfying $\Delta_j \subset \widehat{\Delta}$, $\ell_n(\Delta_j) = \ell_n(\Delta)$, $V_n(\Delta_j) = V_n(\Delta)$ for $1 \leqslant j \leqslant m$. Define $r_n(\Delta)$ to be the integer r so that $\Delta_r = \Delta$.

For convenience, we call the triple

$$\mathcal{C}_n(\Delta) := (\ell_n(\Delta); V_n(\Delta); r_n(\Delta))$$

the *nth characteristic vector* of Δ, or simply *characteristic vector* of Δ. The vector $\mathcal{C}_n(\Delta)$ contains the information about the length and neighborhood relation of Δ.

Define $W_n(\Delta) = (b_1, \ldots, b_k)$, where

$$b_j = \#\{\sigma \in \mathcal{A}_n,\ \rho^{-n}(a - S_\sigma(0)) = a_j\}, \quad j = 1, \ldots, k.$$

Here $a_1, \ldots, a_k$ are defined as in (2.1). We call $W_n(\Delta)$ the *nth multiplicity vector of* Δ. Denote $N_n(\Delta) = \|W_n(\Delta)\| := \sum_{i=1}^k b_i$. We call $N_n(\Delta)$ the *nth multiplicity* of Δ. One may check directly that

$$\begin{aligned} N_n(\Delta) &= \#\left\{\sigma \in \mathcal{A}_n : S_\sigma\left((0, 1)\right) \cap \Delta \neq \emptyset\right\} \\ &= \#\left\{\sigma \in \mathcal{A}_n : S_\sigma\left([0, 1]\right) \supset \Delta\right\}. \end{aligned} \tag{2.2}$$

2.2. Symbolic expressions of net intervals, and products of matrices for multiplicity vectors

We first consider the symbolic expressions of net intervals. Denote by Ω the collection of all possible distinct characteristic vectors, i.e.,

$$\Omega = \{\mathcal{C}_n(\Delta) : \ n \geqslant 0, \ \Delta \in \mathcal{F}_n\}. \tag{2.3}$$

For $\alpha \in \Omega$, we write for simplicity

$$\ell(\alpha) = \ell_n(\Delta), \ V(\alpha) = V_n(\Delta), \ v(\alpha) = v_n(\Delta), \ r(\alpha) = r_n(\Delta) \tag{2.4}$$

if $\alpha = \mathcal{C}_n(\Delta)$ for some $\Delta \in \mathcal{F}_n$.

The following lemma is our starting point.

Lemma 2.1. *The set Ω is finite.*

To prove the above result, we need the following result, which is based on an algebraic property of Pisot numbers.

Lemma 2.2. *There is a finite set B such that for any integer $n > 0$ and any $\sigma, \sigma' \in \mathcal{A}_n$,*

$$\textit{either} \qquad \rho^{-n}|S_\sigma(0) - S_{\sigma'}(0)| > 1 \qquad \textit{or} \qquad \rho^{-n}|S_\sigma(0) - S_{\sigma'}(0)| \in B. \tag{2.5}$$

Proof. Set $\alpha_1 = \rho^{-1}$ and let $\alpha_2, \ldots, \alpha_d$ denote the algebraic conjugates of α_1. Since α_1 is a Pisot number, we have $|\alpha_i| < 1$ for $2 \leqslant i \leqslant d$. It is proved in [22, Lemma 1.51] that for $P(x)$ a polynomial with integer coefficients and height $L = \max\{|a_i| : a_i$ is a coefficient of $P(x)\}$, if $P(\alpha_1) \neq 0$, then

$$|P(\alpha_1)| \geqslant L^{-d+1} \prod_{i=2}^{d} ||\alpha_i| - 1| \,. \tag{2.6}$$

Denote by B the set

$$\left\{\rho^{-n}|S_\sigma(0) - S_{\sigma'}(0)| \leqslant 1 : \ n \in \mathbb{N}, \ \sigma, \sigma' \in \mathcal{A}_n\right\}.$$

We claim that B is a finite set of cardinality less than

$$\frac{1-\rho}{\rho} \cdot \frac{2^{d-1}}{\prod_{i=2}^{d} ||\alpha_i| - 1|} + 1.$$

Assume the claim is not true, then by the Pigeon Hole principle there exist $t_1, t_2 \in B$ with

$$0 < |t_1 - t_2| < \frac{\rho}{1-\rho} \cdot \frac{\prod_{i=2}^{d} ||\alpha_i| - 1|}{2^{d-1}}.$$

However one may check that $t_1 - t_2 = \frac{\rho}{1-\rho} P(\alpha_1)$ for some polynomial $P(x)$ with height not more than 2, which leads to a contradiction with (2.6). $\square$

Proof of Lemma 2.1. It suffices to prove the finiteness of $\{\ell_n(\Delta) : n \geqslant 0, \Delta \in \mathcal{F}_n\}$, $\{V_n(\Delta) : n \geqslant 0, \Delta \in \mathcal{F}_n\}$ and $\{r_n(\Delta) : n \geqslant 0, \Delta \in \mathcal{F}_n\}$, respectively. For simplicity, we only prove that of $\{V_n(\Delta) : n \geqslant 0, \Delta \in \mathcal{F}_n\}$. To prove this, take any $\Delta = [a, b] \in \mathcal{F}_n$ and $e \in V_n(\Delta)$. Then by the definition of Δ_n, there exists $\sigma \in \mathcal{A}_n$ such that $S_\sigma(0) \in (a - \rho^n, a]$ and $e = \rho^{-n}(a - S_\sigma(0))$. It follows that $e \in B$ whenever $a \in P_n^0$, and $1 - e \in B$ whenever $a \in P_n(1)$, where B is defined as in Lemma 2.2. By the finiteness of B, the set $\{V_n(\Delta) : n \geqslant 0, \Delta \in \mathcal{F}_n\}$ is finite. $\square$

Now we present an elementary but important fact about characteristic vectors.

Lemma 2.3. *For a given $\Delta \in \mathcal{F}_n (n \geqslant 0)$, let $\Delta_1, \ldots, \Delta_k$ (ranked in the increasing order) be all the elements in $\mathcal{F}_{n+1}$ which are subintervals of Δ. Then the number k, the characteristic vectors $\mathcal{C}_{n+1}(\Delta_i)$ $(1 \leqslant i \leqslant k)$ are determined by $\ell_n(\Delta)$ and $V_n(\Delta)$ (thus they are determined by $\mathcal{C}_n(\Delta)$).*

Proof. Let $\Delta = [a, b] \in \mathcal{F}_n$. Write $V_n(\Delta) = (a_1, \ldots, a_{v_n(\Delta)})$.

To determine the subintervals of Δ which belong to $\mathcal{F}_{n+1}$, we first determine the points in $[a, b] \cap P_{n+1}$. Assume $\sigma = j_1 \ldots j_{n+1} \in \mathcal{A}_{n+1}$ such that $S_\sigma(0)$ or $S_\sigma(1)$ belongs to the interval (a, b). Then $S_\sigma((0, 1)) \cap (a, b) \neq \emptyset$, and consequently $S_{\hat{\sigma}}(0, 1) \cap (a, b) \neq \emptyset$, where $\hat{\sigma} = j_1 \ldots j_n \in \mathcal{A}_n$. Hence $S_{\hat{\sigma}}(0) \in \{a - \rho^n a_i : 1 \leqslant i \leqslant v_n(\Delta)\}$ and therefore

$$S_\sigma(0) \in \left\{a - \rho^n a_i + \rho^n \varepsilon : 1 \leqslant i \leqslant v_n(\Delta), \ \varepsilon = 0 \text{ or } 1\right\}$$

and

$$S_\sigma(1) \in \left\{a - \rho^n a_i + \rho^n \varepsilon + \rho^{n+1} : 1 \leqslant i \leqslant v_n(\Delta), \ \varepsilon = 0 \text{ or } 1\right\}.$$

This implies that

$$\begin{aligned}(a, b) \cap P_{n+1} = &\ (a, a + \rho^n \ell_n(\Delta)) \\ &\cap \left\{a - \rho^n a_i + \rho^n \varepsilon + \rho^{n+1}\delta : 1 \leqslant i \leqslant v_n(\Delta), \ \varepsilon, \delta \in \{0, 1\}\right\}.\end{aligned}$$

Denote by $a+\rho^n c_j$ $(1\leqslant j\leqslant u)$ all the elements of $[a,b]\cap P_{n+1}$ ranked in the increasing order. The above equality shows that the points c_j $(1\leqslant j\leqslant u)$ are determined completely by $\ell_n(\Delta)$ and $V_n(\Delta)$ (independent of a and n).

Let $\Delta_1,\ldots,\Delta_k$ (ranked in the increasing order) be all the elements in $\mathcal{F}_{n+1}$ which are subintervals of Δ. Then Δ_i $(1\leqslant i\leqslant k)$ are exact the intervals in the following collection:

$$\left\{[a+\rho^n c_j, a+\rho^n c_{j+1}]:\ 1\leqslant j\leqslant u-1\right\},$$

which is determined by $\ell_n(\Delta)$ and $V_n(\Delta)$.

Recall that

$$\begin{aligned}&\{S_\sigma(0):\sigma\in\mathcal{A}_{n+1},\ \ S_\sigma((0,1))\cap(a,b)\neq\emptyset\}\\&\quad\subset\{a-\rho^n a_i+\rho^n\varepsilon:\ 1\leqslant i\leqslant v_n(\Delta),\ \ \varepsilon=0 \text{ or } 1\}.\end{aligned}$$

By the definition of characteristic vector and the analysis in the preceding paragraph, we know that the vectors $\mathcal{C}_{n+1}(\Delta_i)$ $(1\leqslant i\leqslant k)$ are determined by $\ell_n(\Delta)$ and $V_n(\Delta)$. $\square$

Remark 2.4. In fact, the proof of Lemma 2.3 provides an algorithm to determine the elements of Ω. To see this, for $n\geqslant 0$ let Ω_n denote the collection of all possible kth characteristic vectors for $k\leqslant n$. It is clear that $\Omega_0=\{(0;1;0)\}$. Using the method in the proof of Lemma 2.3, one can determine $\Omega_1,\Omega_2,\ldots$ recursively. Furthermore, Ω equals Ω_n if $\Omega_{n+1}=\Omega_n$.

In the following, we would like to use a finite sequence of characteristic vectors to identify a net interval. For each $\Delta\in\mathcal{F}_n$ $(n\geqslant 0)$, we list the intervals

$$\Delta^0,\Delta^1,\ldots,\Delta^n$$

such that $\Delta^n=\Delta$, and Δ^j $(j=0,\ldots,n-1)$ is the unique element in $\mathcal{F}_j$ such that $\Delta^j\supset\Delta^{j+1}$. The sequence

$$\mathcal{C}_0(\Delta^0),\mathcal{C}_1(\Delta^1),\ldots,\mathcal{C}_n(\Delta^n)$$

is called the *symbolic expression of* Δ.

For a given $\Delta\in\mathcal{F}_n(n\geqslant 0)$, let $\Delta_1,\ldots,\Delta_k$ (ranked in the increasing order) be all the elements in $\mathcal{F}_{n+1}$ which are subintervals of Δ. The introduction of the third term in a characteristic vector guarantees that $C_{n+1}(\Delta_j)$ $(1\leqslant j\leqslant k)$ are distinct with each other. By induction, we have

Lemma 2.5. *For any $\Delta_1,\Delta_2\in\mathcal{F}_n(n\geqslant 1)$ with $\Delta_1\neq\Delta_2$, the symbolic expression of Δ_1 is different from that of Δ_2.*

Now we are going to define a natural map ξ from Ω to Ω^*, where Ω^* denotes the collection of all finite words over Ω. For any $\alpha \in \Omega$, pick n and $\Delta \in \mathcal{F}_n$ such that $\alpha = \mathcal{C}_n(\Delta)$. Let $\Delta_1, \ldots, \Delta_k$ (ranked in the increasing order) be all the elements in $\mathcal{F}_{n+1}$ which are subintervals of Δ. Write $\alpha_j = \mathcal{C}_{n+1}(\Delta_j)$ for $1 \leqslant j \leqslant k$. By Lemma 2.3, the word $\alpha_1 \ldots \alpha_k$ depend only on α (independent of the choice of n and Δ). We define ξ by

$$\xi(\alpha) = \alpha_1 \ldots \alpha_k. \tag{2.7}$$

The above ξ is called the *transition map*.

Define a 0–1 matrix A on $\Omega \times \Omega$ in the following way:

$$A_{\alpha,\beta} = \begin{cases} 1 & \text{if } \beta \text{ is a letter of } \xi(\alpha), \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

A word $\beta_1 \ldots \beta_n \in \Omega^*$ is called an *admissible word* if $A_{\beta_j,\beta_{j+1}} = 1$ for $1 \leqslant j < n$.

Remark 2.6. The proof of Lemma 2.3 also provides an algorithm to obtain ξ and A.

For our convenience, denote by $\gamma_0 = \mathcal{C}_0([0,1])$. Combining Lemma 2.5 and the above definitions, we have

Lemma 2.7. *Any $\Delta \in \mathcal{F}_n (n \geqslant 0)$ can be identified (via its symbolic expression) as an admissible word in Ω^* of length $n+1$ starting from the letter γ_0.*

In the remaining part of this subsection, we show that the multiplicity vector of any net interval can be expressed as a product of some transition matrices, according to the symbolic expression of this net interval.

Lemma 2.8. *For any $\Delta \in \mathcal{F}_n$ $(n \geqslant 1)$, denote by $\widehat{\Delta}$ the unique element in $\mathcal{F}_{n-1}$ so that $\widehat{\Delta} \supset \Delta$. There is a $v_{n-1}(\widehat{\Delta}) \times v_n(\Delta)$ matrix $T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta))$ which depends only on $\mathcal{C}_{n-1}(\widehat{\Delta})$ and $\mathcal{C}_n(\Delta)$ such that*

$$W_n(\Delta) = W_{n-1}(\widehat{\Delta}) T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)).$$

Proof. Assume $\Delta = [a,b]$ and $\widehat{\Delta} = [c,d]$. Write $V_n(\Delta) = (a_1, \ldots, a_{v_n(\Delta)})$ and $V_{n-1}(\widehat{\Delta}) = (c_1, \ldots, c_{v_{n-1}(\widehat{\Delta})})$. Also write $W_n(\Delta) = (q_1, \ldots, q_{v_n(\Delta)})$ and $W_{n-1}(\widehat{\Delta}) = (u_1, \ldots, u_{v_{n-1}(\widehat{\Delta})})$. By the definition of $W_n(\Delta)$ and $W_{n-1}(\widehat{\Delta})$, we have

$$q_i = \# \left\{ \sigma \in \mathcal{A}_n : \ \rho^{-n}(a - S_\sigma(0)) = a_i \right\}, \qquad i = 1, \ldots, v_n(\Delta)$$

and

$$u_j = \# \left\{ \sigma' \in \mathcal{A}_{n-1} : \ \rho^{-n}(c - S_{\sigma'}(0)) = c_j \right\}, \qquad j = 1, \ldots, v_{n-1}(\widehat{\Delta}).$$

Observe that if $\sigma = j_1 \dots j_n \in \mathcal{A}_n$ satisfies $\rho^{-n}(a - S_\sigma(0)) \in V_n(\Delta)$, i.e., $0 \leqslant a - S_\sigma(0) < \rho^{-n}$, then $0 \leqslant c - S_{\sigma^*}(0) \leqslant \rho^{n-1}$ for $\sigma^* = j_1 \dots j_{n-1}$, and thus $\rho^{-n+1}(c - S_{\sigma^*}(0)) \in V_{n-1}(\widehat{\Delta})$. Now define a $v_{n-1}(\widehat{\Delta}) \times v_n(\Delta)$ matrix $T = (t_{j,i})$ by

$$t_{j,i} = \begin{cases} 1, & \exists\, \varepsilon \in \{0,1\} \text{ so that } c - \rho^{n-1} c_j + \rho^{n-1}\varepsilon = a - \rho^n a_i, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $t_{j,i} = 1$ if and only if there is $\sigma = i_1 \dots i_n \in \mathcal{A}_n$ such that $S_\sigma(0) = a - \rho^n a_i$ and $S_{i_1 \dots i_{n-1}}(0) = c - \rho^{n-1} c_j$. By the last observation, we have

$$\begin{aligned} &\#\left\{\sigma \in \mathcal{A}_n : \ \rho^{-n}(a - S_\sigma(0)) = a_i\right\} \\ &\quad = \sum_{j=1}^{v_{n-1}(\widehat{\Delta})} t_{j,i} \cdot \#\left\{\sigma' \in \mathcal{A}_{n-1} : \ \rho^{-n}(c - S_{\sigma'}(0)) = c_j\right\}. \end{aligned}$$

That is, $q_i = \sum_{j=1}^{v_{n-1}(\widehat{\Delta})} t_{j,i} u_j$. Therefore we have $W_n(\Delta) = W_{n-1}(\widehat{\Delta})T$. This completes the proof. $\square$

The above result, together with the fact $W_0([0,1]) = 1$, yields immediately

Theorem 2.9. *There exists a family of non-negative matrices* $\{T(\alpha,\beta) : \ \alpha, \beta \in \Omega, \ A_{\alpha,\beta} = 1\}$, *such that for any* $\Delta \in \mathcal{F}_n$,

$$W_n(\Delta) = T(\gamma_0, \gamma_1) \dots T(\gamma_{n-1}, \gamma_n),$$

where $\gamma_0 \dots \gamma_n$ *is the symbolic expression of* Δ.

For convenience we call the above $T(\alpha,\beta)$'s the *transition matrices*.

2.3. Distributions of μ on net intervals

In this subsection we analyze the distributions of μ on net intervals. We start from the following lemma.

Lemma 2.10. *Let* Δ *be an nth net interval. Write* $\ell_n(\Delta) = \ell$, $V_n(\Delta) = (a_1, \dots, a_v)$ *and* $W_n(\Delta) = (b_1, \dots, b_v)$. *Then*

(i) *there exists a constant* $C > 0$ *such that* $C\rho^n \leqslant |\Delta| \leqslant \rho^n$, *where* $|\Delta|$ *denotes the length of* Δ;

(ii) $\mu(\Delta) = 2^{-n} \sum_{i=1}^{v} b_i \mu([a_i, a_i + \ell])$;

(iii) *there exists a constant* $D > 0$ *such that*

$$D 2^{-n} N_n(\Delta) \leqslant \mu(\Delta) \leqslant 2^{-n} N_n(\Delta).$$

Proof. Let $\Delta = [a, b]$. By the definition of the characteristic vector, we have $|\Delta| = \rho^n \ell$. Since Ω is finite, we have $\ell \geqslant C$ for some constant $C > 0$. Note that Δ is always contained in $S_\sigma([0, 1])$ for some $\sigma \in \mathcal{A}_n$, it follows that $|\Delta| \leqslant \rho^n$. This completes the proof of (i).

To see (ii), we iterate (1.2) n times and have

$$\mu(\Delta) = 2^{-n} \sum_{\sigma \in \mathcal{A}_n} \mu\left(S_\sigma^{-1}(\Delta)\right).$$

Since μ is a non-atomic measure supported on $[0, 1]$, we have

$$\begin{aligned} \mu(\Delta) &= 2^{-n} \sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0,1) \cap \Delta \neq \emptyset} \mu\left(S_\sigma^{-1}(\Delta)\right) \\ &= 2^{-n} \sum_{i=1}^{v} \sum_{\sigma \in \mathcal{A}_n:\ \rho^{-n}(a - S_\sigma(0)) = a_i} \mu\left(S_\sigma^{-1}(\Delta)\right) \\ &= 2^{-n} \sum_{i=1}^{v} b_i \mu([a_i, a_i + \ell]). \end{aligned}$$

Part (iii) follows from (ii) and the finiteness of Ω. $\square$

The following lemma is used to compare the distributions of μ on two adjacent nth net intervals.

Lemma 2.11. *Suppose that Δ_1 and Δ_2 are two adjacent nth net intervals ($n \geqslant 1$). Then*

$$\frac{1}{n+1} N_n(\Delta_1) \leqslant N_n(\Delta_2) \leqslant (n+1) N_n(\Delta_1). \tag{2.9}$$

Proof. We prove the statement by induction.

One may verify (2.9) directly for the case $n = 1$, since there are exact three first net intervals with multiplicities 1, 2, and 1, respectively. Now assume that (2.9) holds for $n \leqslant k$. In the following we will show that (2.9) holds for $n = k + 1$.

Suppose that Δ_1, Δ_2 are two adjacent $(k+1)$th net intervals, where Δ_1 lies on the left hand side of Δ_2. We will consider the following two possible cases separately:

(a) Δ_1, Δ_2 are contained in the same kth net interval U.
(b) Δ_1, Δ_2 are contained in two adjacent kth net intervals U_1, U_2, respectively.

In case (a), by (2.2) we have

$$N_k(U) \leqslant N_{k+1}(\Delta_j) \leqslant 2 N_k(U), \quad j = 1, 2$$

and thus

$$\tfrac{1}{2}\, N_{k+1}(\Delta_1) \leqslant N_{k+1}(\Delta_2) \leqslant 2N_{k+1}(\Delta_1).$$

Therefore (2.9) holds for Δ_1, Δ_2 whenever $n = k + 1$.

In case (b), let us define

$$\begin{aligned}
\mathcal{D}_1 &= \{\omega \in \mathcal{A}_k : \ S_\omega([0,1]) \supset U_1, \text{ and they share the same right end-point}\},\\
\mathcal{D}_2 &= \{\omega \in \mathcal{A}_k \backslash \mathcal{D}_1 : \ S_\omega([0,1]) \supset U_1\},\\
\mathcal{D}_3 &= \{\omega \in \mathcal{A}_k : \ S_\omega([0,1]) \supset U_2, \text{ and they share the same left end-point}\},\\
\mathcal{D}_4 &= \{\omega \in \mathcal{A}_k \backslash \mathcal{D}_3 : \ S_\omega([0,1]) \supset U_2\}.
\end{aligned}$$

From (2.2) and the definition of net interval, we have

$$\begin{aligned}
&\mathcal{D}_2 = \mathcal{D}_4,\\
&N_k(U_1) = \#\mathcal{D}_1 + \#\mathcal{D}_2, \qquad N_k(U_2) = \#\mathcal{D}_3 + \#\mathcal{D}_4,\\
&\#\mathcal{D}_1 + \#\mathcal{D}_2 \leqslant N_{k+1}(\Delta_1) \leqslant \#\mathcal{D}_1 + 2\#\mathcal{D}_2,\\
&\#\mathcal{D}_3 + \#\mathcal{A}_4 \leqslant N_{k+1}(\Delta_2) \leqslant \#\mathcal{D}_3 + 2\#\mathcal{D}_4.
\end{aligned}$$

According to the above relations and the assumption

$$\frac{1}{k+1}\, N_k(U_1) \leqslant N_k(U_2) \leqslant (k+1) N_k(U_1),$$

we have

$$\frac{1}{k+2}\, N_{k+1}(\Delta_1) \leqslant N_{k+1}(\Delta_2) \leqslant (k+2) N_{k+1}(\Delta_1).$$

This completes the proof. □

As a corollary of the above two lemmas, we have

Corollary 2.12. *There exists a positive constant C such that*

$$\tfrac{1}{nC}\, \mu(\Delta_1) \leqslant \mu(\Delta_2) \leqslant nC\, \mu(\Delta_1), \tag{2.10}$$

for any $n \geqslant 1$ and any two adjacent nth net intervals Δ_1, Δ_2. Furthermore for a fixed point $x \in [0,1]$,

$$\lim_{n\to\infty} \frac{\log \mu([x - \rho^n, x + \rho^n])}{\log \mu(I_n(x))} = 1,$$

where $I_n(x)$ is an nth net interval containing x.

