The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers (II)

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Abstract

For integer $k \geq 2$, let $\rho = \lambda_k$ be the positive root of the polynomial $x^k + x^{k-1} + \cdots + x - 1$. We show that the infinitely convolved Bernoulli measure μ_ρ (the distribution of random series $\sum_{n=0}^{+\infty}(1-\rho)\rho^n\epsilon_n$, where the coefficients ϵ_n take independently the values 0 and 1 with probability $\frac{1}{2}$) is a locally infinitely-generated self-similar measure without overlap. This result turns out to be essential in the study of local properties of μ_ρ. It provides a direct way to obtain the explicit formula for the Hausdorff dimension of μ_ρ and to analyze the multifractal structure of μ_ρ. The multifratal spectrum of $\mu_{\lambda_k}(k \geq 3)$ are obtained completely.

Contents

1 Introduction

For $0 < \rho < 1$, let μ_ρ be the distribution of the random series $\sum_{n=0}^{+\infty}(1-\rho)\rho^n\epsilon_n$ where the coefficients ϵ_n take independently the values 0 and 1 with probability $\frac{1}{2}$. The measure μ_ρ is termed as "infinitely convolved Bernoulli measure" or simply "Bernoulli convolution" since it is the infinite convolution product of $\frac{1}{2}(\delta_0 + \delta_{(1-\rho)\rho^n})$. These measures have been studied for more than 60 years, revealing many connections with harmonic analysis, algebraic number theory, dynamical systems, and Hausdorff dimension estimation(for a good survey, see e.g. Peres, Schlag and Solomyak [PSS] or Alexander and Yorke [AY]). One can easily see that for $0 < \rho < \frac{1}{2}$ the measure μ_ρ is supported on a Cantor set of zero Lebesgue measure and thus μ_ρ is totally singular with respect to the Lebesgue measure; for $\rho = \frac{1}{2}$, μ_ρ is just the Lebesgue measure restriction on $[0,1]$. However for $\frac{1}{2} < \rho < 1$, μ_ρ is only partially understood still now. Jesson and Wintner [JW] proved that μ_ρ is either absolute continuous or totally singular with respect to the Lebesgue measure. Wintner [Win] showed that μ_ρ is absolutely continuous for $\rho = 2^{-1/n}$($n = 2, 3, \cdots$), and Garsia [G1] found some other algebraic integers for which μ_ρ is absolutely continuous. Moreover, Erdős [Er1] proved that μ_ρ is absolutely continuous for almost all ρ closed enough to one. He conjectured that the result should be true for almost all $\frac{1}{2} < \rho < 1$. Solomyak [Sol] has recently proved this conjecture to be true (see [PS] for a shorter proof). On the other hand, Erdős showed that μ_ρ is totally singular if ρ is the reciprocal of a Pisot number (an algebraic integer is a Pisot number provided that all of its conjugates are less than one in modulus. For example the positive root of polynomial $x^n - x^{n-1} \cdots - x - 1$ is Pisot number for each integer $n \geq 2$. The reader may refer to [Sa] and [BDGPS] for further information about Pisot numbers.)

Recently, a lot of interests have been focused on considering the Hausdorff dimension and multifractal structure of μ_ρ when ρ is the reciprocal of a Pisot number. Before citing the relative works, we give here some basic notations and backgrounds. Let ν be a Borel measure on $\mathbb{R}$, the Hausdorff dimension of ν is defined by

$$\dim_H \nu = \inf\{\dim_H A : \ A \text{ Borel}, \ \nu(\mathbb{R}\backslash\mathbb{A}) = \nvdash\}.$$

For $x \in \mathbb{R}$, the local dimension of ν at x is given by

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$$d(\nu,x)=\lim_{r\downarrow 0}\frac{\log \nu([x-r,x+r])}{\log r}$$

if the limit exists. For $\alpha\geq 0$, let $K_\alpha=\{x\in\mathbb{R}:\ d(\nu,x)=\alpha\}$. For certain measure the set K_α may be non-empty and fractal over a range of α, and when this happens ν is often termed a *multifractal measure*. The main purpose of multifractal analysis is to study the multifractal spectrum or singular spectrum of ν defined by $f(\alpha)=\dim_H K_\alpha$. The multifractal measures and multifractal spectra were first proposed by physicist to study the scaling behavior of physical measures on strange attractors, diffusion-limited aggregates, *etc* (see e.g. Mandelbrot [Man], Frish and Paris [FP], Halsey *et al* [Ha]). Now multifractal analysis has become a strong tool to describe residence measures on the attractors of dynamical systems, turbulence in fluids, rainfall distribution, mass distribution in the universe, and many other phenomena. In order to determine the function $f(\alpha)$, one can consider the L^q-spectrum of ν for each $q\in\mathbb{R}$, which is defined by