3. Fractal indices in terms of products of matrices

In this chapter we study some fractal indices about the Bernoulli convolution μ and the limited Rademacher function f associated with a given Pisot reciprocal ρ. We will re-express those indices as the limits in terms of products of transition matrices.

3.1. Local dimensions of μ

In this subsection we are concerned with the local dimensions of μ. Recall that for any $x \in [0, 1]$, the local upper and lower dimensions of μ at x, denoted as $\overline{d}(\mu, x)$ and $\underline{d}(\mu, x)$, respectively, are defined by

$$\overline{d}(\mu, x) = \limsup_{r\to 0} \frac{\log \mu([x-r, x+r])}{\log r}, \qquad \underline{d}(\mu, x) = \liminf_{r\to 0} \frac{\log \mu([x-r, x+r])}{\log r}.$$

A simple argument shows that

$$\begin{aligned}\overline{d}(\mu, x) &= \limsup_{n\to\infty} \frac{\log \mu([x-\rho^n, x+\rho^n])}{n\log \rho},\\ \underline{d}(\mu, x) &= \liminf_{n\to\infty} \frac{\log \mu([x-\rho^n, x+\rho^n])}{n\log \rho}.\end{aligned}$$

This combining Corollary 2.12 yields

Lemma 3.1. *For any $x \in [0, 1]$, we have*

$$\overline{d}(\mu, x) = \limsup_{n\to\infty} \frac{\log \mu(I_n(x))}{n\log \rho}, \qquad \underline{d}(\mu, x) = \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{n\log \rho},$$

where $I_n(x)$ denotes an nth net interval containing x.

Let Ω and A be constructed as in Section 2. Denote by $\Omega_A^{\mathbb{N}}$ the collection of all admissible words of infinite length, i.e.,

$$\Omega_A^{\mathbb{N}} = \left\{ y = (y_i)_{i=1}^{\infty} : \ y_i \in \Omega, A_{y_i, y_{i+1}} = 1 \right\}.$$

We use $[\gamma_0]$ to denote the sub-collection of all admissible words of infinite length starting from γ_0, the characteristic vector of the 0th net interval $[0, 1]$. This is

$$[\gamma_0] = \{ y = (y_i) \in \Omega_A^{\mathbb{N}} : \ y_1 = \gamma_0 \}.$$

There is a natural projection π from $[\gamma_0]$ to the interval $[0,1]$ defined by

$$\pi(y) = \bigcap_{n=1}^{\infty} \varDelta(y_1 \cdots y_{n+1}), \qquad y = (y_i), \tag{3.1}$$

where $\varDelta(y_1 \cdots y_{n+1})$ denote the nth net interval with the symbolic expression $y_1 \cdots y_{n+1}$.

Let $\{T(\alpha,\beta),\ \ \alpha,\beta \in \Omega,\ \ A_{\alpha,\beta} = 1\}$ be the class of transition matrices given as in Theorem 2.9. Write for shortly $T_{\alpha_1\alpha_2\cdots\alpha_{n+1}} := T(\alpha_1,\alpha_2)\cdots T(\alpha_n,\alpha_{n+1})$. Then we have

Theorem 3.2. *For any $y = (y_i) \in [\gamma_0]$, we have*

$$\overline{d}(\mu, \pi(y)) = -\frac{\log 2}{\log \rho} + \limsup_{n\to\infty} \frac{\log \|T_{y_1\cdots y_{n+1}}\|}{n \log \rho},$$

the lower dimension $\underline{d}(\mu, \pi(y))$ can be obtained by taking the lower limit.

Proof. Let $x = \pi(y)$. Then $\varDelta(y_1 \cdots y_{n+1})$ is an nth net interval containing x. By Theorem 2.9,

$$N_n(\varDelta(y_1 \cdots y_{n+1})) = \|W_n(\varDelta(y_1 \cdots y_{n+1}))\| = \|T_{y_1\cdots y_{n+1}}\|.$$

Hence by Lemma 2.10(iii), $\mu(\varDelta(y_1 \cdots y_{n+1})) \approx 2^{-n}\|T_{y_1\cdots y_{n+1}}\|$. Using Lemma 3.1, we obtain the desired result. $\square$

3.2. L^q-spectrum of μ

In this subsection, we express the L^q-spectrum of μ as a limit in terms of products of transition matrices.

Recall for any $q \in \mathbb{R}$, the L^q-spectrum $\tau(q)$ of μ is defined as

$$\tau(q) = \liminf_{\delta\to 0} \frac{\log \sup \sum_i \mu([x_i - \delta, x_i + \delta])^q}{\log \delta},$$

where the superium is taken over all the families of disjoint intervals $[x_i - \delta, x_i + \delta]$ with $x_i \in [0,1]$. We will show that

Theorem 3.3. *For any $q \in \mathbb{R}$, we have*

$$\tau(q) = -\frac{q \log 2}{\log \rho} + \liminf_{n\to\infty} \frac{\log \sum \|T_{\alpha_1\cdots\alpha_{n+1}}\|^q}{n \log \rho}, \tag{3.2}$$

where the summation is taken over all admissible words $\alpha_1 \cdots \alpha_{n+1}$ *of length* $n+1$ *with* $\alpha_1 = \gamma_0$.

Proof. Suppose Δ is an nth net interval with the symbolic expression $\alpha_1 \cdots \alpha_{n+1}$. Then by Theorem 2.9 and Lemma 2.10, $\mu(\Delta) \approx 2^{-n}\|T_{\alpha_1\cdots\alpha_{n+1}}\|$. It follows that the right hand side of (3.2) equals

$$R(q) := \liminf_{n\to\infty} \frac{\log \sum_{\Delta\in\mathcal{F}_n} \mu(\Delta)^q}{n \log \rho}.$$

In the following we show $\tau(q) = R(q)$.

Let C be the constant in Lemma 2.10. Then $C\rho^n \leqslant |\Delta| \leqslant \rho^n$ for any $\Delta \in \mathcal{F}_n$. Now fix n and take $\delta = \frac{1}{2}C\rho^n$. For each $\Delta \in \mathcal{F}_n$, construct an interval Δ' with $\Delta' \subset \Delta$ such that $|\Delta'| = 2\delta$ and $\mu(\Delta') \geqslant \frac{1}{4}C\mu(\Delta)$. The intervals Δ''s are disjoint and satisfy

$$\sum_{\Delta\in\mathcal{F}_n} \mu(\Delta)^q \leqslant \begin{cases} (\frac{1}{4}c)^{-q} \sum_{\Delta\in\mathcal{F}_n} \mu(\Delta')^q & \text{if } q \geqslant 0, \\ \sum_{\Delta\in\mathcal{F}_n} \mu(\Delta')^q & \text{otherwise,} \end{cases}$$

which implies $\tau(q) \leqslant R(q)$.

Now let us show the reverse inequality. For any small $\delta > 0$, let n be the integer satisfying $\rho^n < \delta \leqslant \rho^{n-1}$. Suppose that $[x_i - \delta, x_i + \delta]_i$ is a family of disjoint intervals with $x_i \in [0,1]$. Then for each i, $[x_i - \delta, x_i + \delta]$ intersects at most $\frac{2}{C\rho} + 1$ many nth net intervals. It follows that when $q \geqslant 0$,

$$\begin{aligned} \sum_i \mu([x_i - \delta, x_i + \delta])^q &\leqslant \sum_i \left(\sum_{\Delta\in\mathcal{F}_n,\ \Delta\cap[x_i-\delta,x_i+\delta]\neq\emptyset} \mu(\Delta) \right)^q \\ &\leqslant (2C^{-1}\rho^{-1} + 1)^q \sum_i \sum_{\Delta\in\mathcal{F}_n,\ \Delta\cap[x_i-\delta,x_i+\delta]\neq\emptyset} \mu(\Delta)^q \\ &\leqslant 2(2C^{-1}\rho^{-1} + 1)^q \sum_{\Delta\in\mathcal{F}_n} \mu(\Delta)^q, \end{aligned} \tag{3.3}$$

where the last inequality uses the fact that each nth net interval intersects at most two distinct intervals $[x_i - \delta, x_i + \delta]$. Note that for each i, the interval $[x_i - \delta, x_i + \delta]$ contains at least one mth net intervals. This implies

$$\sum_i \mu([x_i - \delta, x_i + \delta])^q \leqslant \sum_{\Delta\in\mathcal{F}_n} \mu(\Delta)^q, \qquad \forall\ q < 0. \tag{3.4}$$

The inequality $\tau(q) \geqslant R(q)$ follows from (3.3) and (3.4). $\square$

3.3. *The Hausdorff dimension of the graph of f*

Let $\Gamma(f)$ denote the graph of the limited Rademacher function *f*. In [53, p. 184], Przytycki and Urbański gave a formula of the Hausdorff dimension of $\Gamma(f)$, which is based on the McMullen's formula on the Hausdorff dimension of a class of self-affine sets [42,51]. For $n \in \mathbb{N}$, let $a_{n,1}, \ldots, a_{n,s_n}$ (ranked in the increasing order) be all the distinct points in $\{S_\sigma(0) : \ \sigma \in \mathcal{A}_n\}$. For $j = 1, \ldots, s_n$, denote by

$$d_{n,j} = \#\{\sigma \in \mathcal{A}_n : \ S_\sigma(0) = a_{n,j}\}.$$

The formula given by Przytycki and Urbański is just

$$\dim_{\mathrm{H}} \Gamma(f) = \lim_{n\to\infty} \frac{\log \sum_{j=1}^{s_n} (d_{n,j})^{-\log\rho/\log 2}}{-n\log\rho}. \tag{3.5}$$

Based on the above formula, we have

Theorem 3.4. *The Hausdorff dimension of the graph of f satisfies*

$$\dim_{\mathrm{H}} \Gamma(f) = \lim_{n\to\infty} \frac{\log \sum \|T_{\alpha_1\cdots\alpha_{n+1}}\|^{-\log\rho/\log 2}}{-n\log\rho}, \tag{3.6}$$

where the summation is taken over all admissible words $\alpha_1\cdots\alpha_{n+1}$ *of length* $n+1$ *with* $\alpha_1 = \gamma_0$.

Proof. Denote $u = -\log\rho/\log 2$. Then $0 < u < 1$. Set

$$m = \sup\{v_n(\Delta) : \ \Delta \in \mathcal{F}_n, n \in \mathbb{N}\},$$

where $v_n(\Delta)$ is the dimension of the multiplicity vector $W_n(\Delta)$. By the finiteness of Ω, we have $0 < m < \infty$.

Now fix *n*. Take $\Delta = [c,d] \in \mathcal{F}_n$. Write $W_n(\Delta) = (b_1, \ldots, b_{v_n(\Delta)})$. Then we have

$$\frac{1}{m} \sum_{i=1}^{v_n(\Delta)} b_i^u \leqslant \|W_n(\Delta)\|^u \leqslant \sum_{i=1}^{v_n(\Delta)} b_i^u. \tag{3.7}$$

By the definition of multiplicity vector, the set $\{b_i : \ 1 \leqslant i \leqslant v_n(\Delta)\}$ is equal to

$$\bigcup_{j:\ a_{n,j} \in (c-\rho^n, c]} \{d_{n,j}\}.$$

Therefore we have

$$\frac{1}{m}\sum_{j:\ a_{n,j}\in(c-\rho^n,c]} d_{n,j}^u \leqslant \|W_n(\Delta)\|^u \leqslant \sum_{j:\ a_{n,j}\in(c-\rho^n,c]} d_{n,j}^u. \tag{3.8}$$

Observe that for each $1\leqslant j < s_n$, there is at least one and at most C^{-1} many distinct $\Delta=[c,d]\in\mathcal{F}_n$ satisfying $0\leqslant c-a_{n,j}<\rho^n$, where C is the constant in Lemma 2.10. Taking the summation over $\Delta\in\mathcal{F}_n$ in (3.8), we have

$$\frac{1}{m}\sum_{j=1}^{s_n} d_{n,j}^u \leqslant \sum_{\Delta\in\mathcal{F}_n}\|W_n(\Delta)\|^u \leqslant C^{-1}\sum_{j=1}^{s_n} d_{n,j}^u.$$

This combining (3.5) and Theorem 2.9 yields the desired result. $\square$

3.4. The box dimension of the level sets of f

Recall that the level set of f at t, denoted as L_t, is defined by

$$L_t=\{x\in[0,1]:\ f(x)=t\}. \tag{3.9}$$

Let the projection $\pi:[\gamma_0]\to[0,1]$ be defined as in (3.1). In this subsection we prove

Theorem 3.5. *For any* $y=(y_i)_{i=1}^{\infty}\in[\gamma_0]$, *we have*

$$\overline{\dim}_{\mathrm{B}} L_{\pi(y)}=\limsup_{n\to\infty}\frac{\log\|T_{y_1\ldots y_{n+1}}\|}{n\log 2},$$

$$\underline{\dim}_{\mathrm{B}} L_{\pi(y)}=\liminf_{n\to\infty}\frac{\log\|T_{y_1\ldots y_{n+1}}\|}{n\log 2}.$$

where $\overline{\dim}_{\mathrm{B}}$, $\underline{\dim}_{\mathrm{B}}$ *denote the upper and lower box dimension.*

As a corollary of Theorems 3.5 and 3.2, we have

Corollary 3.6. *For any* $t\in[0,1]$, *we have*

$$\overline{\dim}_{\mathrm{B}} L_t=1+\frac{\log\rho}{\log 2}\cdot\underline{d}(\mu,t),\qquad \underline{\dim}_{\mathrm{B}} L_t=1+\frac{\log\rho}{\log 2}\cdot\overline{d}(\mu,t).$$

To prove Theorem 3.5, we first give a lemma.

Lemma 3.7. *For $\sigma = a_1 \cdots a_n \in \mathcal{A}_n$, let I_σ denote the nth 2-adic interval $\left[\sum_{i=1}^n a_i 2^{-i}, \sum_{i=1}^n a_i 2^{-i} + 2^{-n}\right)$. Then the image of I_σ under f is the interval*

$$J_\sigma := \left[(1-\rho)\sum_{i=1}^n a_i \rho^{i-1},\ (1-\rho)\sum_{i=1}^n a_i \rho^{i-1} + \rho^n\right).$$

Proof. Recall that the Rademacher function R is defined on $\mathbb{R}$ with period 1, taking values 0 and 1 on $[0, 1/2)$ and $[1/2, 1)$, respectively. If $x \in I_\sigma$, it can be checked directly that

$$R(2^{i-1}x) = a_i, \qquad i = 1, \ldots, n.$$

Therefore

$$\begin{aligned} f(x) &= (1-\rho)\sum_{i=1}^\infty R(2^{i-1}x)\rho^{i-1} \\ &= (1-\rho)\sum_{i=1}^n a_i \rho^{i-1} + (1-\rho)\sum_{j=n+1}^\infty R(2^{j-1}x)\rho^{j-1}. \end{aligned}$$

Hence $f(x) \in J_\sigma$.

In the following we show that for any $t \in J_\sigma$, there exists $x \in I_\sigma$ such that $f(x) = t$. To prove this, set $z = \frac{t}{1-\rho} - \sum_{i=1}^n a_i \rho^{i-1}$. It is clear $z \in \left[0, \frac{\rho^n}{1-\rho}\right)$ and $t = (1-\rho)\left(\sum_{i=1}^n a_i \rho^{i-1} + z\right)$. Let us consider the following ρ^{-1}-expansion

$$z = \sum_{i=n+1}^\infty a_i \rho^{i-1}, \tag{3.10}$$

where the 0–1 coefficients a_i $(i \geqslant n+1)$ are defined by induction as follows:

$$a_{n+1} = \begin{cases} 1 & \text{if } z \geqslant \rho^n, \\ 0 & \text{otherwise,} \end{cases}$$

and if $a_{n+1}, \ldots, a_{n+k}$ are defined well, then

$$a_{n+k+1} = \begin{cases} 1 & \text{if } z \geqslant (\sum_{i=n+1}^{n+k} a_i \rho^{i-1}) + \rho^{n+k}, \\ 0 & \text{otherwise.} \end{cases}$$

The sequence $\{a_i\}_{i \geqslant n+1}$ constructed as above satisfies the following property: for any $m \in \mathbb{N}$, there is $k > m$ such that $a_k = 0$. Assume this is not true, i.e., there is k_0 such that $a_k = 1$ for all $k > k_0$. Since $z < \rho^n/(1-\rho)$, $a_i \neq 1$ for some $i \geqslant n+1$. Hence we can assume $a_{k_0} = 0$. That means

$$z < \sum_{i=n+1}^{k_0-1} \rho^{i-1} + \rho^{k_0-1}.$$

However,

$$\begin{aligned} z &= \sum_{i=n+1}^{k_0-1} \rho^{i-1} + \sum_{j=k_0+1}^{\infty} \rho^{j-1} \\ &= \sum_{i=n+1}^{k_0-1} \rho^{i-1} + \frac{\rho^{k_0}}{1-\rho} \\ &> \sum_{i=n+1}^{k_0-1} \rho^{i-1} + \rho^{k_0-1}, \end{aligned}$$

which leads to a contradiction. Now define $x = \sum_{i=1}^{\infty} a_i 2^{-i}$. Since $\{a_i\}_{i \geqslant n+1}$ satisfies the above property, we have

$$R(2^{i-1}x) = a_i, \qquad i = 1, 2, \ldots.$$

Therefore $t = f(x)$. $\square$

Proof of Theorem 3.5. For $y = (y_i)_{i=1}^{\infty}$, let $t = \pi(y)$. Set

$$N(n,t) = \#\left\{\sigma = a_1 \ldots a_n \in \mathcal{A}_n : \ \exists x \in I_{a_1 \ldots a_n} \text{ such that } f(x) = t\right\},$$

where $I_{a_1 \ldots a_n} = \left[\sum_{i=1}^n a_i 2^{-i}, \sum_{i=1}^n a_i 2^{-i} + 2^{-n}\right)$. By the definition of box dimension, we have

$$\overline{\dim}_{\mathrm{B}}\, L_t = \limsup_{n\to\infty} \frac{\log N(n,t)}{-n \log 2}, \quad \underline{\dim}_{\mathrm{B}}\, L_t = \liminf_{n\to\infty} \frac{\log N(n,t)}{-n \log 2}.$$

By Lemma 3.7, we have

$$N(n,t) = \#\{\sigma \in \mathcal{A}_n : t \in S_\sigma([0,1))\}.$$

In the following we show that if Δ is an n-net interval containing t, then

$$\frac{1}{n+1} N(n,t) \leqslant \|W_n(\Delta)\| \leqslant (n+1) N(n,t). \tag{3.11}$$

Table 2
Elements in Ω

$\alpha \in \Omega$	Labelled as
$(1; 0; 1)$	1
$(\rho; 0; 1)$	2
$(1-\rho; (0, \rho); 1)$	3
$(\rho; 1-\rho; 1)$	4
$(\rho; (0, 1-\rho); 1)$	5
$(2\rho-1; 1-\rho; 1)$	6
$(1-\rho; (0, \rho); 2)$	7

In fact if $t = 1$, we may check directly that $N(n,t) = \|W_n(\Delta)\| = 1$ and thus (3.11) holds. Now we assume $t \in [0, 1)$. In this case, there is a unique nth net interval $\Delta_1 = [c, d]$ such that $t \in [c, d)$; furthermore for $\sigma \in \mathcal{A}_n$,

$$t \in S_\sigma([0,1)) \Longleftrightarrow 0 \leqslant c - S_\sigma(0) < \rho^n.$$

This implies that

$$\|W_n(\Delta_1)\| = N(n,t).$$

However if Δ_2 is another nth interval containing t, then Δ_2 and Δ_1 are adjacent and thus by Lemma 2.11,

$$\frac{1}{n+1}\|W_n(\Delta_1)\| \leqslant \|W_n(\Delta_2)\| \leqslant (n+1)\|W_n(\Delta_1)\|.$$

Hence (3.11) holds for $t \in [0, 1)$. By Theorem 2.9, we obtain the desired result. $\square$

4. The golden ratio case

In this section we consider the concrete case $\rho = \frac{\sqrt{5}-1}{2}$, the reciprocal of the golden ratio.

4.1. The symbolic expressions and transition matrices

Let Ω be the collection of all possible characteristic vectors. By a direct check (see Remark 2.4) there are exact 7 elements in Ω. We list them out Table 2.

For our convenience, each element in Ω is labelled by a digit from 1 to 7 as in the above table. Especially, the characteristic vector $(1; 0; 1)$ of the 0th net interval $[0, 1]$, is labelled as 1. Without confusion, we write directly

$$\Omega = \{1, 2, \dots, 7\}. \tag{4.1}$$

The transition map ξ, defined as in (2.7), is given by

$$\begin{aligned}
\xi(1) &= 234,\\
\xi(2) &= 23,\\
\xi(3) &= 5,\\
\xi(4) &= 34,\\
\xi(5) &= 367,\\
\xi(6) &= 3,\\
\xi(7) &= 5
\end{aligned}$$

and the 0–1 matrix A, which is induced by ξ, is the following:

$$A = \begin{bmatrix} 0&1&1&1&0&0&0\\ 0&1&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&1&1&0&0&0\\ 0&0&1&0&0&1&1\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0 \end{bmatrix}. \tag{4.2}$$

The transition matrices $T(i, j)$ constructed as in the proof of Lemma 2.8, are listed as follows:

$$\begin{cases}
T(1,2) = 1, & T(1,3) = [1,1], \ T(1,4) = 1,\\
T(2,2) = 1, & T(2,3) = [1,1],\\
T(3,5) = \begin{bmatrix} 1&0\\ 0&1 \end{bmatrix}, & \\
T(4,3) = [1,1], & T(4,4) = 1,\\
T(5,3) = \begin{bmatrix} 1&1\\ 0&1 \end{bmatrix}, & T(5,6) = \begin{bmatrix} 1\\ 1 \end{bmatrix}, \ T(5,7) = \begin{bmatrix} 1&0\\ 1&1 \end{bmatrix},\\
T(6,3) = [1,1], & \\
T(7,5) = \begin{bmatrix} 1&0\\ 0&1 \end{bmatrix}. &
\end{cases} \tag{4.3}$$

For any $n \geqslant 1$, denote by $\Omega_{A,n}$ the collection of all admissible words of length n, i.e.,

$$\Omega_{A,n} = \left\{ i_1 \ldots i_n \in \Omega^* : \ A_{i_j, i_{j+1}} = 1, \ 1 \leqslant j < n \right\},$$

where Ω^* denotes the collection of all finite words over Ω. By Lemma 2.7 and Theorem 2.9, each nth net interval Δ corresponds to a unique word $i_1 \ldots i_{n+1}$ in $\Omega_{A,n+1}$ with

$i_1 = 1$; and the multiplicity vector $W_n(\Delta)$ satisfies

$$W_n(\Delta) = T_{i_1 \ldots i_{n+1}} := T(i_1, i_2)T(i_2, i_3) \ldots T(i_n, i_{n+1}).$$

To analyze the structure of $\Omega_{A,n}$, as well as the above products of matrices, we present the following simple but important fact:

Lemma 4.1. (i) $\widehat{\Omega} := \{3, 5, 6, 7\}$ *is an essential subclass of* Ω. *That is,* $\{\beta \in \Omega : A_{\alpha,\beta} = 1\} \subset \widehat{\Omega}$ *for any* $\alpha \in \widehat{\Omega}$; *and for any* $\alpha, \beta \in \widehat{\Omega}$, *there exist* $\gamma_1, \ldots, \gamma_n \in \widehat{\Omega}$ *such that* $\gamma_1 = \alpha$, $\gamma_n = \beta$ *and* $A_{\gamma_i, \gamma_{i+1}} = 1$ *for* $1 \leqslant i \leqslant n - 1$.