$$\tau(q)=\liminf_{\delta\downarrow 0}\frac{\log\sup\sum_i \nu([x_i-\delta,x_i+\delta])}{\log\delta},$$

where the supremum takes over all the families of disjoint intervals $[x_i-\delta,x_i+\delta]_i$ with $x_i\in$supp(ν). Then it is asserted ([FP], [Ha], [HP]) and proved in certain cases (see e.g. [CLP], [CM], [EM], [Lo], [O], [R], [RM]) the multifractal spectra $f(\alpha)$ are equal to the Legendre transformation of the L^q-spectrum $\tau(q)$, that is

$$f(\alpha)=\inf\{q\alpha-\tau(q):\ q\in\mathbb{R}\}. \tag{1.1}$$

The relationship (1.1) is called the *multifractal formalism*. For the definitions of various dimensions (Hausdorff dimension, upper box-counting dimension $\overline{\dim}_B$, packing dimension $\dim_{\mathcal{P}}$) and further properties of L^q-spectrum, multifractal formalism, see e.g. the books [Fal], [Mat], [Pe].

Przytycki and Urbanski [PU] proved that $\dim_H\mu_\rho<1$ if ρ is the reciprocal of a Pisot number. For the golden ratio $\lambda=\frac{\sqrt{5}-1}{2}$, several people obtained the explicit formula and numeral estimates of $\dim_H\mu_\lambda$. Alexander and Zagier [AZ] found a formula for $\dim_H\mu_\lambda$ by analyzing the "Fibonacci graph", and used it to show that $0.99557<\dim_H\mu_\lambda<0.99574$. Using another different proof, Sidrov and Vershik [SV] re-obtained the Alexander-Zagier formula. Ledrappier and Porzio[LP1], and independently, Lalley [La] gave another theoretical formula for $\dim_H\mu_\lambda$ by expressing $\dim_H\mu_\lambda$ as the top Lyapunov exponent of certain random matrix products. Ngai [Ng] also found a different explicit formula for $\dim_H\mu_\lambda$ by showing that for any boundedly supported Borel measure ν, $\dim_H\nu$ is equal to the derivative of the L^q-spectrum of ν at $q=1$ if it exists. We should mention that Lau and Ngai [LN2] obtained the explicit formula for the L^q-spectrum of μ_λ for $q>0$ and proved its differentiability on this range. By a direct study of the Ruelle-Perron-Frobenius operator associated to the random unbounded matrix product, Porzio [Po] proved the L^q-spectrum of μ_λ is differential on $(-\frac{1}{2}, +\infty)$. Recently, by introducing an algebraic method to analyze the local properties of μ_λ, Feng [Fe] obtained the explicit formula for the L^q-spectrum of μ_λ for all $q \in \mathbb{R}$, and proved its smoothness on the whole line except for one point $q_0 < -2$; He also obtained the explicit formula for the L^q-spectrum of μ_{λ_k} for $q \in \mathbb{R}$ (λ_k is the positive root of the polynomial $x^k + x^{k-1} + \cdots + x - 1$, $k = 3, 4, \cdots$) with showing its smoothness on the whole line; and obtained the explicit formula of $\dim_H \mu_{\lambda_k}$ by applying Ngai's work. In the following we formulate these results. Set

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_\emptyset = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.2}$$

For any $J = j_1 \cdots j_n \in \{0, 1\}^n$, denote $M_J = M_{j_1} \circ \cdots \circ M_{j_n}$. For any 2×2 non-negative matrix B, denote by $||B|| = (1, 1)B(1, 1)'$.

Theorem 1.1 (Theorem C of [Fe]). **(i)** *Denote $\lambda_2 = \lambda$. For any $q \in \mathbb{R}$, the L^q-spectrum $\tau_{\lambda_2}(q)$ of μ_{λ_2} is equal to*

$$\frac{q \log 2}{\log \lambda_2^{-1}} + \frac{\log \mathbf{x}(2, q)}{\log \lambda_2^{-1}},$$

where

$$\mathbf{x}(2, q) = \sup\{x \geq 0 : \sum_{n=0}^{\infty}(\sum_{|J|=n} ||M_J||^q) x^{2n+3} \leq 1\}.$$