(ii) *The characteristic vector* $\alpha = (2\rho - 1; 1 - \rho; 1)$, *represented by the symbol* 6 *in* $\widehat{\Omega}$, *satisfies* $v(\alpha) = 1$.

For our convenience, denote by F_n the collection of all admissible words of length n starting from 3, i.e.,

$$F_n = \{i_1 \ldots i_n \in \Omega_{A,n} : \ i_1 = 3\}.$$

By Lemma 4.1, $F_n \subset \widehat{\Omega}_{A,n}$.

In the following we give the structure of the words in $\Omega_{A,n}$, as well as the corresponding products of matrices.

Lemma 4.2. *For* $n \geqslant 1$, *suppose* $w = i_1 \ldots i_{n+1} \in \Omega_{A,n+1}$ *with* $i_1 = 1$. *Then all the possible forms of* ω, *as well as* $\|T_\omega\|$, *are the following*:

(1) $\omega = 1\underbrace{2 \ldots 2}_{n}$, *and* $\|T_\omega\| = 1$.

(2) $\omega = 1\underbrace{4 \ldots 4}_{n}$, *and* $\|T_\omega\| = 1$.

(3) $\omega = 1\underbrace{2 \ldots 2}_{k}v$, $0 \leqslant k < n$, $v \in F_{n-k}$, *and* $\|T_\omega\| = \|T_{1v}\|$.

(4) $\omega = 1\underbrace{4 \ldots 4}_{k}v$, $0 \leqslant k < n$, $v \in F_{n-k}$, *and* $\|T_\omega\| = \|T_{1v}\|$.

Now set $\delta_0 = 3$, $\delta_1 = 7$ and

$$M_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad M_\emptyset = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.4}$$

Write for simplicity $M_{i_1 \ldots i_n} = M_{i_1} \ldots M_{i_n}$. Then we have

Lemma 4.3. *Let* $v \in F_n$ *for some* $n \geqslant 1$. *Then all the possible forms of* v, *as well as the value of* $\|T_{1v}\|$, *are the following*:

(1) $n = 1$, $v = 3$. $\|T_{1v}\| = 2$.

(2) $n = 2$, $v = 35$. $\|T_{1v}\| = 2$.
(3) $n = 2k$, $k \geqslant 2$, $v = 35\ \delta_{i_1}5 \ldots \delta_{i_{k-1}}5$ *with* $i_1 \ldots i_{k-1} \in \mathcal{A}_{k-1}$.
$\|T_{1v}\| = \|M_{i_1 \ldots i_{k-1}}\|$.
(4) $n = 2k$, $k \geqslant 2$, $v = 35\ \delta_{i_1}5 \ldots \delta_{i_\ell}5\ 6\eta$, *where* $0 \leqslant \ell \leqslant k-2$, $i_1 \ldots i_\ell \in \mathcal{A}_\ell$,
$\eta \in F_{n-3-2\ell}$. $\|T_{1v}\| = \|M_{i_1 \ldots i_\ell}\| \times \|T_{1\eta}\|$.
(5) $n = 2k+1$, $k \geqslant 1$, $v = 35\ \delta_{i_1}5 \ldots \delta_{i_{k-1}}5\ \delta_{i_k}$, *where* $i_1 \ldots i_k \in \mathcal{A}_k$.
$\|T_{1v}\| = \|M_{i_1 \ldots i_k}\|$.
(6) $n = 2k+1$, $k \geqslant 1$, $v = 35\ \delta_{i_1}5 \ldots \delta_{i_{k-1}}5\ 6$, *where* $i_1 \ldots i_{k-1} \in \mathcal{A}_{k-1}$.
$\|T_{1v}\| = \|M_{i_1 \ldots i_{k-1}}\|$.
(7) $n = 2k+1$, $k \geqslant 2$, $v = 35\ \delta_{i_1}5 \ldots \delta_{i_\ell}5\ 6\eta$, *where* $0 \leqslant \ell \leqslant k-2$, $i_1 \ldots i_\ell \in \mathcal{A}_\ell$,
$\eta \in F_{n-3-2\ell}$. $\|T_{1v}\| = \|M_{i_1 \ldots i_\ell}\| \times \|T_{1\eta}\|$.

4.2. The exponential sum of the products of transition matrices

In this subsection we determine the exact value of

$$E(q) := \lim_{n \to \infty} \left(\sum \|T_{i_1 \ldots i_{n+1}}\|^q \right)^{1/n}$$

for any $q \in \mathbb{R}$, where the summation is taken over all admissible words $i_1 \ldots i_{n+1}$ of length $n+1$ with $i_1 = 1$. By Theorems 3.3 and 3.4, this can be applied to calculate the L^q-spectrum of μ and the Hausdorff dimension of the graph of *f*. We also check the differentiability of $E(q)$, which is necessary in the multifractal analysis of μ.

Let M_0, M_1 be defined as in (4.4). For $q \in \mathbb{R}$, define $u_0(q) = 2^q$ and

$$u_n(q) = \sum_{J \in \mathcal{A}_n} \|M_J\|^q, \qquad n = 1, 2, \ldots \ . \tag{4.5}$$

The main result of this section is the following

Theorem 4.4. (i) *For any* $q \in \mathbb{R}$, *we have* $E(q) = 1/x(q)$, *where*

$$x(q) = \sup\{x \geqslant 0 : \sum_{n=0}^{\infty} u_n(q) x^{2n+3} \leqslant 1\}. \tag{4.6}$$

(ii) *There exists a unique* $q_0 < -2$ *such that* $\sum_{n=0}^{\infty} u_n(q_0) = 1$. *Whenever* $q > q_0$, $x(q)$ *is the positive root of* $\sum_{n=0}^{\infty} u_n(q) x^{2n+3} = 1$, *and it is an infinitely differentiable function of* q *on* $(q_0, +\infty)$. *Whenever* $q \leqslant q_0$, $x(q) = 1$. *Moreover* $x(q)$ *is not differentiable at* $q = q_0$.

We first consider part (i) of Theorem 4.4. For $q \in \mathbb{R}$, define

$$R_n(q) = \sum_{v \in F_n} \|T_{1v}\|^q, \qquad n \geqslant 1,$$

where F_n denotes the collection all admissible words of length n starting from the symbol 3. Set

$$R(q) = \lim_{n\to\infty} (R_n(q))^{1/n}.$$

By Lemma 4.3, we have directly

Lemma 4.5. *$R_1(q) = 2^q = u_0(q)$, $R_2(q) = 2^q = u_0(q)$, $R_3(q) = u_0(q) + u_1(q)$, and for $k \geqslant 2$,*

$$R_{2k}(q) = \left(\sum_{i=0}^{k-2} u_i(q) R_{2k-2i-3}(q)\right) + u_{k-1}(q),$$

$$R_{2k+1}(q) = \left(\sum_{i=0}^{k-2} u_i(q) R_{2k+1-2i-3}(q)\right) + u_{k-1}(q) + u_k(q).$$

Now we prove

Lemma 4.6. *$R(q) = 1/x(q)$ for any $q \in \mathbb{R}$.*

Proof. We divide the proof into two steps.

Step 1: $\limsup_{n\to\infty} (R_n(q))^{1/n} \leqslant 1/x(q)$. Since $\sum_{i=0}^{\infty} u_i(q)x(q)^{3+2i} \leqslant 1$, it follows that for any $k \geqslant 1$,

$$\begin{aligned} x(q)^{-2k} &\geqslant \sum_{i=0}^{k} u_i(q)x(q)^{3+2i-2k} \\ &\geqslant \sum_{i=0}^{k-2} u_i(q)x(q)^{3+2i-2k} + u_{k-1}(q)x(q) \end{aligned} \tag{4.7}$$

and similarly

$$x(q)^{-2k-1} \geqslant \sum_{i=0}^{k-2} u_i(q)x(q)^{3+2i-2k-1} + u_{k-1}(q) + u_k(q)x(q)^2. \tag{4.8}$$

Choose a positive number $C > \max\{1, x(q)^{-2}, x(q)^{-1}\}$ such that

$$R_i(q) < Cx(q)^{-i}, \qquad i = 1, 2, 3.$$

We claim that

$$R_i(q) < Cx(q)^{-i} \tag{4.9}$$

for all $i \in \mathbb{N}$. We show this claim by induction. Suppose (4.9) holds for any $i < 2k$. By Lemma 4.5, (4.7) and (4.8), we have

$$\begin{aligned} R_{2k}(q) &= \left(\sum_{i=0}^{k-2} u_i(q)R_{2k-2i-3}(q)\right) + u_{k-1}(q) \\ &\leqslant C\left(\sum_{i=0}^{k-2} u_i(q)x(q)^{2i+3-2k}\right) + u_{k-1}(q) \\ &\leqslant C\left(\sum_{i=0}^{k-2} u_i(q)x(q)^{2i+3-2k}\right) + Cu_{k-1}(q)x(q) \\ &\leqslant Cx(q)^{-2k} \end{aligned}$$

and

$$\begin{aligned} R_{2k+1}(q) &= \left(\sum_{i=0}^{k-2} u_i(q)R_{2k+1-2i-3}(q)\right) + u_{k-1}(q) + u_k(q) \\ &\leqslant C\left(\sum_{i=0}^{k-2} u_i(q)x(q)^{2i+3-2k-1}\right) + u_{k-1}(q) + u_k(q) \\ &\leqslant C\left(\sum_{i=0}^{k-2} u_i(q)x(q)^{2i+3-2k-1}\right) + Cu_{k-1}(q) + Cu_k(q)x(q)^2 \\ &\leqslant Cx(q)^{-2k-1}. \end{aligned}$$

Therefore (4.9) holds for $i = 2k$ and $i = 2k+1$. This finishes the proof of the claim. Hence the main statement in this step follows.

Step 2: $\limsup_{n\to\infty} (R_n(q))^{1/n} \geqslant 1/x(q)$. To see this, take any $y \in (0,\ x(q)^{-1})$. Since $y^{-1} > x(q)$, there exists a positive integer N such that

$$1 < \sum_{i=0}^{N-2} u_{i,q}y^{-3-2i}.$$

Thus for $k \geqslant N$, we have

$$y^{2k} \leqslant \sum_{i=0}^{k-2} u_i(q)y^{2k-3-2i}, \quad y^{2k+1} \leqslant \sum_{i=0}^{k-2} u_i(q)y^{2k+1-3-2i}. \tag{4.10}$$

Choose a positive number $D < \min\{1, x(q)^{-1}, x(q)^{-2}\}$ such that

$$R_i(q) > Dy^i, \qquad i = 1, \dots, 2N-1.$$

Then by an argument similar to that in Step 1, we have

$$R_i(q) > Dy^i, \qquad \forall i \in \mathbb{N},$$

which implies $\lim_{m\to\infty} \left(R_{n,q}\right)^{1/n} \geqslant y$. Since $y \in (0, 1/x(q))$ is arbitrary, we obtain the desired inequality in this step. $\square$

Proof of (i) of Theorem 4.4. By Lemma 4.6, it suffices to prove $E(q) = R(q)$. Set

$$E_n(q) = \sum \|T_{i_1\ldots i_{n+1}}\|^q,$$

where the summation is taken over all admissible words $i_1 \ldots i_{n+1}$ of length $n+1$ with $i_1 = 1$. Then $E(q) = \lim_{n\to\infty} (E_n(q))^{1/n}$. By Lemma 4.2, we have

$$E_n(q) = R_n(q) + 2\sum_{i=1}^{n-1} R_{n-1}(q) + 2. \tag{4.11}$$

On the other hand, observe that $u_n(q) > \|M_0^n\|^q = (n+2)^q$. It follows that the series $\sum_{n=0}^{\infty} u_n(q)x^{2n+3}$ diverges for $x > 1$. Therefore $0 \leqslant x(q) \leqslant 1$. This combining Lemma 4.6 and (4.11) yields the desired result. $\square$

In the following, we prove part (ii) of Theorem 4.4. We first present two propositions, for which the proofs will be given later.

Proposition 4.7. *There is a non-empty open interval $U \subset (-\infty, 0)$ such that*

$$1 < \sum_{n=0}^{\infty} u_n(q) < \infty, \qquad \sum_{n=0}^{\infty} nu_n(q) < \infty \tag{4.12}$$

for each $q \in U$.

Proposition 4.8. *Suppose that q is a real number satisfying $\sum_{n\geqslant 0} u_n(q) = +\infty$. Then for any integer L there exists $0 < y < 1$ such that*

$$L < \sum_{n=0}^{\infty} u_n(q)y^n < +\infty.$$

Proof of part (ii) of Theorem 4.4. We divide the proof into four steps.

Step 1: *There exists a real number $q_0 < 0$ such that $\sum_{n=0}^{\infty} u_n(q_0) = 1$.* To see this, we denote

$$F(q) = \sum_{n=0}^{\infty} u_n(q).$$

By Proposition 4.7, there exists $t < 0$ such that $1 < F(t) < \infty$. Since $u_n(q)$ is an increasing positive function of q for each n, the series $\sum_{n=1}^{\infty} u_n(q)$ converges uniformly on $(-\infty, t)$. Thus $F(q)$ is a continuous function on $(-\infty, t)$. The existence of q_0 follows from the Intermediate Value Theorem and the fact $\lim_{q\to-\infty} F(q) = 0$, which we will prove below.

For any $n \in \mathbb{N}$ and $q < q' < t$, we have

$$\frac{u_n(q)}{u_n(q')} \leqslant \max_{J\in\mathcal{A}_n} \|M_J\|^{q-q'} \leqslant 2^{q-q'},$$

which implies $F(q)/F(q') \leqslant 2^{q-q'}$ and thus $\lim_{q\to-\infty} F(q) = 0$.

Step 2: *Whenever* $q \leqslant q_0$, $x(q) = 1$; *and whenever* $q > q_0$, $x(q)$ *is the positive root of* $\sum_{n=0}^{\infty} u_n(q)x^{2n+3} = 1$. First assume $q \leqslant q_0$. In this case $\sum_{n=0}^{\infty} u_{n,q} \leqslant 1$, and thus $x(q) \geqslant 1$. Since $u_n(q) > \|M_0^n\|^q = (n+1)^q$, we have $\sum_{n\geqslant 0} u_n(q)x^{2n+3} = \infty$ for $x > 1$, which implies $x(q) \leqslant 1$. Therefore $x(q) = 1$.

Now assume $q > q_0$. Under this assumption we have either

$$1 < \sum_{n=0}^{\infty} u_n(q) < \infty$$

or

$$\sum_{n=0}^{\infty} u_n(q) = \infty.$$

In the first case, $\sum_{n\geqslant 0} u_n(q)x^{2n+3}$ is a continuous function of x on $(0, 1)$ and thus there exists $y \in (0, 1)$ satisfying $\sum_{n=0}^{\infty} u_n(q)y^{2n+3} = 1$, hence $x(q) = y$. In the second case, by Proposition 4.8, there exists $0 < t_1 < t_2 < 1$ such that

$$1 < \sum_{n\geqslant 0} u_n(q)t_1^{2n} < +\infty, \qquad t_1^{-3} < \sum_{n\geqslant 0} u_n(q)t_2^{2n} < \infty.$$

Thus $1 < \sum_{n=0}^{\infty} u_n(q)t_2^{2n+3} < \infty$. Therefore $\sum_{n=0}^{\infty} u_n(q)x^{2n+3}$ is a continuous function of x on $(0, t_2)$. Combining it with (4.6), we see that $x(q)$ satisfies $\sum_{n=0}^{\infty} u_n(q)$ $x(q)^{2n+3} = 1$.

Step 3: $x(q)$ *is infinitely differentiable on* $(q_0, +\infty)$. *Moreover*

$$x'(q) = -\frac{\sum_{n=0}^{\infty}\left(\sum_{J\in\mathcal{A}_n} \|M_J\|^q \log\|M_J\|\right)x(q)^{2n+3}}{\sum_{n=0}^{\infty} u_n(q)(2n+3)x(q)^{2n+2}}, \qquad \forall\ q > q_0. \tag{4.13}$$

To see it, define

$$F(q, x) = \sum_{n=0}^{\infty} u_n(q)x^{2n+3}.$$

Fix $q_1 \in (q_0, +\infty)$. As we have shown in step 2, there exists a real number $y > x(q_1)$ such that $1 < F(q_1, y) < +\infty$. Take a real number z so that $x(q_1) < z < y$, and take q_2 such that

$$q_2 > q_1, \qquad 4^{q_2 - q_1} < \frac{y}{z}.$$

Note that for any integer $n \geqslant 0$,

$$\frac{u_n(q_2)}{u_n(q_1)} \leqslant \max_{J \in \mathcal{A}_n} \|M_J\|^{q_2 - q_1} \leqslant 4^{n(q_2 - q_1)}.$$

Therefore for any $q < q_2$ and $0 < x < z$, we have

$$\begin{aligned} F(q, x) &\leqslant \sum_{n=0}^{\infty} u_n(q_2) z^{2n+3} \\ &\leqslant \sum_{n=0}^{\infty} u_n(q_1) y^{2n+3} 4^{n(q_2 - q_1)} \left(\frac{z}{y}\right)^{2n+3} < +\infty, \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{du_n(q)}{dq} x^{2n+3} &= \sum_{n \geqslant 0} \Big(\sum_{J \in \mathcal{A}_n} \|M_J\|^q \log \|M_J\| \Big) x^{2n+3} \leqslant \sum_{n=0}^{\infty} u_n(q) (\log 4^n) x^{2n+3} \\ &\leqslant \sum_{n=0}^{\infty} u_n(q_1) y^{2n+3} (\log 4^n) 4^{n(q_2 - q_1)} \left(\frac{z}{y}\right)^{2n+3} < +\infty \end{aligned}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(q)(3 + 2n) x^{2n+2} &< \sum_{n=0}^{\infty} u_n(q_2)(3 + 2n) z^{2n+2} \\ &\leqslant \frac{1}{z} \sum_{n=0}^{\infty} u_n(q_1) y^{2n+3} (3 + 2n) 4^{n(q_2 - q_1)} \left(\frac{z}{y}\right)^{2n+3} < +\infty. \end{aligned}$$

The above three inequalities imply that $F(q, x)$ is well defined and differentiable on $(-\infty, q_2) \times (0, z)$. Furthermore using similar discussions we can show that $F(q, x)$ is infinitely differentiable on $(-\infty, q_2) \times (0, z)$. Thus by the Implicit Function Theorem, $x(q)$ is infinitely differentiable on a neighborhood of q_1. Since q_1 is taken arbitrarily on $(q_0, +\infty)$, $x(q)$ is infinitely differentiable on $(q_0, +\infty)$. Formula (4.13) follows by a direct calculation.

Step 4: *$x(q)$ is not differentiable at $q = q_0$*. Note that $x'(q_0-) = 0$, we need to prove $x'(q_0+) < 0$. To see it, notice that for $q > q_0$,

$$\sum_{n=0}^{\infty} u_n(q)x(q)^{2n+3} - \sum_{n=0}^{\infty} u_n(q_0)x(q_0)^{2n+3} = 0.$$

Thus we have

$$\begin{aligned}\frac{x(q)-x(q_0)}{q-q_0} &= -\frac{\sum_{n=0}^{\infty} \frac{u_n(q)-u_n(q_0)}{q-q_0} \cdot x(q_0)^{2n+3}}{\sum_{n\geqslant 0} u_n(q)\Big(x(q)^{2n+2} + x(q)^{2n+1}x(q_0) + \cdots + x(q_0)^{2n+2}\Big)} \\ &= -\frac{\sum_{n=0}^{\infty} \frac{u_n(q)-u_n(q_0)}{q-q_0}}{\sum_{n=0}^{\infty} u_n(q)\Big(x(q)^{2n+2} + x(q)^{2n+1} + \cdots + x(q) + 1\Big)}.\end{aligned}$$

Since $\sum_{n=0}^{\infty} u_n(q)(2n+3) < +\infty$ on a neighborhood of q_0 (by Proposition 4.7), we obtain the following formula by taking $q \downarrow q_0$:

$$x'(q_0+) = -\frac{\sum_{n\geqslant 0}(\sum_{J\in\mathcal{A}_n} \|M_J\|^{q_0} \log\|M_J\|)}{\sum_{n=0}^{\infty} u_n(q_0)\cdot(2n+3)} < 0. \qquad \square$$

In what follows, we give the proofs of Propositions 4.7 and 4.8. First we give some simple lemmas. For any positive integer k and positive $n_1, \ldots, n_k$, we define for simplicity

$$a(n_1, \ldots, n_k) = (1,0) \prod_{i=0}^{k} \left(M_{\varepsilon_i}\right)^{n_i} (1,0)^T$$

and

$$b(n_1, \ldots, n_k) = \left\| \prod_{i=0}^{k} \left(M_{\varepsilon_i}\right)^{n_i} \right\|,$$

where $\varepsilon_i = 0$ if i is odd, and $\varepsilon_i = 1$ if i is even.

Lemma 4.9. *Let $b(n_1, \ldots, n_k)$ be defined as above, then*

(i) $b(n_1, \ldots, n_k) \leqslant (1+n_1)\ldots(1+n_{k-1})(2+n_k)$.

(ii) $b(n_1, \ldots, n_{2k}) \geqslant (1+n_1n_2)\ldots(1+n_{2k-1}n_{2k})$ *and*
$b(n_1, \ldots, n_{2k+1}) \geqslant (1+n_1n_2)\ldots(1+n_{2k-1}n_{2k})(1+n_{2k+1})$.

Proof. Part (i) follows directly from the inequality

$$(1,1)M_\varepsilon^n \leqslant (n+1, n+1), \qquad \varepsilon = 0 \text{ or } 1.$$

To see part (ii), it suffices to notice that

$$\begin{aligned}(M_0^{n_1}M_1^{n_2})\ldots(M_0^{n_{2k-1}}M_1^{n_{2k}}) &= \begin{pmatrix} 1+n_1n_2 & n_1 \\ n_2 & 1 \end{pmatrix} \ldots \begin{pmatrix} 1+n_{2k-1}n_{2k} & n_{2k-1} \\ n_{2k} & 1 \end{pmatrix} \\ &\geqslant \begin{pmatrix} (1+n_1n_2)\ldots(1+n_{2k-1}n_{2k}) & * \\ * & * \end{pmatrix}\end{aligned}$$

and

$$M_0^{n_1}M_1^{n_2}\ldots M_1^{n_{2k}}M_0^{n_{2k+1}} \geqslant \begin{pmatrix} (1+n_1n_2)\ldots(1+n_{2k-1}n_{2k}) & * \\ * & * \end{pmatrix}\begin{pmatrix} 1 & n_{2k+1} \\ 0 & 1 \end{pmatrix}. \qquad \square$$

Since any element in $\{0,1\}^n$ can be uniquely written as $\varepsilon_1^{n_1}\ldots\varepsilon_k^{n_k}$, or $(1-\varepsilon_1)^{n_1}\ldots(1-\varepsilon_k)^{n_k}$ with $n_1+\cdots+n_k = n$, we obtain the following lemma immediately.

Lemma 4.10. *For each $q \in \mathbb{R}$, we have*

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(q) &= 2^q + 2\sum_{n\geqslant 1} b(n)^q + 2\sum_{l\geqslant 2}\sum_{n_1,\ldots,n_l\geqslant 1} b(n_1,\ldots,n_l)^q \\ &= 2^q + 2\sum_{n\geqslant 1} b(n)^q + 2\sum_{l\geqslant 1}\sum_{n_1,\ldots,n_{2l}\geqslant 1} b(n_1,\ldots,n_{2l})^q \\ &\quad +2\sum_{l\geqslant 1}\sum_{n_1,\ldots,n_{2l+1}\geqslant 1} b(n_1,\ldots,n_{2l+1})^q\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} nu_n(q) &= 2\sum_{n\geqslant 1} nb(n)^q + 2\sum_{l\geqslant 1}\sum_{n_1,\ldots,n_{2l}\geqslant 1}(n_1+\cdots+n_{2l})b(n_1,\ldots,n_{2l})^q \\ &\quad +2\sum_{l\geqslant 1}(n_1+\cdots+n_{2l+1})\sum_{n_1,\ldots,n_{2l+1}\geqslant 1} b(n_1,\ldots,n_{2l+1})^q.\end{aligned}$$

Proof of Proposition 4.7. By Lemmas 4.9 and 4.10, we have for any $q<0$,

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(q) &\geqslant 2^q+2\sum_{n\geqslant 1}(2+n)^q+2\sum_{l\geqslant 2}\sum_{n_1,\ldots,n_l\geqslant 1}(1+n_1)^q\ldots(1+n_{l-1})^q(2+n_l)^q \\ &= 2^q+2\left(\sum_{n\geqslant 1}(2+n)^q\right)\left(1+\sum_{l\geqslant 1}\left(\sum_{n\geqslant 1}(1+n)^q\right)^l\right) \end{aligned}\tag{4.14}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} nu_n(q) &\leqslant 2\sum_{n\geqslant 1} n(2+n)^q+2\sum_{k\geqslant 1}\sum_{n_1,\ldots,n_{2k}\geqslant 1}(n_1+\cdots+n_{2k})(1+n_1n_2)^q\ldots(1+n_{2k-1}n_{2k})^q \\ &\quad +2\sum_{k\geqslant 1}\sum_{n_1,\ldots,n_{2k+1}\geqslant 1}(n_1+\cdots+n_{2k+1})(1+n_1n_2)^q\ldots(1+n_{2k-1}n_{2k})^q n_{2k+1}^q \\ &= 2\sum_{n\geqslant 1} n(2+n)^q+2\sum_{k\geqslant 1}\sum_{n_1,\ldots,n_{2k}\geqslant 1}2kn_1(1+n_1n_2)^q\ldots(1+n_{2k-1}n_{2k})^q \\ &\quad +2\sum_{k\geqslant 1}\sum_{n_1,\ldots,n_{2k+1}\geqslant 1}2kn_1(1+n_1n_2)^q\ldots(1+n_{2k-1}n_{2k})^q n_{2k+1}^q \\ &\quad +2\sum_{k\geqslant 1}\sum_{n_1,\ldots,n_{2k+1}\geqslant 1}n_{2k+1}(1+n_1n_2)^q\ldots(1+n_{2k-1}n_{2k})^q n_{2k+1}^q \\ &= 2\sum_{n\geqslant 1} n(2+n)^q+\left(\sum_{n_1,n_2\geqslant 1}n_1(1+n_1n_2)^q\right)\left(\sum_{k\geqslant 1}4k\left(\sum_{m_1,m_2\geqslant 1}(1+m_1m_2)^q\right)^{k-1}\right) \\ &\quad +\left(\sum_{n_1,n_2\geqslant 1}n_1(1+n_1n_2)^q\right)\left(\sum_{n\geqslant 1}n^q\right)\left(\sum_{k\geqslant 1}4k\left(\sum_{m_1,m_2\geqslant 1}(1+m_1m_2)^q\right)^{k-1}\right) \\ &\quad +2\left(\sum_{n\geqslant 1}n^{q+1}\right)\left(\sum_{k\geqslant 1}\left(\sum_{m_1,m_2\geqslant 1}(1+m_1m_2)^q\right)^{k}\right). \end{aligned}\tag{4.15}$$

By (4.14) and (4.15), to prove the proposition, it suffices to construct an interval U such that for each $q\in U$, one has

$$\sum_{n\geqslant 1}n^{q+1}<\infty,\qquad \sum_{n_1,n_2\geqslant 1}n_1(1+n_1n_2)^q<\infty,\qquad \sum_{m_1,m_2\geqslant 1}(1+m_1m_2)^q<1 \tag{4.16}$$

and

$$2^q + 2 \sum_{n \geqslant 1} (2+n)^q + 2 \cdot \Big(1 + \sum_{n \geqslant 1} n^q\Big) \cdot \Big(\sum_{l \geqslant 1} \Big(\sum_{n_1, n_2 \geqslant 1} (1 + n_1 n_2)^q \Big)^l \Big) > 1. \tag{4.17}$$

To do this, denote by ζ the Riemann–Zeta function, that is

$$\zeta(x) = \sum_{n \geqslant 1} n^{-x}, \qquad x > 1.$$

Let θ_0 be the positive root of $x^2 + 2x - \frac{9}{8} = 0$, i.e., $\theta_0 \approx 0.45774$. We set

$$U = \Big(-\zeta^{-1}(\tfrac{16}{11}), -\zeta^{-1}(1+\theta_0) \Big) \approx (-2.2599, -2.2543).$$

Now suppose $q \in U$. Since $q < -2$, we can see $\sum_{n \geqslant 1} n^{q+1} < \infty$ and $\sum_{n_1, n_2 \geqslant 1} n_1 (1 + n_1 n_2)^q < \infty$. Moreover,

$$\begin{aligned} \sum_{n_1, n_2 \geqslant 1} (1 + n_1 n_2)^q &= 2 \sum_{n \geqslant 1} (1+n)^q - 2^q + \sum_{n_1, n_2 \geqslant 2} (1 + n_1 n_2)^q \\ &< 2 \sum_{n \geqslant 1} (1+n)^q - 2^q + \Bigg(\sum_{n \geqslant 2} n^q \Bigg)^2 \\ &= 2 \big(\zeta(-q) - 1 \big) - 2^q + \big(\zeta(-q) - 1 \big)^2 \\ &< 2\theta_0 - \frac{1}{8} + \theta_0^2 = 1 \end{aligned}$$

and

$$\begin{aligned} &2^q + 2 \Bigg(\sum_{n \geqslant 1} (2+n)^q \Bigg) \Bigg(1 + \sum_{l \geqslant 1} \Bigg(\sum_{n \geqslant 1} (1+n)^q \Bigg)^l \Bigg) \\ &= 2^q + 2 \cdot \frac{\zeta(-q) - 1 - 2^q}{2 - \zeta(-q)} = 1 + \frac{(3 - 2^q)\zeta(-q) - 4}{2 - \zeta(-q)} \\ &> 1 + \frac{(3 - 2^{-2})\zeta(-q) - 4}{2 - \zeta(-q)} > 1 + \frac{(3 - 2^{-2}) \cdot \frac{16}{11} - 4}{2 - \zeta(-q)} = 1. \end{aligned}$$

Thus (4.16) and (4.17) hold when $q \in U$, which completes the proof of the proposition. $\square$

Proof of Proposition 4.8. First we assume $q > 0$. In this case, $u_n(q) > 1$ for $n \geqslant 0$, and therefore $\sum_{n\geqslant 0} u_{n,q} = +\infty$. Moreover, the sequence $\{u_{n,q}\}_n$ is sub-multiplicative, this is $u_{m+n}(q) \leqslant u_m(q)u_n(q)$. Thus

$$\lim_{n\to+\infty} u_n(q)^{1/n} = \inf_{n\geqslant 1} u_n(q)^{1/n}.$$

Denote by r_q the value of above limit, then $1 \leqslant r_q < \infty$ and $u_n(q) \geqslant r_q^n$ for $n \geqslant 1$. Hence $\lim_{x\to r_q^{-1}} \sum_{n=0}^{\infty} u_n(q)x^n = +\infty$, which implies the desired result since the series $\sum_{n=0}^{\infty} u_n(q)y^n$ converges on $(0, r_q^{-1})$.

In the remaining part we assume $q < 0$ and $\sum_{n=0}^{\infty} u_n(q) = \infty$. Note that for any integers $n_1, n_2, \ldots, n_l$ and $m_1, m_2, \ldots, m_s$, we have

$$a(n_1, n_2, \ldots, n_l) \leqslant b(n_1, n_2, \ldots, n_l)$$

and

$$a(n_1, n_2, \ldots, n_l)a(m_1, m_2, \ldots, m_s) \leqslant a(n_1, n_2, \ldots, n_l, m_1, m_2, \ldots, m_s), \tag{4.18}$$

where $m_1, m_2, \ldots, m_s$ are positive integers. It is not hard to show that

$$a(n_1, n_2, \ldots, n_l) \geqslant \frac{1}{4} b(n_1, n_2, \ldots, n_l), \text{ if } l \text{ is even.} \tag{4.19}$$

(To see this, denote

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = (M_0^{n_1} M_1^{n_2}) \ldots (M_0^{n_{l-1}} M_1^{n_l})$$

for even integer l. Then by induction on l, one can verify that among the x_i's, x_1 is the greatest and x_4 the smallest.)

For any integer $L \geqslant 1$, take an integer $y(L) \geqslant L \cdot 4^{-q}$ and define $p = 2^{y(L)}$. Now for any $0 < x < 1$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} u_n(q)x^n \\
&= 2^q + 2 \cdot \sum_{j=1}^{2p-1} \sum_{n_1,\ldots,n_j \geqslant 1} b(n_1, n_2, \ldots, n_j)^q \cdot x^{n_1+\cdots+n_j} \\
&\quad + 2 \cdot \sum_{j=0}^{2p-1} \sum_{k=1}^{+\infty} \sum_{n_1,\ldots,n_{2kp+j} \geqslant 1} b(n_1, \ldots, n_{2kp+j})^q \cdot x^{n_1+\cdots+n_{2kp+j}}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant 2^q + 2 \cdot \sum_{j=1}^{2p-1} \sum_{n_1,\ldots,n_j \geqslant 1} b(n_1, n_2, \ldots, n_j)^q \cdot x^{n_1+\cdots+n_j} \\
&\quad + 2 \cdot \sum_{j=0}^{2p-1} \sum_{k=1}^{+\infty} \sum_{n_1,\ldots,n_{2kp+j} \geqslant 1} a(n_1, \ldots, n_{2kp+j})^q \cdot x^{n_1+\cdots+n_{2kp+j}} \\
&\leqslant 2^q + 2 \cdot \sum_{j=1}^{2p-1} \sum_{n_1,\ldots,n_j \geqslant 1} b(n_1, n_2, \ldots, n_j)^q \cdot x^{n_1+\cdots+n_j} \\
&\quad + 2 \cdot \sum_{j=0}^{2p-1} \sum_{k=1}^{+\infty} \sum_{n_1,\ldots,n_{2kp+j} \geqslant 1} a(n_1, \ldots, n_{2kp})^q \\
&\quad \times a(n_{2kp+1}, \ldots, n_{2kp+j})^q x^{n_1+\cdots+n_{2kp+j}} \\
&\leqslant 2^q + 2 \cdot \sum_{j=1}^{2p-1} \sum_{n_1,\ldots,n_j \geqslant 1} b(n_1, n_2, \ldots, n_j)^q \cdot x^{n_1+\cdots+n_j} \\
&\quad + 2 \left(\sum_{j=0}^{2p-1} \sum_{n_1,\ldots,n_j \geqslant 1} a(n_1, \ldots, n_j)^q \; x^{n_1+\cdots+n_j} \right) \\
&\quad \times \left(\sum_{k=1}^{+\infty} \left(\sum_{n_1,\ldots,n_{2p} \geqslant 1} a(n_1, \ldots, n_{2p})^q \; x^{n_1+\cdots+n_{2p}} \right)^k \right). \qquad (4.20)
\end{aligned}
$$

Since $a(n_1, n_2, \ldots, n_l), b(n_1, n_2, \ldots, n_l)$ are polynomials about $n_1, n_2, \ldots, n_l$ and $0 < x < 1$, it follows

$$\sum_{n_1,\ldots,n_l \geqslant 1} a(n_1, \ldots, n_l)^q \; x^{n_1+\cdots+n_l} < \infty,$$

$$\sum_{n_1,\ldots,n_l \geqslant 1} b(n_1, \ldots, n_l)^q \; x^{n_1+\cdots+n_l} < \infty$$

for any positive integer l. Therefore by (4.20), we have

$$\sum_{n \geqslant 0} u_n(q) x^n < \infty$$

if $\sum_{n_1,\ldots,n_{2p} \geqslant 1} a(n_1, \ldots, n_{2p})^q \; x^{n_1+\cdots+n_{2p}} < 1$.

Since $\sum_{n=0}^{\infty} u_n(q) = \infty$, by (4.20) we have

$$\sum_{n_1,\ldots,n_{2p}\geqslant 1} a(n_1,\ldots,n_{2p})^q \geqslant 1 \quad \text{or} = +\infty.$$

Hence there exists $0 < z \leqslant 1$ such that

$$\sum_{n_1,\ldots,n_{2p}\geqslant 1} a(n_1,\ldots,n_{2p})^q z^{n_1+\cdots+n_{2p}} = 1.$$

Moreover,

$$\sum_{n=0}^{\infty} u_n(q)x^n < \infty, \qquad \forall x \in (0,z). \tag{4.21}$$

For $l = 2, 2^2, \ldots, p$, by (4.18) we obtain

$$\sum_{n_1,\ldots,n_{2p}\geqslant 1} a(n_1,\ldots,n_{2p})^q z^{n_1+\cdots+n_{2p}} \leqslant \left(\sum_{n_1,\ldots,n_l\geqslant 1} a(n_1,\ldots,n_l)^q z^{n_1+\cdots+n_l} \right)^{2p/l},$$

which implies

$$\sum_{n_1,\ldots,n_l\geqslant 1} a(n_1,\ldots,n_l)^q z^{n_1+\cdots+n_l} \geqslant 1.$$

Thus by (4.19), we have

$$\sum_{n_1,\ldots,n_l\geqslant 1} b(n_1,\ldots,n_l)^q z^{n_1+\cdots+n_l} \geqslant 4^q, \qquad l = 2, 2^2, \ldots, p.$$

Therefore

$$\begin{aligned} &\lim_{x\to z-} \sum_{n=0}^{\infty} u_n(q)x^n \\ &\quad\geqslant 2^q + 2\cdot \sum_{j=1}^{2p-1} \sum_{n_1,\ldots,n_j\geqslant 1} b(n_1,\ldots,n_j)^q \cdot z^{n_1+\cdots+n_j} \\ &\quad\geqslant 2^q + 2\cdot y(L)\cdot 4^q \\ &\quad\geqslant 2^q + 2L, \end{aligned}$$

which finishes the proof. $\square$

4.3. The Hausdorff dimension of the graph of f

Combining Theorems 3.4 and 4.4, we have directly

Theorem 4.11. *The Hausdorff dimension of the graph of f satisfies*

$$\dim_{\mathrm{H}} \Gamma(f) = \frac{\log x(-\log \rho / \log 2)}{\log \rho},$$

where $x(-\log \rho/\log 2)$ is the positive root of

$$\sum_{n=0}^{\infty} \sum_{J \in \mathcal{A}_n} \|M_J\|^{-\log \rho / \log 2}\, x^{2n+3} = 1.$$

4.4. The L^q-spectrum of μ

Combining Theorems 3.3 and 4.4, we have directly

Theorem 4.12. *For any $q \in \mathbb{R}$, we have*

$$\tau(q) = -\frac{q \log 2}{\log \rho} - \frac{\log x(q)}{\log \rho},$$

where $x(q)$ satisfies (4.6). *There exists a unique $q_0 < -2$ such that*

$$\sum_{n=0}^{\infty} \sum_{J \in \mathcal{A}_n} \|M_J\|^{q_0} = 1.$$

Whenever $q > q_0$, $x(q)$ is the positive root of

$$\sum_{n=0}^{\infty} \sum_{J \in \mathcal{A}_n} \|M_J\|^{q} x^{2n+3} = 1,$$

and it is an infinitely differentiable function of q on $(q_0, +\infty)$. Whenever $q \leqslant q_0$, $x(q) = 1$. Moreover $x(q)$ is not differentiable at $q = q_0$.

4.5. The Hausdorff dimension of μ

The Hausdorff dimension of μ has been considered by many authors (e.g. see [1,2,36,43,58]). A computable theoretical formula of $\dim_{\mathrm{H}} \mu$ was first given by Ledrappier and Porzio [36]. In this section, we state another theoretical formula obtained by Ngai [43] based on the following result.

Theorem 4.13 (*Ngai [43]*). *Suppose that ν is a Borel probability measure on $\mathbb{R}$ with bounded support, and furthermore its L^q-spectrum $\tau(\nu, q)$ is differentiable at $q = 1$. Then the local dimension $d(\nu, x)$ is equal to $\tau'(\nu, 1)$ for ν almost all $x \in \mathbb{R}$, and the Hausdorff dimension of ν is also equal to $\tau'(\nu, 1)$.*

The above result has also been obtained (in a generalized form) by Heurteaux [23] and Olsen [45]. As we have showed in Theorem 4.12, the L^q-spectrum $\tau(q)$ of μ is differentiable at $q = 1$. A direct calculation of $\tau'(1)$ through the formula of $\tau(q)$ given in Theorem 4.12 yields

Theorem 4.14 (*Ngai [43]*). *The Hausdorff dimension of μ satisfies*

$$\dim_{\mathrm{H}} \mu = \tau'(1) = -\frac{\log 2}{\log \rho} + \frac{\sum_{n=0}^{\infty} 2^{-2n-3} \sum_{J \in \mathcal{A}_n} \|M_J\| \log \|M_J\|}{9 \log \rho}.$$

In Remark 4.26, we will provide another approach to the above formula.

4.6. Local dimensions of μ

In this section we present results on the range and almost all value of the local dimensions of μ. Recall that $\overline{d}(\mu, x), \underline{d}(\mu, x)$ and $d(\mu, x)$ are used to denote the upper local dimension, the lower local dimension and local dimension, respectively, and $\mathcal{R}(\mu)$ the range of $d(\mu, x)$, i.e.,

$$\mathcal{R}(\mu) := \{y \in \mathbb{R} : \; d(\mu, x) = y \text{ for some } x \in [0, 1]\}.$$

The main results of this section are the following:

Theorem 4.15. $\mathcal{R}(\mu) = \left[-\frac{\log 2}{\log \rho} - \frac{1}{2}, \; -\frac{\log 2}{\log \rho}\right].$

Theorem 4.16. *For μ almost all $x \in [0, 1]$,*

$$d(\mu, x) = \tau'(1) = -\frac{\log 2}{\log \rho} + \frac{\sum_{n=0}^{\infty} 2^{-2n-3} \sum_{J \in \mathcal{A}_n} \|M_J\| \log \|M_J\|}{9 \log \rho}.$$

Theorem 4.17. *For $\mathcal{L}$ almost all $x \in [0, 1]$,*

$$d(\mu, x) = -\frac{\log 2}{\log \rho} + \frac{7\sqrt{5} - 15}{10 \log \rho} \cdot \sum_{n=0}^{\infty} \left(\frac{3 - \sqrt{5}}{2}\right)^{n+1} \sum_{J \in \mathcal{A}_n} \log \|M_J\|.$$

Theorem 4.15 was first stated and proved by Hu [24]. In his proof, Hu used a complicated combinatoric method. In this section we will give a new proof, which is also valid to deal with the parameter λ_k ($k = 3, 4, \ldots$).

Theorem 4.16 is just a combination of Ngai's two results—Theorems 4.13 and 4.14. Theorem 4.17 is new, and it can be used to determine the almost all value of the box dimensions of level sets of f. In what follows, we will prove Theorems 4.15 and 4.17, respectively.