There exists a unique $q_0 < -2$ such that $\sum_{n=0}^{\infty}(\sum_{|J|=n} ||M_J||^{q_0}) = 1$. When $q > q_0$, $\mathbf{x}(2, q)$ is the positive root of $\sum_{n=0}^{\infty}(\sum_{|J|=n} ||M_J||^q) x^{2n+3} = 1$, and it is an infinitely differentiable function of q on $(q_0, +\infty)$. When $q \leq q_0$, $\mathbf{x}(2, q) = 1$. Moreover $\mathbf{x}(2, q)$ is not differentiable at $q = q_0$,

$$x'(2, q_0-) = 0, \; x'(2, q_0+) = -\frac{\sum_{n\geq 0}(\sum_{|J|=n} ||M_J||^{q_0} \log ||M_J||)}{\sum_{n\geq 0} u_{n,q_0} \cdot (2n+3)} \in (-\infty, 0).$$

(ii) *For any integer $k \geq 3$ and any real number q, the L^q-spectrum $\tau_{\lambda_k}(q)$ of the Bernoulli convolution μ_{λ_k} is equal to*

$$\frac{q \log 2}{\log \lambda_k^{-1}} + \frac{\log \mathbf{x}(k, q)}{\log \lambda_k^{-1}},$$

where $0 < \mathbf{x}(k, q) < \lambda_{k-1}$, and $\mathbf{x}(k, q)$ satisfies that

$$\frac{1 - 2x^{k-1} + x^k}{1 - 2x + x^k} \cdot \sum_{n=0}^{\infty}(\sum_{|J|=n} ||M_J||^q) x^{kn+k+1} = 1.$$

Moreover $\mathbf{x}(k, q)$ is an infinitely differentiable function on the whole line.

(iii) *For any integer $k \geq 2$, the Hausdorff dimension of the Bernoulli convolution μ_{λ_k} satisfies that*

$$\dim_H \mu_{\lambda_k} = -\frac{\log 2}{\log \lambda_k} + \left(\frac{2^k - 3}{2^k - 1}\right)^2 \cdot \frac{\sum_{n=0}^{\infty} 2^{-kn-k-1} \sum_{|J|=n} ||M_J|| \log ||M_J||}{\log \lambda_k}.$$

Recently Lau and Ngai [LN1] concerned whether the multifractal formalism holds for μ_ρ if ρ is the reciprocal of Pisot number. By some rigorous ϵ-δ arguments, they proved that the multifractal formalism of μ_ρ does hold over the following range of α:

$$\{\alpha \geq 0 : \ \tau'(q) \text{ exists and is equal to } \alpha \text{ for some } q > 0\} \ .$$

Using above result they showed the multifractal formalism of μ_λ holds for $\alpha = \tau'(q)$, $q > 0$; Based on the previous work [LP2] joint with Ledrappier, Porzio [Po] extended this range to $\alpha = \tau'(q)$, $-\frac{1}{2} < q < +\infty$.

In the present paper, we will show that the multifractal formalism of μ_λ holds for all $\alpha = \tau'(q)$, $q \in \mathbb{R}\backslash\{\shortmid\nVdash\}$, where q_0 is given as in Theorem 1.1; Moreover, the multifractal formalism of μ_{λ_k} $(k = 3, 4, \cdots)$ holds for all $\alpha = \tau'(q)$, $q \in \mathbb{R}$.

Our multifractal analysis is based on a remarkable fact: the measure μ_λ is a locally infinitely-generated self-similar measure without overlap (so do the measure μ_{λ_k}, $k = 3, 4, \cdots$). To make it more precisely, recall that μ_ρ is a self-similar measure for the iterated function system $\{\rho x, \rho x + 1 - \rho\}$ with the probability weight $(\frac{1}{2}, \frac{1}{2})$, this is, μ_ρ satisfies that

$$\mu_\rho = \frac{1}{2}\mu_\rho \circ \phi_{0,\rho}^{-1} + \frac{1}{2}\mu_\rho \circ \phi_{1,\rho}^{-1} \tag{1.3}$$

where $\phi_{0,\rho}(x) = \rho x$, $\phi_{1,\rho} = \rho x + 1 - \rho$(for a proof, see e.g. Theorem 4.3 of Lau [L1]). The measure μ_ρ $(\rho > \frac{1}{2})$ can not be easily understood since $\{\ \phi_{0,\rho}, \ \phi_{1,\rho}\}$ does not satisfy the open set condition, that is, there exists no non-empty open set U such that $\phi_{0,\rho}(U) \subset U$, $\phi_{1,\rho}(U) \subset U$ and $\phi_{0,\rho}(U) \cap \phi_{1,\rho}(U) = \emptyset$. However, in this paper, we can construct for each integer $k \geq 2$ a sequence of disjoint intervals $\{I_i\}_i$ such that μ_{λ_k} is supported on $\cup_i I_i$ (i.e., $\mu_{\lambda_k}(\ \cup_i I_i) = 1$), and the restriction of μ_{λ_k} on each interval I_i is an infinitely-generated self-similar measure without overlap (for details see Theorem 1.2). This fact is very important for us to understand the local properties of μ_{λ_k} $(k \geq 2)$.