Proof of Theorem 4.15. Denote

$$\Omega_A^{\mathbb{N}} = \left\{ (i_n)_{n=1}^{\infty} : \; i_n \in \Omega,\, A_{i_n, i_{n+1}} = 1 \text{ for } n \geqslant 1 \right\},$$

where Ω, A are defined as in (4.1) and (4.2). By Theorem 3.2, to determine $\mathcal{R}(\mu)$ it suffices to determine the range, denoted as $\mathcal{R}_T$, of the limits

$$\lim_{n \to \infty} \frac{\log \|T_{i_1 \ldots i_{n+1}}\|}{n},$$

where $(i_n)_{n=1}^{\infty} \in \Omega_A^{\mathbb{N}}$ with $i_1 = 1$. Denote $\delta_0 = 3$ and $\delta_1 = 7$. Define a sequence of sets $(\mathcal{B}_i)_{i=0}^{\infty}$ of admissible words by

$$\mathcal{B}_0 = \{356\},$$

$$\mathcal{B}_n = \left\{ 35\delta_{i_1} 5 \ldots \delta_{i_n} 56 : \; i_1 \ldots i_n \in \mathcal{A}_n \right\}, \qquad n = 1, 2, \ldots$$

and define

$$\mathcal{B} = \bigcup_{n=0}^{\infty} \mathcal{B}_n. \tag{4.22}$$

It can be checked directly that $1\omega_1 \ldots \omega_n$ is an admissible word starting from 1 for any $\omega_1, \ldots, \omega_n \in \mathcal{B}$. Moreover, by Lemma 4.3,

$$\|T_{1\omega_1 \ldots \omega_n}\| = \prod_{i=1}^{n} \|T_{1\omega_i}\|.$$

Therefore we have

$$\mathcal{R}_T \supset [x_0, y_0]$$

with

$$x_0 = \inf_{\omega \in \mathcal{B}} \frac{\log \|T_{1\omega}\|}{|\omega|}, \qquad y_0 = \sup_{\omega \in \mathcal{B}} \frac{\log \|T_{1\omega}\|}{|\omega|},$$

where $|\omega|$ denotes the length of ω. At the end of the proof, we will show that

$$x_0 = 0, \qquad y_0 = -\frac{\log \rho}{2}. \tag{4.23}$$

Now we use (4.23) to prove

$$\mathcal{R}_T = \left[0,\ -\frac{\log \rho}{2}\right]. \tag{4.24}$$

Observe that for each $n \geqslant 1$ and $\gamma = i_1 \dots i_{n+1} \in \Omega_{A,n+1}$ with $i_1 = 1$, there exists a word γ' of length at most 3 such that $\gamma\gamma'$ can be written as

$$1u^\ell w_1 \dots \omega_m$$

with $\ell, m \geqslant 0$, $u = 2$ or 4, $\omega_1, \dots, \omega_m \in \mathcal{B}$, and thus

$$T_{\gamma\gamma'} = \prod_{i=1}^{m} \|T_{1\omega_i}\|.$$

Hence we have

$$0 \leqslant \frac{\log \|T_{i_1 \dots i_{n+1}}\|}{n+3} \leqslant \frac{\sum_{i=1}^m \log \|T_{1\omega_i}\|}{\sum_{i=1}^m |\omega_i|} \leqslant -\frac{\log \rho}{2}.$$

Letting $n \to \infty$, we obtain $\mathcal{R}_T \subset \left[0, -\frac{\log \rho}{2}\right]$. Thus (4.24) holds and

$$\mathcal{R}(\mu) = \left[-\frac{\log 2}{\log \rho} - \frac{1}{2},\ -\frac{\log 2}{\log \rho}\right].$$

Now we turn to prove (4.23). Note that for each $n \geqslant 1$,

$$(35)^n 6 \in \mathcal{B}, \qquad \|T_{1(35)^n 6}\| = \|M_0^n\| = n + 2.$$

It follows that

$$\inf_n \frac{\log \|T_{1(35)^n 6}\|}{2n+1} = 0,$$

and thus $x_0 = 0$.

On the other hand, for $\omega = 35\delta_{i_1}5\ldots\delta_{i_n}56 \in \mathcal{B}_n$, we have

$$\|T_{1\omega}\| = \|M_{i_1}\ldots M_{i_n}\|. \tag{4.25}$$

Observe that

$$\|M_{i_1\ldots i_n}\| < \|M_{1-i_1}\ldots M_{1-i_{j-1}}M_{1-i_j}M_{i_{j+1}}M_{i_{j+2}}\ldots M_{i_n}\|$$

if $i_j = i_{j+1}$ for some $1 \leqslant j < n$. It follows that the maximum value of the right hand side of (4.25) is attained when $i_1, \ldots, i_n$ take $0, 1$ alternatively. By a direct calculation, this maximum value is equal to

$$\frac{2+\rho+(-1)^n\rho^{2n+4}}{1+\rho^2}\,\rho^{-n}.$$

Therefore

$$\begin{aligned} y_0 &= \sup_{n\geqslant 0}\frac{\log\left(\frac{2+\rho+(-1)^n\rho^{2n+4}}{1+\rho^2}\,\rho^{-n}\right)}{2n+3} \\ &= \sup_{n\geqslant 0}\frac{\log\left(\frac{2+\rho+(-1)^n\rho^{2n+4}}{1+\rho^2}\right)+\log\rho^{-n}}{3+2n}. \end{aligned}$$

Since

$$\frac{1}{3}\log\left(\frac{2+\rho+(-1)^n\rho^{2n+4}}{1+\rho^2}\right) \leqslant \frac{1}{3}\log\left(\frac{2+\rho+\rho^4}{1+\rho^2}\right) = \frac{1}{3}\log 2$$

and

$$\frac{1}{2n}\log\rho^{-n} = -\frac{\log\rho}{2},$$

we have

$$y_0 = \max\left\{\frac{\log 2}{3},\quad -\frac{\log\rho}{2}\right\} = -\frac{\log\rho}{2},$$

which finishes the proof. □

Proof of Theorem 4.17. For any $i \in \Omega = \{1, 2, \ldots, 7\}$, denote by ℓ_i the relative length (i.e., the first term) of the characteristic vector labelled by i. By Table 4.1, we have

$$\ell_1 = 1, \quad \ell_2 = \ell_4 = \ell_5 = \rho, \quad \ell_3 = \ell_7 = 1-\rho, \quad \ell_6 = 2\rho - 1.$$

Denote $\widehat{\Omega} = \{3, 5, 6, 7\}$. As we presented in Lemma 4.1, $\widehat{\Omega}$ is an essential subclass of Ω. Let $\left(\widehat{\Omega}_A^{\mathbb{N}}, \sigma\right)$ be a subshift space of finite type defined by

$$\widehat{\Omega}_A^{\mathbb{N}} = \left\{(i_n)_{n=1}^{\infty} :\ i_n \in \widehat{\Omega},\ A_{i_n, i_{n+1}} = 1 \text{ for } n \geqslant 1\right\},$$

and $\sigma\left((i_n)_{n=1}^{\infty}\right) = (i_{n+1})_{i=1}^{\infty}$. Especially define

$$[3] = \left\{(i_n)_{n=1}^{\infty} \in \widehat{\Omega}_A^{\mathbb{N}} :\ i_1 = 3\right\}.$$

For any $\omega \in [3]$, it is clear $1\omega \in \Omega_A^{\mathbb{N}}$. Let π be the projection defined as in (3.1). We define a map $\hat{\pi} :\ [3] \to \Delta(13)$ by

$$\hat{\pi}(\omega) = \pi(1\omega), \qquad \forall \omega \in [3],$$

where $\Delta(13)$ denotes the 1st net interval with the symbolic expression 13. By a direct check, $\Delta(13) = [1 - \rho, \rho]$.

Now we would like to define a Markov measure η on $\widehat{\Omega}_A^{\mathbb{N}}$, such that the measure $\eta \circ \hat{\pi}^{-1}$ on $\Delta(13)$ only differs from the Lebesgue measure $\mathcal{L}$ on $\Delta(13)$ by a constant factor. Construct a matrix $P = (P_{i,j})_{i,j \in \widehat{\Omega}}$ by

$$P_{i,j} = \begin{cases} \frac{\rho \ell_j}{\ell_i} & \text{if } A_{i,j} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

One can check directly that P is a primitive probability matrix. Let $\mathbf{p} = (p_i)_{i \in \widehat{\Omega}}$ be the probability vector satisfying $\mathbf{p}P = \mathbf{p}$. A direct calculation shows that

$$p_3 = \tfrac{\rho}{2\rho+1} = \tfrac{5-\sqrt{5}}{10}, \quad p_5 = \tfrac{1}{2\rho+1} = \tfrac{\sqrt{5}}{5},$$
$$p_6 = \tfrac{2\rho-1}{2\rho+1} = \tfrac{5-2\sqrt{5}}{5}, \; p_7 = \tfrac{1-\rho}{2\rho+1} = \tfrac{3\sqrt{5}-5}{10}.$$

Define η to be the $(\mathbf{p}, P)$ Markov measure on $\widehat{\Omega}_A^{\mathbb{N}}$, i.e., η is the unique Borel probability measure on $\widehat{\Omega}_A^{\mathbb{N}}$ satisfying

$$\eta([i_1 i_2 \ldots i_n]) = p_{i_1} P_{i_1, i_2} \ldots P_{i_{n-1}, i_n}$$

for any $n \geqslant 2$ and any cylinder set

$$[i_1 i_2 \ldots i_n] := \left\{(x_j)_{j=1}^{\infty} \in \widehat{\Omega}_A^{\mathbb{N}} :\ x_j = i_j \text{ for } 1 \leqslant j \leqslant n\right\}.$$

The measure η is σ-invariant and ergodic. The reader is referred to [59] for more information about Markov measures. Now we claim that

$$\eta \circ \hat{\pi}^{-1}(A) = \frac{p_3}{\rho \ell_3} \mathcal{L}(A), \qquad \forall \text{ Borel set } A \subset \Delta(13). \tag{4.26}$$

To prove the claim, note that for any $i_1 \ldots i_n \in \widehat{\Omega}_{A,n}$ with $i_1 = 3$, the net interval $\Delta(1i_1 \ldots i_n)$ has the length $\rho^n \ell_{i_n}$, while

$$\begin{aligned} \eta([i_1 \ldots i_n]) &= p_{i_1} P_{i_1,i_2} \ldots P_{i_{n-1},i_n} = p_{i_1} \prod_{j=1}^{n-1} \frac{\rho \ell_{i_{j+1}}}{\ell_{i_j}} \\ &= p_{i_1} \rho^{n-1} \frac{\ell_{i_n}}{\ell_{i_1}} = \frac{p_3}{\rho \ell_3} \rho^n \ell_{i_n}. \end{aligned} \tag{4.27}$$

It follows that (4.26) holds for $A = \Delta(1i_1 \ldots i_n)$. A standard argument using the monotone class theorem yields (4.26). Now we prove

$$\begin{aligned} &\lim_{n\to\infty} \frac{\log \|T_{1x_1\ldots x_n}\|}{n} \\ &= \sum_{v \in \mathcal{B}} \eta([6v]) \log \|T_{1v}\| \qquad (4.28) \\ &= \frac{7\sqrt{5}-15}{10} \sum_{n=0}^{\infty} \left(\frac{3-\sqrt{5}}{2} \right)^{n+1} \sum_{J \in \mathcal{A}_n} \log \|M_J\| \qquad (4.29) \end{aligned}$$

for η almost all $(x_j)_{j=1}^{\infty} \in [3]$, where $\mathcal{B}$ is defined as in (4.22). To show this, define

$$E := \left\{ (x_i)_{i=1}^{\infty} \in \widehat{\Omega}_A^{\mathbb{N}} : \ \exists\, m_j \uparrow \infty, \ \text{such that } \lim_{j\to\infty} \frac{m_j}{m_{j+1}} = 1, \ x_{m_j} = 6 \right\}.$$

Since η is ergodic and $\eta([6]) > 0$, by the Birkhoff ergodic theorem we have $\eta(E) = 1$. Now for each $\omega = (x_i)_{i=1}^{\infty} \in E \cap [3]$, let $m_j = m_j(\omega)$ ($j \in \mathbb{N}$) be the increasing sequence of all integers k satisfying $x_k = 6$. Write ω as

$$\omega = \omega_1 \circ \omega_2 \circ \ldots \circ \omega_n \circ \ldots,$$

where $\omega_1 = (x_i)_{i=1}^{m_1}$, $\omega_2 = (x_i)_{i=m_1+1}^{m_2}, \ldots, \omega_n = (x_i)_{i=m_{n-1}+1}^{m_n}, \ldots$. It is clear that $\omega_i \in \mathcal{B}$ for $i \geqslant 1$. Furthermore,

$$\begin{aligned} \|T_{1\omega_1\ldots\omega_n}\| &= \prod_{i=1}^{n} \|T_{1\omega_i}\| \\ &= \|T_{1\omega_1}\| \times \left(\prod_{v \in \mathcal{B},\ |v| \leqslant m_n} \|T_{1v}\|^{\sum_{j=0}^{m_n-|v|} \chi_{[6v]}(\sigma^j \omega)} \right), \end{aligned} \tag{4.30}$$

where $\chi_{[6v]}$ denotes the characteristic function on $[6v]$. Hence

$$\frac{\log \|T_{1\omega_1\ldots\omega_n}\|}{m_n} = \frac{\log \|T_{1\omega_1}\|}{m_n} + \sum_{v\in\mathcal{B}, |v|\leqslant m_n} \frac{1}{m_n} \sum_{j=0}^{m_n-|v|} \chi_{[6v]}(\sigma^j \omega) \log \|T_{1v}\|. \tag{4.31}$$

Since η is ergodic, by the Birkhoff ergodic theorem, for each $v \in \mathcal{B}$,

$$\lim_{n\to\infty} \frac{1}{m_n} \sum_{j=0}^{m_n-|v|} \chi_{[6v]}(\sigma^j \omega) = \eta([6v]) \tag{4.32}$$

for η almost all $\omega \in E \cap [3]$. Combining (4.31) and (4.32) we obtain that

$$\liminf_{n\to\infty} \frac{\log \|T_{1\omega_1\ldots\omega_n}\|}{m_n} \geqslant \sum_{v\in\mathcal{B}} \eta([6v]) \log \|T_{1v}\| \tag{4.33}$$

for η almost all $\omega \in E \cap [3]$. Observe that $\|T_{1\omega_1\ldots\omega_n}\| \leqslant 4^{m_n}$, thus the right hand side of (4.33) converges with an upper bound $\log 4$. To state the inequality for upper limits, we fix $l \in \mathbb{N}$. For any $k > l$ and $v = 35\delta_{i_1}5\ldots\delta_{i_k}56 \in \mathcal{B}$, we have

$$\|T_{1v}\| \leqslant \|T_{5\delta_{i_1}5\ldots\delta_{i_l}5}\| \cdot \|T_{5\delta_{i_2}5\ldots\delta_{i_{l+1}}5}\| \ldots \|T_{5\delta_{i_{k-l+1}}5\ldots\delta_{i_k}5}\|.$$

Denote

$$\mathcal{C}_l = \left\{5\delta_{i_1}\ldots\delta_{i_l}5 : \ i_1\ldots i_l \in \mathcal{A}_l\right\}$$

By (4.30), we have

$$\begin{aligned}\frac{\log \|T_{1\omega_1\ldots\omega_n}\|}{m_n} \leqslant\ & \frac{\log \|T_{1\omega_1}\|}{m_n} + \sum_{v\in\mathcal{B}, |v|\leqslant 2l+3} \frac{1}{m_n} \sum_{j=0}^{m_n-|v|} \chi_{[6v]}(\sigma^j \omega) \log \|T_{1v}\| \\ & + \sum_{v'\in\mathcal{C}_l} \frac{1}{m_n} \sum_{j=0}^{m_n-2l} \chi_{[v']}(\sigma^j \omega) \log \|T_{v'}\|\end{aligned}$$

whenever $m_n > 2l + 3$. Thus by the Birkhoff ergodic theorem,

$$\begin{aligned}\limsup_{n\to\infty} \frac{\log \|T_{1\omega_1\ldots\omega_n}\|}{m_n} \leqslant\ & \sum_{v\in\mathcal{B},\ |v|\leqslant 2l+3} \eta([6v]) \log \|T_{1v}\| \\ & + \sum_{v'\in\mathcal{C}_l} \eta([v']) \log \|T_{v'}\|\end{aligned} \tag{4.34}$$

for η almost all $\omega \in E \cap [3]$. To estimate the second sum in (4.34), we observe that for each $v' = 5\delta_{i_1}5\ldots\delta_{i_l}5$,

$$\begin{aligned}
\eta([v']) &= p_5 P_{5,\delta_{i_1}} P_{\delta_{i_1},5} \ldots P_{\delta_{i_l},5}\\
&= \frac{p_5}{p_3 P_{3,5} P_{\delta_{i_l},5} P_{5,6}} \cdot p_3 P_{3,5} P_{5,\delta_{i_1}} P_{\delta_{i_1},5} \ldots P_{5,\delta_{i_l}} P_{\delta_{i_l},5} P_{5,6}\\
&= \frac{p_5}{p_3 P_{3,5} P_{\delta_{i_l},5} P_{5,6}} \eta([35\delta_{i_1}5\ldots\delta_{i_l}56])\\
&\leqslant \max\left\{\frac{p_5}{p_3 P_{3,5} P_{3,5} P_{5,6}}, \frac{p_5}{p_3 P_{3,5} P_{7,5} P_{5,6}}\right\} \times \eta([35\delta_{i_1}5\ldots\delta_{i_l}56])
\end{aligned}$$

and

$$\log \|T_{v'}\| \leqslant \log \|T_{135\delta_{i_1}5\ldots\delta_{i_l}56}\|.$$

Hence

$$\begin{aligned}
\sum_{v' \in \mathcal{C}_l} \eta([v']) \log \|T_{v'}\| &\leqslant \max\left\{\frac{p_5}{p_3 P_{3,5} P_{3,5} P_{5,6}}, \frac{p_5}{p_3 p_{3,5} P_{7,5} P_{5,6}}\right\}\\
&\qquad \times \sum_{v \in \mathcal{B}, |v| = 2l+3} \eta([v]) \log \|T_{1v}\|. \qquad (4.35)
\end{aligned}$$

Since $\sum_{v \in \mathcal{B}} \eta([6v]) \log \|T_{1v}\|$ converges and

$$\eta([6v]) = \frac{p_6 P_{6,3}}{p_3} \eta([v]), \qquad \forall\, v \in \mathcal{B},$$

the right hand side of (4.35) tends to 0 as $l \to \infty$. Thus combining (4.33) and (4.34) we obtain

$$\lim_{n \to \infty} \frac{\log \|T_{1\omega_1\ldots\omega_n}\|}{m_n} = \sum_{v \in \mathcal{B}} \eta([6v]) \log \|T_{1v}\|$$

for η almost all $\omega \in E \cap [3]$. Thus (4.28) holds from the facts $\lim_{n\to\infty} \frac{m_{n+1}}{m_n} = 1$ for $\omega \in E \cap [3]$ and $\eta(E) = 1$. Since for any $v = 35\delta_{i_1}5\ldots\delta_{i_n}56 \in \mathcal{B}$,

$$\eta([6v]) = \frac{p_6 P_{6,3}}{p_3} \eta([v])$$

$$= \frac{p_6 P_{6,3}}{p_3} \frac{p_3}{\rho \ell_3} \rho^{2n+3} \ell_6 \quad \text{(by (4.27))}$$

$$= \frac{(2\rho-1)\rho^{2n+3}}{2\rho+1}$$

and $\|T_{1v}\| = \|M_{i_1 \dots i_n}\|$, we get (4.29).

Combining (4.26) and (4.29), we have for $\mathcal{L}$ almost all $y \in \Delta(13)$,

$$\lim_{n\to\infty} \frac{\log \|T_{1x_1\dots x_n}\|}{n} = \frac{7\sqrt{5}-15}{10} \sum_{n=0}^{\infty} \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} \sum_{J\in\mathcal{A}_n} \log \|M_J\|, \tag{4.36}$$

where $1x_1 \dots x_n$ is the symbolic expression of an nth net interval containing y. A similar argument shows that (4.36) holds for $\mathcal{L}$ almost all $y \in \Delta(1u^l 3)$, where $u = 2$ or 4, and $l \in \mathbb{N}$. One may check that the intervals $\Delta(13)$, $\Delta(1u^l 3)$ are disjoint and

$$\mathcal{L}(\Delta(13)) + \sum_{l=1}^{\infty} \sum_{u\in\{2,4\}} \mathcal{L}(\Delta(1u^l 3)) = 1.$$

Hence (4.36) holds for $\mathcal{L}$ almost all $y \in [0,1]$ with $1x_1 \dots x_n \dots \in \pi^{-1}(y)$. This together with Theorem 3.2 proves Theorem 4.17. □

4.7. The box dimension and Hausdorff dimension of the level sets of f

In this section we prove the following theorems:

Theorem 4.18. *For $\mathcal{L}$ almost all $t \in [0,1]$, we have*

$$\dim_{\mathrm{B}} L_t = \frac{7\sqrt{5}-15}{10\log 2} \sum_{n=0}^{\infty} \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} \sum_{J\in\mathcal{A}_n} \log \|M_J\|,$$

where the level set L_t is defined as in (3.9).

Theorem 4.19. *For $\mathcal{L}$ almost all $t \in [0,1]$, we have*

$$\dim_{\mathrm{H}} L_t = \frac{7\sqrt{5}-15}{10\log 2} \sum_{n=0}^{\infty} \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} \sum_{J\in\mathcal{A}_n} \log \|M_J\|. \tag{4.37}$$

Theorem 4.18 follows directly from Theorem 4.17 and Corollary 3.6. To prove Theorem 4.19, we need a dimensional result about *homogeneous Moran sets*.

Let $\{n_k\}_{k\geqslant 1}$ be a sequence of positive integers and $\{c_k\}_{k\geqslant 1}$ be a sequence of positive numbers satisfying $n_k \geqslant 2$, $0 < c_k < 1$, $n_1 c_1 \leqslant \delta$ and $n_k c_k \leqslant 1$ ($k \geqslant 2$), where δ is some positive number. Let

$$D = \bigcup_{k\geqslant 0} D_k \quad \text{with} \quad D_0 = \{\emptyset\},\ D_k = \{(i_1, \ldots, i_k);\ \ 1\leqslant i_j \leqslant n_j,\ \ 1 \leqslant j \leqslant k\}.$$

If $\gamma = (\gamma_1, \ldots, \gamma_k) \in D_k$, $\tau = (\tau_1, \ldots, \tau_m) \in D_m$, we define

$$\gamma * \tau = (\gamma_1, \ldots, \gamma_k, \tau_1, \ldots, \tau_m).$$

Suppose that J is an interval of length δ. A collection $\mathcal{F} = \{J_\gamma :\ \gamma \in D\}$ of subintervals of J is said to have a *homogeneous Moran structure* if it satisfies

(1) $J_\emptyset = J$;

(2) For any $k \geqslant 0$ and $\gamma \in D_k$, $J_{\gamma*1}, J_{\gamma*2}, \ldots, J_{\gamma*n_{k+1}}$ are subintervals of J_γ and $J_{\gamma*i} \bigcap J_{\gamma*j} = \emptyset$ $(i \neq j)$;

(3) For any $k \geqslant 1$ and any $\gamma \in D_{k-1}$, $1 \leqslant j \leqslant n_k$, we have

$$\frac{|J_{\gamma*j}|}{|J_\gamma|} = c_k,$$

where $|A|$ denotes the length of A.

If $\mathcal{F}$ is such a collection, $F := \bigcap_{k\geqslant 1} \bigcup_{\gamma\in D_k} J_\gamma$ is called a *homogeneous Moran set* determined by $\mathcal{F}$. We refer the readers to [19,21] for more information about homogeneous Moran sets. For some applications of homogeneous Moran sets in the dimension theory of dynamical systems, see [10,11,17,50]. For the purpose of the present paper, we only need the following simplified version of a result contained in [21], whose simpler proof was given in [10, Proposition 3].

Proposition 4.20. *For the homogeneous Moran set F defined above, we have*

$$\dim_{\mathrm{H}} F \geqslant \liminf_{n\to\infty} \frac{\log n_1 n_2 \ldots n_k}{-\log c_1 c_2 \ldots c_{k+1} n_{k+1}}.$$

Proof of Theorem 4.19. We only prove (4.37) for $\mathcal{L}$ almost all $t \in \Delta(13)$. A similar argument can extend it for $\mathcal{L}$ almost all $t \in \Delta(1u^l 3)$ with $u = 2$ or 4, $l \geqslant 1$.

The upper bound is easy to see since we have always $\dim_{\mathrm{H}} L_t \leqslant \dim_{\mathrm{B}} L_t$. Now let us consider the lower bound. We use some notations (the map $\hat{\pi}$, the measure η, the cylinder set [3], the set E and the sequence $m_j = m_j(\omega)$, etc.) introduced in the proof of Theorem 4.17. For any $\omega \in E \cap [3]$, write

$$\omega = \omega_1 \circ \omega_2 \circ \ldots$$

with $\omega_i \in \mathcal{B}$. We claim that the level set $L_{\hat{\pi}(\omega)}$ is a homogeneous Moran set with respect to $n_k = \|T_{1\omega_k}\|$ and $c_k = 2^{-|\omega_k|}$. Therefore by Proposition 4.20,

$$\begin{aligned}\dim_{\mathrm{H}} L_{\hat{\pi}(\omega)} &\geqslant \liminf_{k\to\infty} \frac{\log(\|T_{1\omega_1}\| \times \cdots \times \|T_{1\omega_k}\|)}{\log(2^{|\omega_1|+\cdots+|\omega_{k+1}|} \times \|T_{1\omega_{k+1}}\|^{-1})}\\ &\geqslant \liminf_{k\to\infty} \frac{\log(\|T_{1\omega_1}\| \times \cdots \times \|T_{1\omega_k}\|)}{\log(2^{|\omega_1|+\cdots+|\omega_{k+1}|})}\\ &= \liminf_{k\to\infty} \frac{\log(\|T_{1\omega_1\dots\omega_k}\|)}{m_{k+1}\log 2}\\ &= \liminf_{k\to\infty} \frac{\log(\|T_{1\omega_1\dots\omega_k}\|)}{m_k\log 2}.\end{aligned}$$

Combining it with (4.33), we obtain the lower bound of $\dim_{\mathrm{H}} L_{\hat{\pi}(\omega)}$ for η almost all $\omega \in E \cap [3]$. By the equivalence of η and $\mathcal{L}$, we get the lower bound of $\dim_{\mathrm{H}} L_t$ for $\mathcal{L}$ almost all $t \in \Delta(13)$.

Now we prove the claim about the homogeneous Moran structure of $L_{\hat{\pi}(\omega)}$. For $k \in \mathbb{N}$, write $\Delta(1\omega_1 \dots \omega_{m_k}) = [c_k, d_k]$ and define

$$D_k := \left\{a_1 \dots a_{m_k} \in \mathcal{A}_{m_k} : \ \pi(1\omega) \in I_{a_1\dots a_{m_k}}\right\},$$

where $I_{a_1\dots a_{m_k}} = \left[\sum_{i=1}^{m_k} a_i 2^{-i}, \sum_{i=1}^{m_k} a_i 2^{-i} + 2^{m_k}\right)$. By Lemma 3.7,

$$\begin{aligned}D_k &= \left\{a_1 \dots a_{m_k} \in \mathcal{A}_{m_k} : S_{a_1\dots a_{m_k}}([0,1)) \supset [c_k, d_k)\right\}\\ &= \left\{a_1 \dots a_{m_k} \in \mathcal{A}_{m_k} : 0 \leqslant c_k - S_{a_1\dots a_{m_k}}(0) \leqslant \rho^{m_k}\right\}.\end{aligned}$$

Observe that the characteristic vector of $\Delta(1\omega_1 \dots \omega_{m_k})$, represented by the symbol 6, is $(2\rho - 1; 1-\rho; 1)$. It follows that the points $S_{a_1\dots a_{m_k}}(0)$ are the same for all $a_1 \dots a_{m_k} \in \mathcal{A}_{m_k}$. For $\gamma = a_1 \dots a_{m_k} \in D_k$, define

$$J_\gamma = \left[\sum_{i=1}^{m_k} a_i 2^{-i}, \sum_{i=1}^{m_k} a_i 2^{-i} + 2^{m_k}\right].$$

Then $L_{\pi(1\omega)} = \bigcap_{k=1}^{\infty} \bigcup_{\gamma \in D_k} J_\gamma$ is a Moran set with $n_k = \|T_{1\omega_k}\|$ and $c_k = 2^{-|\omega_k|}$. This finishes the proof. $\square$

4.8. The infinite similarity of μ

We know that μ is a self-similar measure generated by the similitudes S_0 and S_1 (see (1.2)). However, S_i $(i = 1, 2)$ have overlaps (i.e., they do not satisfy the open set condition). In this section, we show that μ is a locally infinitely generated self-similar measure without overlaps. More precisely, let $\mathcal{B}$ be defined as in (4.22), and denote

$$\mathcal{D} = \left\{1u^l v : \ u = 2 \text{ or } 4, \ l \geqslant 0, \ v \in \mathcal{B}\right\}.$$

It is clear that ωv is an admissible word starting from the letter 1 for any $\omega \in \mathcal{D}$ and $v \in \mathcal{B}$. Denote by $g_{\omega,v}$ the unique similitude $cx + d$ with $c > 0$ so that $g_{\omega,v}(\Delta(\omega)) = \Delta(\omega v)$, where $\Delta(\omega)$ denotes the net interval with respect to the admissible word ω. We have the following

Theorem 4.21. *μ is supported on $\bigcup_{\omega\in\mathcal{D}} \Delta(\omega)$. For any fixed $\omega \in \mathcal{D}$,*

$$\mu_\omega = \sum_{v\in\mathcal{B}} 2^{-|v|} \|T_{1v}\| \cdot \mu_\omega \circ g_{\omega,v}^{-1}, \tag{4.38}$$

where μ_ω denotes the restriction of μ on $\Delta(\omega)$, that is, $\mu_\omega(A) = \mu(A \cap \Delta(\omega))$ for any Borel set $A \subset \mathbb{R}$. Furthermore,

$$g_{\omega,v}(\Delta(\omega)) = \Delta(\omega v) \subset \Delta(\omega), \qquad \Delta(\omega v) \cap \Delta(\omega v') = \emptyset \ \ for \ \ v \neq v'. \tag{4.39}$$

To prove the above theorem, we need the following lemma.

Lemma 4.22. (i) $\mu[1-\rho, \rho] = \frac{1}{3}$.
(ii) $\sum_{v\in\mathcal{B}} \rho^{|v|} = 1$.
(iii) $\sum_{v\in\mathcal{B}} 2^{-|v|} \|T_{1v}\| = 1$.

Proof. By (1.2), we have

$$\mu(A) = \tfrac{1}{2}\, \mu\left(S_0^{-1}(A)\right) + \tfrac{1}{2}\, \mu\left(S_1^{-1}(A)\right)$$

for any Borel set $A \subset \mathbb{R}$. Taking A to be $[0, 1-\rho]$ and $[\rho, 1]$, respectively, we have

$$\mu([0, 1-\rho]) = \tfrac{1}{2}\, \mu([0, 1-\rho]) + \tfrac{1}{2}\, \mu([1-\rho, \rho])$$

and

$$\mu([\rho, 1]) = \tfrac{1}{2}\, \mu([1-\rho, \rho]) + \tfrac{1}{2}\, \mu([\rho, 1]).$$

These two equalities imply $\mu([0, 1-\rho]) = \mu([1-\rho, \rho]) = \mu([\rho, 1]) = \frac{1}{3}$. Hence (i) follows.

To see (ii), observe that each string in $\mathcal{B}$ is of length $3+2n$ $(n\geqslant 0)$, and for each $n\geqslant 0$ there are exact 2^n different strings in $\mathcal{B}$ having length $3+2n$. Therefore

$$\begin{aligned}\sum_{v\in\mathcal{B}}\rho^{|v|} &= \sum_{n=0}^{\infty}2^n\rho^{3+2n}\\ &= \rho^3\sum_{n=0}^{\infty}(2\rho^2)^n = \frac{\rho^3}{1-2\rho^2} = 1.\end{aligned}$$

To see (iii), we have

$$\begin{aligned}\sum_{v\in\mathcal{B}}2^{-|v|}\|T_{1v}\| &= 2^{-3}\cdot 2+\sum_{n=1}^{\infty}\sum_{i_1\ldots i_n\in\mathcal{A}_n}2^{-3-2n}(1,1)M_{i_1}\ldots M_{i_n}(1,1)^T\\ &= 2^{-2}+\sum_{n=1}^{\infty}2^{-3-2n}(1,1)(M_0+M_1)^n(1,1)^T\\ &= 2^{-2}+\sum_{n\geqslant 1}2^{-3-2n}(1,1)\begin{bmatrix}2&1\\1&2\end{bmatrix}^n(1,1)^T\\ &= 2^{-2}+\sum_{n=1}^{\infty}2^{-3-2n}\cdot 2\cdot 3^n\\ &= 2^{-2}+2^{-2}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = 1. \qquad \square\end{aligned}$$

Proof of Theorem 4.21. It is not hard to see that the net intervals $\Delta(\omega)$ $(\omega\in\mathcal{D})$ are disjoint. Now we show that $\mu\left(\bigcup_{\omega\in\mathcal{D}}\Delta(\omega)\right)=1$. For each $v\in\mathcal{B}$, by (ii) of Lemmas 2.10 and 4.22 we have

$$\begin{aligned}\mu(\Delta(1u^lv)) &= 2^{-|v|-l}\cdot\|T_{1u^lv}\|\cdot\mu([1-\rho,\rho])\\ &= 2^{-|v|-l}\cdot\|T_{1v}\|\cdot\mu([1-\rho,\rho])\\ &= \tfrac{1}{3}\cdot 2^{-|v|-l}\cdot\|T_{1v}\|, \qquad \forall u=2 \text{ or } 4,\ l\geqslant 1.\end{aligned}$$

Therefore

$$\mu\left(\bigcup_{\omega\in\mathcal{D}}\Delta(\omega)\right) = \sum_{\omega\in\mathcal{D}}\mu(\Delta(\omega))$$

$$= \sum_{v\in\mathcal{B}} \frac{1}{3}\cdot 2^{-|v|}\cdot \|T_{1v}\| + 2\sum_{v\in\mathcal{B}}\sum_{n=1}^{\infty} \frac{1}{3}\cdot 2^{-|v|-n}\cdot \|T_{1v}\|$$

$$= \sum_{v\in\mathcal{B}} 2^{-|v|}\cdot \|T_{1v}\| = 1.$$

This is, μ is supported on $\bigcup_{\omega\in\mathcal{D}} \Delta(\omega)$. Similarly, we have

$$\mu\left(\bigcup_{v_1,\ldots,v_n\in\mathcal{B}} \Delta(\omega v_1\ldots v_n)\right) = \mu(\Delta(\omega)), \qquad \forall\ n\in\mathbb{N},\ \ \omega\in\mathcal{D}.$$

For any $v, v_1,\ldots,v_n \in \mathcal{B}$, we claim that $g_{\omega,v}(\Delta(\omega v_1\ldots v_n)) = \Delta(\omega v v_1\ldots v_n)$. To see this, for two given intervals $[a,b]$ and $[c,d]$ with $[a,b]\supset[c,d]$ we call the ratio $\frac{c-a}{b-a}$ *the relative place* of $[c,d]$ in $[a,b]$. Since $g_{\omega,v}$ is a similitude with positive ratio, the relative place of $g_{\omega,v}(\Delta(\omega v_1\ldots v_n))$ in $g_{\omega,v}(\Delta(\omega)) = \Delta(\omega v)$ is the same as that of $\Delta(\omega v_1\ldots v_n)$ in $\Delta(\omega)$. To prove our claim, it suffices to show that the relative place of $\Delta(\omega v v_1\ldots v_n)$ in $\Delta(\omega v)$ is the same as that of $\Delta(\omega v_1\ldots v_n)$ in $\Delta(\omega)$, and the length of $g_{\omega,v}(\Delta(\omega v_1\ldots v_n))$ equals which of $\Delta(\omega v v_1\ldots v_n)$. The second fact is easy to see. Let us show the first one. By the structure of net intervals, the relative place of $\Delta(\omega v v_1\ldots v_n)$ in $\Delta(\omega v)$ is determined completely by $v_1\ldots v_n$ and the end letter of v, and the relative place of $\Delta(\omega v_1\ldots v_n)$ in $\Delta(\omega)$ is determined completely by $v_1\ldots v_n$ and the end letter of ω. Since the end letter of v is the same as that of ω, the first fact follows.

Now let us turn to prove (4.38) for a fixed $\omega\in\mathcal{D}$. For $n\in\mathbb{N}$ and $v_1,\ldots,v_n\in\mathcal{B}$, by the claim, we have $g_{\omega,v_1}(\Delta(\omega v_2\ldots v_n)) = \Delta(\omega v_1\ldots v_n)$. For $v\in\mathcal{B}$ and $v\neq v_1$, it is clear that $g_{\omega,v}(\Delta(\omega))\cap\Delta(\omega v_1\ldots v_n) = \emptyset$ since $g_{\omega,v}(\Delta(\omega)) = \Delta(\omega v)$. Therefore

$$\begin{aligned}
&\sum_{v\in\mathcal{B}} 2^{-|v|}\|T_{1v}\|\cdot \mu_\omega\circ g_{\omega,v}^{-1}(\Delta(\omega v_1\ldots v_n))\\
&\quad= 2^{-|v_1|}\|T_{1v_1}\|\cdot \mu_\omega\circ g_{\omega,v_1}^{-1}(\Delta(\omega v_1\ldots v_n))\\
&\quad= 2^{-|v_1|}\|T_{1v_1}\|\cdot \mu_\omega(\Delta(\omega v_2\ldots v_n))\\
&\quad= \mu_\omega(\Delta(\omega v_1\ldots v_n)).
\end{aligned}$$

A standard argument using the monotone class yields (4.38). $\square$

Remark 4.23. For each $\omega\in\mathcal{D}$, μ_ω is equivalent to a Bernoulli shift measure. To see this, consider the one-side shift space $(\mathcal{B}^{\mathbb{N}},\sigma)$ over $\mathcal{B}$. Endow this shift space with the product measure η, which satisfies

$$\eta([v]) = 2^{-|v|}\|T_{1v}\|, \qquad \forall v\in\mathcal{B}.$$

Define the projection π_ω from $\mathcal{B}^{\mathbb{N}}$ to $\Delta(\omega)$ by

$$\pi_\omega\left((v_n)_{n=1}^{\infty}\right) = \bigcap_{n\geqslant 1} \Delta(\omega v_1 \ldots v_n).$$

Theorem 4.21 implies that

$$\eta = \frac{1}{\mu(\Delta(\omega))}\,\mu_\omega \circ \pi_\omega,$$

or equivalently,

$$\mu_\omega = \mu(\Delta(\omega)) \cdot \eta \circ \pi_\omega^{-1}.$$

4.9. The multifractal formalism for μ

For $\alpha \geqslant 0$, define $K(\alpha) = \{x \in \mathbb{R} : \ d(\mu, x) = \alpha\}$. By Theorem 4.15,

$$K(\alpha) \neq \emptyset \Longleftrightarrow \alpha \in \mathcal{R}(\mu) = \left[-\frac{\log 2}{\log \rho} - \frac{1}{2}, -\frac{\log 2}{\log \rho}\right].$$

The main purpose of multifractal analysis is to study $\dim_{\mathrm{H}} K(\alpha)$, which is called the multifractal spectrum of μ. We refer the readers to the books [9,50] and papers [4,5,12,23,44,54] for more information about the multifractal analysis of measures. In this section, we prove

Theorem 4.24. *Let q_0 be defined as in Theorem* 4.4. *Then for any $q \neq q_0$, we have*

$$\dim_{\mathrm{H}} K(\alpha(q)) = \inf_{t\in\mathbb{R}}\{\alpha(q)t - \tau(t)\} = \alpha(q)q - \tau(q), \tag{4.40}$$

where $\tau(t)$ is the L^t-spectrum of μ, and $\alpha(q) = \tau'(q)$.

Remark 4.25. Riedi and Mandelbrot [54] studied the multifractal structure of self-similar measures with infinitely many non-overlapping generators. They verified the validity of the multifractal formalism under some additional assumptions on the decay speed of contraction ratios and probability weights (see [54, Theorem 10]). However our measure μ in Theorem 4.24 (and measures in Theorem 5.14) does not satisfy their assumptions. By the way, the reader may see [40] for more details about dimensions and measures in infinite iterated function systems.

Proof of Theorem 4.24. The upper bound

$$K(\alpha) \leqslant \inf_{t\in\mathbb{R}}\{\alpha t - \tau(t)\}, \qquad \forall \alpha \in \mathcal{R}(\mu)$$

is generic, not depending on the special property of μ (see e.g., [4, Theorem 1] or [32, Theorem 4.1]). In the following we prove the lower bound.

First assume $q < q_0$. By Theorem 4.12, $\alpha(q) = -\frac{\log 2}{\log \rho}$ and $-\tau(q) + \alpha(q)q = 0$. Since $K(\alpha(q)) \neq \emptyset$, we have

$$\dim_{\mathrm{H}} K(\alpha(q)) \geqslant 0 \geqslant \inf_{t \in \mathbb{R}} \{\alpha(q)t - \tau(t)\}.$$

From now on, we assume $q > q_0$. Let $\mathcal{B}$ be defined as in (4.22). First we define a probability product measure $\widehat{\eta}_q$ on the shift space $(\mathcal{B}^{\mathbb{N}}, \sigma)$ with the weights $\widetilde{p}_v = p_v^q r_v^{-\tau(q)}$ for each $v \in \mathcal{B}$, where

$$p_v = 2^{-|v|} \| T_{1v} \|, \qquad r_v = \rho^{|v|}.$$

Theorem 4.12 implies $\sum_{v \in \mathcal{B}} \widetilde{p}_v = 1$. It is well known that $\widehat{\eta}_q$ is a σ-invariant ergodic measure. Denote $v_0 = 356$ and define the projection $\pi = \pi_{v_0}$ from $\mathcal{B}^{\mathbb{N}}$ to the net interval $V(1v_0)$ by

$$\pi\left((v_n)_{n=1}^{\infty}\right) = \bigcap_{n=1}^{\infty} V(1v_0v_1 \ldots v_n).$$

The map π is continuous and injective (not surjective). Define

$$\eta_q = \widehat{\eta}_q \circ \pi^{-1}.$$

Then

$$\eta_q(V(1v_0v_1 \ldots v_n)) = \widetilde{p}_{v_1} \ldots \widetilde{p}_{v_n}. \tag{4.41}$$

Note that

$$\mu(V(1v_0v_1 \ldots v_n)) = \mu(V(1v_0)) \cdot p_{v_1} \ldots p_{v_n}, \tag{4.42}$$

$$|V(1v_0v_1 \ldots v_n)| = |V(1v_0)| \cdot r_{v_1} \ldots r_{v_n}. \tag{4.43}$$

Since $\widehat{\eta}_q$ is σ-invariant and ergodic, by the Birkhoff ergodic theorem, there exists a Borel measurable set $G_q \subset \mathcal{B}^{\mathbb{N}}$ with $\widehat{\eta}_q(G_q) = 1$ such that for each $\omega = (v_n)_{n=1}^{\infty} \in G_q$,

$$\lim_{n\to\infty} \frac{\log \eta_q(V(1v_0v_1 \ldots v_n))}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \widetilde{p}_{\sigma^j \omega|1}$$

$$
\begin{aligned}
&= \sum_{v\in\mathcal{B}} \widetilde{p}_v \log \widetilde{p}_v \\
&= \sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)} \log\left(p_v r_v^{-\tau(q)}\right), \qquad (4.44)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\log \mu_\rho(V(1v_0v_1\ldots v_n))}{n} &= \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log p_{\sigma^j\omega|1} \\
&= \sum_{v\in\mathcal{B}} \widetilde{p}_v \log p_v \\
&= \sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)} \log p_v \qquad (4.45)
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{n\to+\infty} \frac{\log |V(1v_0v_1\ldots v_n)|}{n} &= \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log r_{\sigma^j\omega|1} \\
&= \sum_{v\in\mathcal{B}} \widetilde{p}_v \log r_v \\
&= \sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)} \log r_v, \qquad (4.46)
\end{aligned}
$$

where $\omega|1 = v_1$ for $\omega = (v_n)_{n=1}^\infty \in \mathcal{B}^{\mathbb{N}}$. The integrality of functions $\omega \mapsto \widetilde{p}_{\omega|1}$, $\omega \mapsto p_{\omega|1}$ and $\omega \mapsto r_{\omega|1}$, or equivalently, the finiteness of $\sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)} \log(p_v r_v^{-\tau(q)})$, $\sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)} \log p_v$ and $\sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)} \log r_v$ come from the following inequality:

$$
\sum_{n=0}^{\infty} n \sum_{J\in\mathcal{A}_n} \|M_J\|^q x(q)^{2n+3} < \infty, \qquad \forall q > q_0
$$

which was implied in Step 3 at the proof of part (ii) of Theorem 4.4.

Now fix $\omega = (v_n)_{n=1}^\infty \in G_q$. By (4.43) we have

$$
V(1v_0v_1\ldots v_n) \subset \left[\pi(\omega) - |V(1v_0)|\cdot r_{v_1}\ldots r_{v_n},\ \pi(\omega) + |V(1v_0)|\cdot r_{v_1}\ldots r_{v_n}\right] \qquad (4.47)
$$

for all $n \in \mathbb{N}$. On the other hand, we know that the end letter of the string v_{n+1} is 6. Changing from v_{n+1} this letter to 3 and 7, we get two words $\mathbf{j}_{n+1}$ and $\mathbf{j}'_{n+1}$ respectively. The three net intervals $V(1v_0v_1\ldots v_n\mathbf{j}_{n+1})$, $V(1v_0v_1\ldots v_nv_{n+1})$ and $V(1v_0v_1\ldots v_n\mathbf{j}'_{n+1})$ lie in $V(1v_0v_1\ldots v_n)$ in an increasing order with no overlaps. Furthermore, the first and the third intervals have length larger than the second one. Therefore

$$\begin{aligned}
&V(1v_0v_1\dots v_n)\\
&\quad\supset \left[\pi(\omega)-|V(1v_0v_1\dots v_{n+1})|,\ \pi(\omega)+|V(1v_0v_1\dots v_{n+1})|\right]\\
&\quad= [\pi(\omega)-|V(1v_0)|\cdot r_{v_1}\dots r_{v_{n+1}},\ \pi(\omega)+|V(1v_0)|\cdot r_{v_1}\dots r_{v_{n+1}}] \qquad (4.48)
\end{aligned}$$

for all $n\in\mathbb{N}$. Now for any small number $r>0$, choose n so that

$$|V(1v_0)|\cdot r_{v_1}\dots r_{v_n}r_{v_{n+1}}\leqslant r<|V(1v_0)|\cdot r_{v_1}\dots r_{v_n}.$$

By (4.47) and (4.48), we have

$$\begin{aligned}
&[\pi(\omega)-r,\pi(\omega)+r]\\
&\quad\subset \left[\pi(\omega)-|V(1v_0)|\cdot r_{v_1}\dots r_{v_n},\ \pi(\omega)+|V(1v_0)|\cdot r_{v_1}\dots r_{v_n}\right]\\
&\quad\subset V(1v_0v_1\dots v_{n-1}), \qquad (4.49)
\end{aligned}$$

and

$$\begin{aligned}
&[\pi(\omega)-r,\pi(\omega)+r]\\
&\quad\supset \left[\pi(\omega)-|V(1v_0)|\cdot r_{v_1}\dots r_{v_{n+1}},\ \pi(\omega)+|V(1v_0)|\cdot r_{v_1}\dots r_{v_{n+1}}\right]\\
&\quad\supset V(1v_0v_1\dots v_{n+1}). \qquad (4.50)
\end{aligned}$$

By (4.49), (4.50), and (4.44)–(4.46) and Theorem 4.4, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{\log\eta_q[\pi(\omega)-r,\pi(\omega)+r]}{\log r} &= \frac{\sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)}\log(p_v^q r_v^{-\tau(q)})}{\sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)}\log r_v}\\
&= -\tau(q)+\alpha(q)q \qquad (4.51)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to+\infty}\frac{\log\mu_\rho[\pi(\omega)-r,\pi(\omega)+r]}{\log r} &= \frac{\sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)}\log p_v}{\sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)}\log r_v}\\
&= \alpha(q). \qquad (4.52)
\end{aligned}$$

Since $\eta_q(\pi(G_q))=\widehat{\eta}_q(G_q)=1$, by (4.51) we obtain

$$\dim_{\mathrm{H}}\eta_q=-\tau(q)+\alpha(q)q. \qquad (4.53)$$

By (4.52) we have $\pi(G_q) \subset K(\alpha(q))$. Therefore

$$\dim_{\mathrm{H}} K(\alpha(q)) \geqslant \dim_{\mathrm{H}} \pi(G_q) \geqslant \dim_{\mathrm{H}} \eta_q = -\tau(q) + \alpha(q)q.$$

This finishes the proof of the lower bound. □

Remark 4.26. The above proof contains another way to prove Theorems 4.16 and 4.17. To see this, let η_q be the measures constructed in the above proof. First take $q = 1$. One can see that η_1 is just equivalent to μ_{v_0}, the restriction of μ on $V(1v_0)$. By (4.51),

$$d(\mu, x) = \left. \frac{\sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)} \log(p_v^q r_v^{-\tau(q)})}{\sum_{v\in\mathcal{B}} p_v^q r_v^{-\tau(q)} \log r_v} \right|_{q=1} = \frac{\sum_{v\in\mathcal{B}} p_v \log p_v}{\sum_{v\in\mathcal{B}} p_v \log r_v}, \quad \mu \text{ a.e. } x \in V(1v_0),$$

where we have used the easily checked fact $\tau(1) = 0$. By a similar argument this equality holds for μ almost all $x \in V(1v)$ for each $v \in \mathcal{B}$. This proves Theorem 4.16 since $\mu\left(\bigcup_{v\in\mathcal{B}} V(1v)\right) = 1$.

Now take $q = 0$. The measure η_0 is equivalent to the restriction of $\mathcal{L}$ on $V(1v_0)$. Formula (4.52) implies that

$$d(\mu, x) = \frac{\sum_{v\in\mathcal{B}} r_v \log p_v}{\sum_{v\in\mathcal{B}} r_v \log r_v}, \qquad \text{for } \mathcal{L} \text{ a.e. } x \in V(1v_0),$$

where we use the fact $\tau(0) = -1$ which was derived from $\mathrm{Supp}(\mu) = [0, 1]$. A similar argument shows this equality also holds for $\mathcal{L}$ almost all $x \in V(1v)$ for each $v \in \mathcal{B}$. This proves Theorem 4.17 since $\mathcal{L}\left(\bigcup_{v\in\mathcal{B}} V(1v)\right) = 1$.

4.10. Biased Bernoulli convolutions

In this subsection we present some results on biased Bernoulli convolutions. For $0 < t < 1$, the self-similar measure μ_t which satisfies

$$\mu_t = t\mu_t \circ S_0^{-1} + (1-t)\mu_t \circ S_1^{-1},$$

is called *the biased Bernoulli convolution* associated with ρ and t, where $S_0 x = \rho x$ and $S_1 x = \rho x + 1 - \rho$.

Theorem 4.27. *Let $t \in (0, 1)$. There exists a family of non-negative matrices $\{\Gamma_t(\alpha, \beta) : \alpha, \beta \in \Omega,\ A_{\alpha,\beta} = 1\}$ such that for any net interval $\Delta(i_1 i_2 \ldots i_{n+1})$,*

$$\mu_t(\Delta(i_1 i_2 \ldots i_{n+1})) = \|\Gamma_t(i_1, i_2)\Gamma_t(i_2, i_3) \ldots \Gamma_t(i_n, i_{n+1})\|.$$

More precisely, these matrices are given by

$$\begin{cases} \Gamma_t(1,2) = \frac{t^2}{1-t+t^2}, & \Gamma_t(1,3) = \left[\frac{t(1-t)^2}{1-t+t^2}, \frac{t^2(1-t)}{1-t+t^2}\right], \quad \Gamma_t(1,4) = \frac{(1-t)^2}{1-t+t^2}, \\ \Gamma_t(2,2) = t, & \Gamma_t(2,3) = [(1-t)^2, (1-t)t], \\ \Gamma_t(3,5) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \\ \Gamma_t(4,3) = [(1-t)t, t^2], & \Gamma_t(4,4) = 1-t, \\ \Gamma_t(5,3) = \begin{bmatrix} t(1-t) & t^2 \\ 0 & t^2 \end{bmatrix}, & \Gamma_t(5,6) = \begin{bmatrix} (1-t)t \\ (1-t)t \end{bmatrix}, \quad \Gamma_t(5,7) = \begin{bmatrix} (1-t)^2 & 0 \\ (1-t)^2 & (1-t)t \end{bmatrix}, \\ \Gamma_t(6,3) = [1-t, t], & \\ \Gamma_t(7,5) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. & \end{cases}$$

Proof. The first part of the result was proved by the author under a more general setting (see [14, Theorem 3.3]). The second part involves the construction of these matrices $\Gamma_t(\alpha, \beta)$'s, while the general method was given in the proof of [14, Lemma 3.2]. $\square$

For any $v = i_1 i_2 \dots i_n \in \mathcal{B}$, define

$$p_{t,v} = \|\Gamma_t(6, i_1)\Gamma_t(i_1, i_2)\Gamma_t(i_2, i_3) \dots \Gamma_t(i_{n-1}, i_n)\|.$$

Then $p_{t,v} = [1-t, t]X_\emptyset[(1-t)t, (1-t)t]^T$ for $v = 356$ and

$$p_{t,v} = [1-t, t]X_{i_1} \dots X_{i_n}[(1-t)t, (1-t)t]^T$$

for $v = 35\ \delta_{i_1}5\ \dots\ \delta_{i_n}5\ 6$, where $\delta_0 = 3$, $\delta_1 = 7$ and

$$X_\emptyset = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X_0 = \begin{bmatrix} t(1-t) & t^2 \\ 0 & t^2 \end{bmatrix}, \quad X_1 = \begin{bmatrix} (1-t)^2 & 0 \\ (1-t)^2 & (1-t)t \end{bmatrix}. \tag{4.54}$$

It is easily checked that for any admissible word ωv with $v \in \mathcal{B}$,

$$\mu_t(\Delta(\omega v)) = p_{t,v}\mu_t(\Delta(\omega)).$$

From the above equality we obtain an analogue of Theorem 4.21:

Theorem 4.28. *μ_t is supported on $\bigcup_{\omega \in \mathcal{D}} \Delta(\omega)$. For any fixed $\omega \in \mathcal{D}$,*

$$\mu_{t,\omega} = \sum_{v \in \mathcal{B}} p_{t,v} \cdot \mu_{t,\omega} \circ g_{\omega,v}^{-1}, \tag{4.55}$$

where $\mu_{t,\omega}$ denotes the restriction of μ_t on $\Delta(\omega)$, i.e., $\mu_\omega(A) = \mu(A \cap \Delta(\omega))$ for any Borel set $A \subseteq \mathbb{R}$.

The following theorem gives the Hausdorff dimension of μ_t.

Theorem 4.29. *For any* $t \in (0,1)$,

$$\dim_{\mathrm{H}} \mu_t = \frac{t-t^2}{2+t-t^2} \cdot \frac{\sum_{n=0}^{\infty} \sum_{|J|=n} Q_J \log Q_J}{\log \rho},$$

where

$$Q_J = [1-t, t] \; X_{j_1} \cdots X_{j_n} \begin{bmatrix} t-t^2 \\ t-t^2 \end{bmatrix}$$

and $X_0, X_1, X_{\emptyset}$ *are defined as in* (4.54).

Proof. Using Theorem 4.28 and a similar argument as in Remark 4.26, we have

$$d(\mu_t, x) = \frac{\sum_{v\in\mathcal{B}} p_{t,v} \log p_{t,v}}{\sum_{v\in\mathcal{B}} p_{t,v} \log \rho^{|v|}}, \quad \mu_t \ a.e. \ x \in [0,1].$$

Thus $\dim_{\mathrm{H}} \mu_t$ equals the right hand side of the above equality. By the definition of $p_{t,v}$ and (4.54), we have

$$\sum_{v\in\mathcal{B}} p_{t,v} \log p_{t,v} = \sum_{n=0}^{\infty} \sum_{|J|=n} Q_J \log Q_J.$$

A further careful calculation (using the fact that $(1,1)^T$ is an eigenvector of $X_0 + X_1$) yields

$$\sum_{v\in\mathcal{B}} p_{t,v} \log \rho^{|v|} = \frac{2+t-t^2}{t-t^2} \; \log \rho,$$

which finishes the proof of the theorem. $\square$

5. The ratio case $\rho = \lambda_k$ $(k \geqslant 3)$

In this section, we fix an integer $k \geqslant 3$ and consider the case $\rho = \lambda_k$, where λ_k is the unique positive root of the polynomial $x^k + x^{k-1} + \cdots + x - 1$. The transition map for this case is different slightly from that for the golden ratio, which leads to some different properties (see e.g. Theorems 1.3 and 1.5).

Table 3
Elements in Ω

$\alpha \in \Omega$	Labelled as
$(1; 0; 1)$	a
$(1-\rho^k; 0; 1)$	b
$(\rho^k; (0, 1-\rho^k); 1)$	c_1
$(\rho^k; (0, 1-\rho^k); 2)$	$\overline{c}_1$
$(\rho^{k-i+1}; (0, 1-\rho^{k-i+1}); 1)$ $(i = 2, 3, \ldots, k)$	c_i $(i = 2, 3, \ldots, k)$
$(1-\rho^k; \rho^k; 1)$	d
$(1-\rho^k-\rho^{k-i}; \rho^k; 1)$ $(i = 1, 2, \ldots, k-1)$	e_i $(i = 1, 2, \ldots, k-1)$
$(1-\rho^k-\rho^{k-i}; \rho^{k-i}; 1)$ $(i = 1, 2, \ldots, k-1)$	f_i $(i = 1, 2, \ldots, k-1)$
$(1-2\rho^k; \rho^k; 1)$	g

5.1. The symbolic expressions and transition matrices

By an inductive discuss, we can determine all the elements in Ω. In Table 3 we list them all, and for simplicity we relabel them by latin and arabic letters.

The map ξ, defined as in (2.7), is given by

$$\begin{aligned}
\xi(a) &= b\, c_1\, d, \\
\xi(b) &= b\, c_1\, e_1, \\
\xi(c_1) &= c_2, \\
\xi(\overline{c}_1) &= c_2, \\
\xi(c_i) &= c_{i+1} \quad (i = 2, \ldots, k-1) \\
\xi(c_k) &= c_1\, g\, \overline{c}_1 \\
\xi(d) &= f_1\, c_1\, d, \\
\xi(e_i) &= f_1\, c_1\, e_{i+1} \quad (i = 1, \ldots, k-2) \\
\xi(e_{k-1}) &= f_1, \\
\xi(f_i) &= f_{i+1}\, c_1\, e_i \quad (i = 1, \ldots, k-2) \\
\xi(f_{k-1}) &= e_1, \\
\xi(g) &= f_1\, c_1\, e_1
\end{aligned}$$

and the corresponding 0–1 matrix $A = (A_{i,j})_{i,j\in\Omega}$, is defined by

$$A_{i,j} = \begin{cases} 1 & \text{if } j \text{ is a letter of } \xi(i), \\ 0 & \text{otherwise.} \end{cases}$$

The transition matrices $T(i,j)$ constructed as in the proof of Lemma 2.8, are listed as follows:

$$\begin{cases} T(a,b)=1, & T(a,c_1)=[1,1], & T(a,d)=1, \\ T(b,b)=1, & T(b,c_1)=[1,1], & T(b,e_1)=1, \\ T(c_m,c_{m+1})=\begin{bmatrix}1&0\\0&1\end{bmatrix}, & (m=1,2,\ldots,k-1), & \\ T(c_k,c_1)=\begin{bmatrix}1&1\\0&1\end{bmatrix}, & T(c_k,g)=\begin{bmatrix}1\\1\end{bmatrix}, & T(c_k,\overline{c}_1)=\begin{bmatrix}1&0\\1&1\end{bmatrix}, \\ T(\overline{c}_1,c_2)=\begin{bmatrix}1&0\\0&1\end{bmatrix}, & & \\ T(d,f_1)=1, & T(d,c_1)=[1,1], & T(d,d)=1, \\ T(e_m,f_1)=1=T(e_m,e_{m+1}), & T(e_m,c_1)=[1,1] & (m=1,\ldots,k-2), \\ T(e_{k-1},f_1)=1, & & \\ T(f_m,f_{m+1})=1=T(f_m,e_1), & T(f_m,c_1)=[1,1] & (m=1,\ldots,k-2), \\ T(f_{k-1},e_1)=1, & & \\ T(g,f_1)=1, & T(g,c_1)=[1,1], & T(g,e_1)=1. \end{cases} \tag{5.1}$$

For any $n \geqslant 1$, denote by $\Omega_{A,n}$ the collection of all admissible words of length n, i.e.,

$$\Omega_{A,n}=\left\{i_1\ldots i_n\in\Omega^*:\ A_{i_j,i_j+1}=1,\ 1\leqslant j<n\right\},$$

where Ω^* denotes the collection of all finite word over Ω. By Lemma 2.7 and Theorem 2.9, each nth net interval Δ corresponds to a unique word $i_1\ldots i_{n+1}$ in $\Omega_{A,n+1}$ with $i_1=1$; and the multiplicity vector $W_n(\Delta)$ satisfies

$$W_n(\Delta)=T_{i_1\ldots i_{n+1}}:=T(i_1,i_2)T(i_2,i_3)\ldots T(i_n,i_{n+1}).$$

The letter g in Ω has the following properties:

(a) For all $i\in\Omega$, there exists some integer n such that g is a letter in the word $\xi^n(i)$.
(b) The characteristic vector represented by g is $(1-2\rho^k;\rho^k,1)$, which satisfies $v(g)=1$.

Due to (b), we have

$$\|T_{\omega_1 g\omega_2}\|=\|T_{\omega_1 g}\|\times\|T_{\omega_2}\|$$

for all $\omega_1,\ \omega_2\in\Omega^*$ with $\omega_1 g\omega_2\in\bigcup_{n=1}^{\infty}\Omega_{A,n}$.

Noting that $\xi(g)=f_1c_1e_1$, we define a collection $\mathcal{B}$ of admissible words by

$$\mathcal{B}=\left\{i_1i_2\ldots i_m\in\Omega_{A,m}:\ m\in\mathbb{N}, i_1\in\{f_1,c_1,e_1\}, i_m=g, i_\ell\neq g \text{ for all } 1\leqslant\ell<m\right\}. \tag{5.2}$$

To describe the structure of $\mathcal{B}$, we define:

$$\begin{aligned} B_0 &:= c_1c_2\dots c_k,\\ B_1 &:= \bar{c}_1c_2\dots c_k,\\ Y_0^{(j)} &:= e_1\dots e_j \quad (j=1,\dots,k-1),\\ Y_1^{(j)} &:= f_1\dots f_j \quad (j=1,\dots,k-1). \end{aligned}$$

From the form of ξ, we can check that

Lemma 5.1. *A word $\omega\in\mathcal{B}$ if and only if it has one of the following forms*:

(1) $\omega = B_{i_1}\dots B_{i_\ell}g$, *where* $\ell\in\mathbb{N}$, $i_1,\dots,i_\ell\in\{0,1\}$ *and* $i_1=0$.
(2) $\omega = Y_{i_1}^{(p_1)}\dots Y_{i_m}^{(p_m)}v$, *where v is of form* (1), $m\in\mathbb{N}$, $1\leqslant p_1,\dots,p_{m-1}\leqslant k-1$, $1\leqslant p_m\leqslant k-2$, $i_s=\frac{1-(-1)^s}{2}$ $(s=1,\dots,m)$ *or* $i_s=\frac{1+(-1)^s}{2}$ $(s=1,\dots,m)$.

Furthermore if ω has the form (1), *then*

$$\|T_\omega\| = \begin{cases} 2 & for\ \ \ell=1,\\ \|M_{i_2\dots i_\ell}\| & for\ \ \ell\geqslant 2; \end{cases}$$

and if ω has the form (2), *then* $\|T_\omega\|=\|T_v\|$.

In the following lemma, we list some basic facts which can be derived from the form of ξ and properties (a) and (b):

Lemma 5.2. (i) $\widehat{\Omega}:=\Omega\backslash\{a,b,d\}$ *is an essential subclass of Ω. The characteristic vector $\alpha=(1-2\rho^k;\rho^k;1)$, represented by the symbol g in $\widehat{\Omega}$, satisfies $v(\alpha)=1$.*

(ii) *For any word η in $\Omega_{A,n}$, there exists a word $v\in\bigcup_{j=1}^{k+2}\Omega_{A,j}$ with the end letter g such that $\eta v\in\Omega_{A,n+|v|}$ and $\|T_{\eta v}\|=\|T_\eta\|$ or $2\|T_\eta\|$.*

(iii) *Let η be any word in $\bigcup_{n=1}^{\infty}\Omega_{A,n}$ with first letter a. Then one can find a word $v\in\bigcup_{n=1}^{k+1}\Omega_{A,n}$ such that the word ηv has one of the following three forms*:

1. $\eta v = a\omega_1\dots\omega_\ell$, *the first letter of ω_1 is c_1*
2. $\eta v = a\underbrace{b\dots b}_{r}\omega_1\dots\omega_\ell$, *the first letter of ω_1 is c_1 or e_1*
3. $\eta v = a\underbrace{d\dots d}_{r}\omega_1\dots\omega_\ell$, *the first letter of ω_1 is c_1 or f_1*

where $\ell,r\in\mathbb{N}$ and $\omega_i\in\mathcal{B}$ for $1\leqslant i\leqslant\ell$.

(iv) *For any $\omega_1,\dots,\omega_\ell\in\mathcal{B}$,*

$$\omega_1\cdots\omega_\ell\in\bigcup_{n=1}^{\infty}\Omega_{A,n},\quad and\quad \|T_{\omega_1\cdots\omega_\ell}\|=\prod_{i=1}^{\ell}\|T_{\omega_i}\|=\prod_{i=1}^{\ell}\|T_{g\omega_i}\|.$$

5.2. The exponential sum of the product of matrices

In this subsection we will determine the value of

$$E(q) := \lim_{n\to\infty} \left(\sum \|T_{i_1\ldots i_{n+1}}\|^q \right)^{1/n}$$

for any $q \in \mathbb{R}$, where the summation is taken over all admissible words $i_1 \ldots i_{n+1}$ of length $n+1$ with $i_1 = a$. More precisely, we prove the following:

Theorem 5.3. *For any $q \in \mathbb{R}$, $E(q) = 1/x(q)$, where $x(q)$ is the unique positive root of*

$$\frac{1-2x^{k-1}+x^k}{1-2x+x^k} \cdot \sum_{n=0}^{\infty} u_n(q) x^{kn+k+1} = 1 \tag{5.3}$$

which lies in the interval $(0, \lambda_{k-1})$. Here $u_n(q)$ is defined as in (4.5). *Furthermore, $x(q)$ is an infinitely differentiable function of q on $\mathbb{R}$.*

To prove the theorem, denote by F_n the collection of all admissible words of length n starting from a letter in $\{c_1, e_1, f_1\}$, i.e.,

$$F_n = \left\{ i_1 \ldots i_n \in \Omega_{A,n} : \ i_1 \in \{c_1, e_1, f_1\} \right\}.$$

For $q \in \mathbb{R}$, define

$$R_n(q) = \sum_{v \in F_n} \|T_{gv}\|^q, \qquad n \geqslant 1.$$

Set

$$R(q) = \lim_{n\to\infty} (R_n(q))^{1/n}.$$

We first determine the value of $R(q)$. To achieve this, we define

$$w_n(q) = \sum_{v \in \mathcal{B}, |v|=n} \|T_{gv}\|^q, \quad n \in \mathbb{N}, \ q \in \mathbb{R}. \tag{5.4}$$

Lemma 5.4. *For any $n \geqslant 2$, we have*

$$R_n(q) = o_n(q) + \sum_{i=1}^{n} w_i(q) R_{n-i}(q),$$

where $o_n(q)$ is a number in $\left(0,\ 2^{|q|} \sum_{i=1}^{k+2} w_{n+i}(q)\right)$.

Proof. Partition F_n into its two subsets $F_n^{(1)}$ and $F_n^{(2)}$, where $F_n^{(1)}$ consists of all the words in F_n which do not contain the letter g and $F_n^{(2)} = F_n \backslash F_n^{(1)}$. By Lemma 5.2(ii), each word $\eta \in F_n^{(1)}$ can be extended to a word in $\mathcal{B}$ by attaching some word $v \in \bigcup_{j=1}^{k+2} \Omega_{A,j}$ with $\|T_\eta\| \leqslant \|T_{\eta v}\| \leqslant 2\|T_\eta\|$. This implies

$$\sum_{v\in F_n^{(1)}} \|T_{gv}\|^q \leqslant 2^{|q|} \sum_{i=1}^{k+2} w_{n+i}(q). \tag{5.5}$$

Observe that each word in $F_n^{(2)}$ can be uniquely written as $\omega\eta$ for some $\omega \in \mathcal{B}$ and $\eta \in F_{n-|\omega|}$. This deduces

$$\sum_{v\in F_n^{(2)}} \|T_{gv}\|^q = \sum_{i=1}^{n} w_i(q) R_{n-i}(q). \tag{5.6}$$

Combining (5.6) with (5.5) yields the desired result. $\square$

According the above lemma, using an argument similar to that in the proof of Lemma 4.6, we have

Lemma 5.5. *$R(q) = 1/x(q)$ for any $q \in \mathbb{R}$, where $x(q)$ is defined by*

$$x(q) = \left\{ x \geqslant 0 : \sum_{n=1}^{\infty} w_n(q) x^n \leqslant 1 \right\}. \tag{5.7}$$

To study the value of $x(q)$ in the above lemma, we define a sequence of integers $\{t_n\}_{n=0}^{\infty}$ by letting $t_0 = 1$ and $\frac{1}{2} t_n$ $(n > 0)$ be the number of different integral solutions of the following conditional Diophantine equation:

$$p_1 + \cdots + p_m = n \text{ with } m \in \mathbb{N},\ 1 \leqslant p_1, \ldots, p_{m-1} \leqslant k-1,\ 1 \leqslant p_m \leqslant k-2.$$

Lemma 5.6. (i) *For any positive integer n,*

$$t_n = 2 \cdot (\underbrace{1, 1, \ldots, 1}_{k-2}, 0) \cdot \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(ii) *For any $x \geqslant 0$,*

$$1 + \sum_{n=1}^{\infty} t_n x^n = \begin{cases} \dfrac{1 - 2x^{k-1} + x^k}{1 - 2x + x^k} & \text{if } 0 \leqslant x < \lambda_{k-1}, \\ +\infty & \text{if } x \geqslant \lambda_{k-1}. \end{cases} \tag{5.8}$$

Proof. Denote by $\frac{1}{2}t_n^{(i)}$ $(1\leqslant i\leqslant k-1)$ the number of integral solutions of the following conditional equation:

$$p_1+\cdots+p_m=n, \quad \text{with } m\in\mathbb{N},\ 1\leqslant p_1,\ldots,p_{m-1}\leqslant k-1,\ p_m=i.$$

Then $t_1^{(1)}=2$ and $t_1^{(i)}=0$ for $2\leqslant i\leqslant k-1$. Furthermore for any $n\geqslant 1$,

$$\begin{cases} t_{n+1}^{(1)}=\sum_{i=1}^{k-1} t_n^{(i)}, \\ t_{n+1}^{(j)}=t_n^{(j-1)} \quad (j=2,\ldots,k-1). \end{cases} \tag{5.9}$$

Hence we have

$$\left(t_n^{(1)},t_n^{(2)},\ldots,t_n^{(k-1)}\right)^T=Q^{n-1}(2,0,\ldots,0)^T,$$

where Q denotes the matrix in the formula for t_n. Observing $t_n=\sum_{i=1}^{k-2} t_n^{(i)}$, we get the desired formula for t_n.

To prove (ii), define a sequence $\{s_n\}_{n=1}^{\infty}$ by $s_1=s_2=\cdots=s_{k-2}=0$, $s_{k-1}=1$ and

$$s_n=s_{n-1}+s_{n-2}+\cdots+s_{n-k-1}, \qquad \forall n\geqslant k.$$

Denote $p(x)=\sum_{n=1}^{\infty} s_n x^n$. Then

$$\begin{aligned} p(x) &= \sum_{i=1}^{k-1} s_i x^i+\sum_{n=k}^{\infty}(s_{n-1}+s_{n-2}+\cdots+s_{n-k+1})x^n \\ &= x^{k-1}+\sum_{i=1}^{k-1} x^i\left(\sum_{n=k-i}^{\infty} s_n x^n\right) \\ &= x^{k-1}+\left(\sum_{i=1}^{k-1} x^i\right)p(x). \end{aligned}$$

This deduces

$$p(x)=\begin{cases} \frac{x^{k-1}}{1-\sum_{i=1}^{k-1} x^i} & \text{if } 0\leqslant x<\lambda_{k-1}, \\ +\infty & \text{for } x\geqslant\lambda_{k-1}. \end{cases}$$

By the definition of s_n we have $Q^{n-1}(1,0,\dots,0)^T = (s_{n+k-2}, s_{n+k-3}, \dots, s_n)^T$. Hence $t_n = 2\sum_{i=1}^{k-2} s_{n+i}$. Therefore

$$\begin{aligned} 1+\sum_{n=1}^{\infty} t_n x^n &= 1+2\sum_{i=1}^{k-2}\sum_{n=1}^{\infty} s_{n+i}x^n \\ &= 1+2\sum_{i=1}^{k-2} x^{-i}\sum_{n=i+1}^{\infty} s_n x^n \\ &= 1+2p(x)\sum_{i=1}^{k-2} x^{-i}. \end{aligned}$$

Combining the above result with the expression of $p(x)$, we obtain the formula in (ii) directly. $\square$

Now we give the exact expression of $x(q)$.

Proposition 5.7. *Let $x(q)$ be defined as in* (5.7) *and $u_n(q)$ defined as in* (4.5). *Then for any $q \in \mathbb{R}$, $x(q)$ is the unique positive root of*

$$\frac{1-2x^{k-1}+x^k}{1-2x+x^k} \cdot \sum_{n=0}^{\infty} u_n(q)x^{kn+k+1} = 1 \tag{5.10}$$

which lies in the interval $(0, \lambda_{k-1})$. Furthermore, $x(q)$ is an infinitely differentiable function of q on $\mathbb{R}$.

Proof. By Lemma 5.1 and the definition of $w_n(q)$ (see (5.4)), we have

$$w_n(q) = \begin{cases} 0 & \text{for } 1\leqslant n < k+1, \\ \sum_{i,j\geqslant 0:\ i+kj+k+1=n} t_i u_j(q) & \text{for } n \geqslant k+1. \end{cases}$$

Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} w_n(q)x^n &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} t_i u_j(q) x^{i+kj+k+1} \\ &= \left(1+\sum_{i=1}^{\infty} t_i x^i\right)\cdot\left(\sum_{n=0}^{\infty} u_n(q)x^{kn+k+1}\right), \end{aligned} \tag{5.11}$$

where $1+\sum_{i=1}^{\infty} t_i x^i$ can be simplified as in (5.8).

To prove that $x(q)$ satisfies (5.10) and is differentiable infinitely, as that in the proof of Theorem 4.4(ii), essentially we only need to show that for any $q \in \mathbb{R}$, there exists

$0 < y < 1$ such that $1 < \sum_{n=1}^{\infty} w_n(q)y^n < \infty$. Assume this fact is not true. Then there exist $q' \in \mathbb{R}$ and $0 < x' < 1$ such that $\sum_{n=1}^{\infty} w_n(q')(x')^n \leqslant 1$ and

$$\sum_{n=1}^{\infty} w_n(q')z^n = +\infty, \quad \forall z > x'. \tag{5.12}$$

By (5.11) and $\sum_{n=1}^{\infty} w_n(q')(x')^n \leqslant 1$, we have

$$x' \in (0,\ \lambda_{k-1}) \text{ and } \sum_{n=0}^{\infty} u_n(q')\ (x')^{kn+k+1} < +\infty.$$

Thus by (5.11) and (5.12), we have

$$\sum_{n=0}^{\infty} u_n(q')z^{kn+k+1} = +\infty \quad \forall z > x', \tag{5.13}$$

which implies $\sum_{n=0}^{\infty} u_n(q') = +\infty$ since $x' < 1$. Therefore by Proposition 4.8, there exists $0 < y < 1$ such that

$$\sum_{n=0}^{\infty} u_n(q')\ (x')^{kn} < \sum_{n=0}^{\infty} u_n(q')\ y^{kn} < +\infty,$$

which leads to a contradiction with (5.13). This finishes the proof of the proposition. □

Proof of Theorem 5.3. By Lemma 5.5 and Proposition 5.7, to prove Theorem 5.3 we only need to show $E(q) = R(q)$ for $q \in R$. Define

$$E_n(q) = \sum \|T_{i_1 \ldots i_{n+1}}\|^q,$$

where the summation is taken over for all words $i_1 \ldots i_{n+1}$ in $\Omega_{A,n+1}$ with $i_1 = a$. Take $\omega_0 = B_0 g$. Then for any $v \in F_{n-k-2}$, $a\omega_0 v \in \Omega_{A,n+1}$. Therefore for any $n \geqslant k+3$,

$$\begin{aligned} E_n(q) &\geqslant \sum_{v \in F_{n-k-2}} \|T_{a\omega_0 v}\|^q \\ &= \sum_{v \in F_{n-k-2}} \|T_{gv}\|^q = R_{n-k-2}(q). \end{aligned}$$

On the other hand, by Lemma 5.2(iii) each word in $\Omega_{A,n+1}$ with the first letter a has one of the following forms

$$av\ (v \in F_n);\quad ab^m v\ (v \in F_{n-m});\quad ad^m v\ (v \in F_{n-m}).$$

Hence,

$$\begin{aligned} E_n(q) &\leqslant \sum_{v\in F_n} \|T_{av}\|^q + \sum_{m=1}^{n} \sum_{v\in F_{n-m}} \|T_{ab^m v}\|^q + \sum_{m=1}^{n} \sum_{v\in F_{n-m}} \|T_{ad^m v}\|^q \\ &= R_n(q) + 2\sum_{m=1}^{n} R_{n-m}(q). \end{aligned}$$

Using the above two inequalities and the fact $R(q) \geqslant 1$, we have $E(q) = R(q)$. This proves the theorem. □

5.3. The Hausdorff dimension of the graph f, the L^q-spectrum and Hausdorff dimension of μ

The main result of this subsection is the following:

Theorem 5.8. *Let $x(q)$ be given as in Theorem* 5.3. *Then*

(i) *The Hausdorff dimension of the graph of f satisfies*

$$\dim_{\mathrm{H}} \Gamma(f) = \frac{\log x(-\log\rho/\log 2)}{\log\rho}.$$

(ii) *For any $q \in \mathbb{R}$, the L^q-spectrum $\tau(q)$ of μ satisfies*

$$\tau(q) = \frac{-q\log 2}{\log\rho} - \frac{\log x(q)}{\log\rho}.$$

Furthermore $\tau(q)$ is infinitely differentiable on $\mathbb{R}$.

(iii) *The Hausdorff dimension of μ is*

$$\dim_{\mathrm{H}} \mu = -\frac{\log 2}{\log\rho} + \left(\frac{2^k-3}{2^k-1}\right)^2 \cdot \frac{\sum_{n=0}^{\infty} 2^{-kn-k-1} \sum_{J\in\mathcal{A}_n} \|M_J\| \log\|M_J\|}{\log\rho}.$$

Proof. Combining Theorem 3.4 with Theorem 5.3 yields (i) directly. Similarly, combining Theorem 3.3 with Theorem 5.3 yields (ii). To show (iii), by Theorem 4.14,

$\dim_{\mathrm{H}} \mu = \tau'(1)$. By a direct computation of $\tau'(1)$, we obtain the formula in (iii) (for shortness we omit the details). $\square$

5.4. Local dimensions of μ

In this subsection we present the following two theorems about the local dimensions of μ.

Theorem 5.9. *The range of the local dimension of μ satisfies*

$$\mathcal{R}(\mu) = \left[\frac{k \log 2}{-(k+1)\log \rho}, \ \frac{\log 2}{-\log \rho} \right].$$

Theorem 5.10. *For $\mathcal{L}$ almost all $x \in [0,1]$,*

$$d(\mu, x) = -\frac{\log 2}{\log \rho} + \frac{\rho^k (1-2\rho^k)^2}{(2-(k+1)\rho^k)} \sum_{n=0}^{\infty} \rho^{kn} \sum_{J \in \mathcal{A}_n} \log \|M_J\|.$$

To prove these theorems, we first give the following lemma.

Lemma 5.11. *Let $\mathcal{B}$ be defined as in* (5.6). *Then*

$$\inf_{v \in \mathcal{B}} \frac{\log \|T_{gv}\|}{|v|} = 0, \qquad \sup_{v \in \mathcal{B}} \frac{\log \|T_{gv}\|}{|v|} = \frac{\log 2}{k+1}.$$

Proof. By Lemma 5.1(1), $(B_0)^n g \in \mathcal{B}$ for $n \in \mathbb{N}$. Note that $\|T_{g(B_0)^n g}\| = \|M_0^n\| = 2+n$ and $|(B_0)^n g| = nk+1$. We have $\lim_{n\to\infty} \frac{\log \|T_{g(B_0)^n g}\|}{|(B_0)^n g|} = 0$, and thus $\inf_{v\in\mathcal{B}} \frac{\log \|T_{gv}\|}{|v|} = 0$.

On the other hand by Lemma 5.1, we have

$$\sup_{v \in \mathcal{B}} \frac{\log \|T_{gv}\|}{|v|} = \max \left\{ \frac{\log 2}{k+1}, \sup_{n \geqslant 1,\ i_1,\dots,i_n \in \{0,1\}} \frac{\log \|M_{i_1\dots i_n}\|}{kn+k+1} \right\}.$$

Note that in the proof of Theorem 4.15 we have obtained

$$\max_{i_1,\dots,i_n \in \{0,1\}} \|M_{i_1\dots i_n}\| = \frac{2+\lambda_2+(\lambda_2)^{2n+4}}{1+(\lambda_2)^2} \ (\lambda_2)^{-n}.$$

Observing that

$$\frac{1}{k+1} \log \left[\frac{2+\lambda_2+(\lambda_2)^{2n+4}}{1+(\lambda_2)^2} \right] \leqslant \frac{1}{k+1} \log \left[\frac{2+\lambda_2+(\lambda_2)^4}{1+(\lambda_2)^2} \right] = \frac{\log 2}{k+1}$$

and

$$\frac{\log(\lambda_2)^{-n}}{nk} = \frac{-\log\lambda_2}{k} < \frac{\log 2}{k+1},$$

we have

$$\sup_{n\geqslant 1,\ i_1,\ldots,i_n\in\{0,1\}} \frac{\log\|M_{i_1\ldots i_n}\|}{kn+k+1} \leqslant \frac{\log 2}{k+1},$$

where we have used the inequality $\frac{a+c}{b+d}\leqslant \max\{\frac{a}{b},\frac{c}{d}\}$ for positive numbers a, b, c, d. Therefore $\sup_{v\in\mathcal{B}} \frac{\log\|T_{gv}\|}{|v|} = \frac{\log 2}{k+1}$. This finishes the proof of the lemma. □

Proof of Theorem 5.9. Using an argument similar to that in the proof of Theorem 4.15, we can prove that $\mathcal{R}(\mu) = \left[-\frac{\log 2}{\log\rho} + \frac{1}{\log\rho}\ \sup_{v\in\mathcal{B}} \frac{\log\|T_{gv}\|}{|v|},\ -\frac{\log 2}{\log\rho} + \frac{1}{\log\rho}\ \inf_{v\in\mathcal{B}} \frac{\log\|T_{gv}\|}{|v|}\right]$. Combining it with Lemma 5.11 yields the theorem. □

Proof of Theorem 5.10. Let $\widehat{\Omega} = \Omega\backslash\{a, b, d\}$. Consider the one-sided shift space $\left(\widehat{\Omega}_A^{\mathbb{N}}, \sigma\right)$, where

$$\widehat{\Omega}_A^{\mathbb{N}} = \left\{(i_n)_{n=1}^{\infty}:\ i_n\in\widehat{\Omega},\ A_{i_n,i_{n+1}} = 1 \text{ for } n\geqslant 1\right\},$$

and $\sigma\left((i_n)_{n=1}^{\infty}\right) = (i_{n+1})_{i=1}^{\infty}$.

Define a probability matrix $P = \left(P_{i,j}\right)_{i,j\in\widehat{\Omega}}$ by

$$P_{i,j} = \begin{cases} \rho\,\frac{\ell_j}{\ell_i} & \text{if } A_{i,j} = 1,\\ 0 & \text{otherwise},\end{cases}$$

where ℓ_i denotes the relative length of the characteristic vector labelled by i. Let $\mathbf{p} = (p_i)_{i\in\widehat{\Omega}}$ be the probability vector such that $\mathbf{p} = \mathbf{p}P$. By a careful calculation, we have

$$p_g = \frac{\rho^k(1-2\rho^k)}{2-(k+1)\rho^k},\ p_{c_2} = \cdots = p_{c_{k-1}} = \frac{\rho^k}{2-(k+1)\rho^k},$$

$$p_{c_1} = \frac{(1-\rho^k)\rho^k}{2-(k+1)\rho^k},\qquad p_{e_i} = p_{f_i} = \frac{\rho^i(1-\rho^k-\rho^{k-i})}{2-(k+1)\rho^k}\quad (i = 1,\ldots,k-1),$$

$$p_{\overline{c}_1} = \frac{\rho^{2k}}{2-(k+1)\rho^k}.$$

Define η to be the $(\mathbf{p}, P)$ Markov measure on $\widehat{\Omega}_A^{\mathbb{N}}$. Using the Birkhoff ergodic theorem and a proof essentially identical to that of Theorem 4.17, we have

$$\lim_{n\to\infty} \frac{\log \|T_{x_1\ldots x_n}\|}{n} = \sum_{v\in\mathcal{B}} \eta([gv]) \log \|T_{gv}\|$$

for η almost all $x = (x_j)_{j=1}^\infty \in \widehat{\Omega}_A^{\mathbb{N}}$, where $[i_1 \ldots i_n]$ denote the n-cylinder set

$$\left\{(x_j)_{j=1}^\infty \in \widehat{\Omega}_A^{\mathbb{N}} : \ x_j = i_j \text{ for } 1 \leqslant j \leqslant n\right\}.$$

Furthermore for $\mathcal{L}$ almost all $x \in [0, 1]$,

$$d(\mu, x) = -\frac{\log 2}{\log \rho} + \frac{1}{\log \rho} \sum_{v\in\mathcal{B}} \eta([gv]) \log \|T_{gv}\|.$$

Now we are going to compute $\sum_{v\in\mathcal{B}} \eta([gv]) \log \|T_{gv}\|$. By the definition of η, we have

$$\eta([i_1 \ldots i_n]) = p_{i_1} \prod_{j=1}^{n-1} \frac{\rho \ell_{i_{j+1}}}{\ell_{i_j}} = \rho^{n-1} \frac{p_{i_1} \ell_{i_n}}{\ell_{i_1}}.$$

Thus for each $v \in \mathcal{B}$, we have $\eta([gv]) = \rho^{|v|} p_g$. By Lemma 5.1 and (5.8), we have

$$\begin{aligned}
\sum_{v\in\mathcal{B}} \eta([gv]) \log \|T_{gv}\| &= \sum_{v\in\mathcal{B}} p_g \rho^{|v|} \log \|T_{gv}\| \\
&= p_g \sum_{i=0}^\infty \sum_{j=0}^\infty \rho^{i+kj+k+1} t_i \sum_{J\in\mathcal{A}_j} \log \|M_J\| \\
&= p_g \rho^{k+1} \frac{1-2\rho^{k-1}+\rho^k}{1-2\rho+\rho^k} \sum_{j=0}^\infty \rho^{kj} \sum_{J\in\mathcal{A}_j} \log \|M_J\| \\
&= \frac{\rho^k (1-2\rho^k)^2}{(2-(k+1)\rho^k)} \sum_{j=0}^\infty \rho^{kj} \sum_{J\in\mathcal{A}_j} \log \|M_J\|.
\end{aligned}$$

This finishes the proof of the theorem. $\square$

5.5. The box dimension and Hausdorff dimension of μ

In this subsection we prove

Theorem 5.12. *For $\mathcal{L}$ almost all $t \in [0,1]$, the Hausdorff dimension and box-counting dimension of t-level set of the limit Rademacher function f with parameter $\rho = \lambda_k$ are equal to*

$$\frac{\rho^k\left(1-2\rho^k\right)^2}{\left(2-(k+1)\rho^k\right)\log 2}\sum_{n=0}^{\infty}\left(\rho^{kn}\sum_{J\in\mathcal{A}_n}\log\|M_J\|\right)$$

Proof. The result for the box-counting dimension follows directly from Corollary 3.6 and Theorem 5.10. The proof for the coincidence of the Hausdorff dimension and the box-counting dimension for almost all t is essentially identical to that of Theorem 4.19 by using the dimensional result about homogeneous Moran sets. □

5.6. The infinite similarity and multifractal formalism for μ

Denote

$$\mathcal{D} = \left\{ i_1 i_2 \ldots i_n \in \Omega_{A,n}:\ n \in \mathbb{N},\ i_1 = a,\ i_n = g,\ i_j \neq g \text{ for } 1 \leqslant j < n \right\}.$$

For any $\omega \in \mathcal{D}$ and $v \in \mathcal{B}$, denote by $\Phi_{\omega,v}$ the unique similitude $cx+d$ with $c>0$ so that $\Phi_{\omega,v}(\Delta(\omega)) = \Delta(\omega v)$, where $\Delta(\omega)$ denotes the net interval with respect to the admissible word ω. Then the following theorem can be proved in the same line as that of Theorem 4.21:

Theorem 5.13. *μ is supported on $\bigcup_{\omega\in\mathcal{D}}\Delta(\omega)$. For any fixed $\omega \in \mathcal{D}$,*

$$\mu_\omega = \sum_{v\in\mathcal{B}} 2^{-|v|}\|T_{gv}\|\cdot\mu_\omega\circ\Phi_{\omega,v}^{-1}, \tag{5.14}$$

where μ_ω denotes the restriction of μ on $\Delta(\omega)$, i.e., $\mu_\omega(A) = \mu(A\cap\Delta(\omega))$ for any Borel set $A \subset \mathbb{R}$. Furthermore,

$$\Phi_{\omega,v}(\Delta(\omega)) = \Delta(\omega v) \subset \Delta(\omega), \qquad \Delta(\omega v)\cap\Delta(\omega v') = \emptyset \ \ for\ \ v \neq v'. \tag{5.15}$$

Using the above theorem and a proof identical to that in Theorem 4.24, we have

Theorem 5.14. *Then for any $q \in \mathbb{R}$, we have*

$$\dim_{\mathrm{H}} K_{\alpha(q)} = \inf_{t\in\mathbb{R}}\{\alpha(q)t - \tau(t)\} = \alpha(q)q - \tau(q), \tag{5.16}$$

where $K(\alpha) = \{x \in \mathbb{R}:\ d(\mu,x) = \alpha\}$, $\tau(t)$ is the L^t-spectrum of μ, and $\alpha(q) = \tau'(q)$.

Remark 5.15. After our work, Olivier et al. [46] also verifies the validity of the multifractal formalism for μ by viewing μ as a weak Gibbs measure.

Acknowledgments

The main results of this paper were announced earlier in [13]. The author thanks Prof. Zhi-Ying Wen, Prof. Yang Wang and Dr. Jun Wu for some helpful discussions. He thanks Prof. Ka-Sing Lau for his careful reading of an earlier version of the manuscript and many suggestions that led to the improvement of the manuscript.

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