Now we formulate our main results of this paper as follows.

Theorem 1.2 **(i)** *Let $\lambda = \frac{\sqrt{5}-1}{2}$. The Bernoulli convolution μ_λ is a locally infinitely self-similar measure in the following sense: μ_λ is supported on the union of a sequence of disjoint intervals $\{I_j\}_j$; and for each j there exists a countable family of similitudes $\{g_{j,i}\}_i$ with contraction ratio $\{r_i\}_i$, and a probability weight $\{p_i\}_i$, such that $g_{j,i}(I_j) \subset I_j$ for each i, $g_{j,i}(I_j) \cap g_{j,i'}(I_j) = \emptyset$ if $i \neq i'$, and*

$$\mu_\lambda^{(j)} = \sum_i p_i \mu_\lambda^{(j)} \circ g_{j,i}^{-1}$$

where $\mu_\lambda^{(j)}$ denote the restriction of μ_λ on the interval I_j, i.e., $\mu_\lambda^{(j)}(\cdot) = \mu_\lambda(I_j \cap \cdot)$.

(ii) *For each integer $k \geq 3$, let λ_k be the positive root of the polynomial $x^k + x^{k-1} + \cdots + x - 1$, then μ_{λ_k} is also a locally infinitely-generated self-similar measure in the above sense..*

Theorem 1.3 **(i)** *Let q_0 defined as in Theorem 1.1(i). Then for each $q \in \mathbb{R}\backslash\{q_0\}$, the multifractal formalism (1.1) of μ_λ holds for $\alpha = \alpha(q) := \tau'(q)$, where $\tau(q)$ is the L^q-spectrum of μ_λ; and $f(\alpha(q)) = -\tau(q) + \alpha(q) \cdot q$.*

(ii) *For each integer $k \geq 3$, the multifractal formalism (1.1) of μ_{λ_k} holds for $\alpha = \alpha(q) := \tau'(q)$, where $\tau(q)$ is the L^q-spectrum of μ_{λ_k}; and $f(\alpha(q)) = -\tau(q) + \alpha(q) \cdot q$.*

Remark 1.4 **(i)** With some additive work Theorem 1.2 can be generalized to the biased Bernoulli convolutions. For $0 < p < 1$ and $1/2 < \rho < 1$, let $\mu_\rho^{(p)}$ denote the p-biased Bernoulli convolution, i.e., the infinite convolution product of $p\delta_0 + (1-p)\delta_{(1-\rho)\rho^n}$. Then for any $k \geq 2$, $\mu_{\lambda_k}^{(p)}$ is a locally self-similar measure without overlap.

(ii) Under some assumptions about the decay speed of contraction ratios and probability weights, Riedi and Mandelbrot [RM] verified the multifractal formalism of infinitely generated self-similar measures without overlap, see Theorem 10 of [RM]. However, our cases don't satisfy those assumptions.

(iii) It is still open question whether or not the multifractal formalism (1.1) of μ_λ holds for $\alpha \in (\tau'(q_0+), \tau'(q_0-))$.

The proof of Theorem 1.2 is based on the precise estimates of the Bernoulli convolution $\mu_{\lambda_k}(k \geq 2)$ on so called "net intervals", which we will discuss in Section 2.

We will give the detailed proofs of Theorem 1.2 and 1.3 for the case $\rho = \lambda$ (see Theorem 3.1 and 4.4). Since the similar proofs will works well for $\rho = \lambda_k$ $(k \geq 3)$, to avoid our paper being too long we only give the generating relations of I-colors and II-colors associated with λ_k $(k \geq 3)$ in the Appendix. To make this paper self-contained, we give the proof of Theorem 1.1(i) in the Appendix.

Our paper is organized as follows. In Section 2, we present the definitions and properties of net intervals, I-colors and II-colors, and give the generating relations of I-colors and II-colors for $\rho = \lambda$. In Section 3, we prove Theorem 1.2 for $\rho = \lambda$. In section 4, we prove Theorem 1.3 for $\rho = \lambda$, meanwhile, we give a direct way to obtain the formula for $\dim_H \mu_\lambda$ (it will be shown to be equal to $\sum_i p_i \log p_i / \sum_i p_i \log r_i$, where $\{p_i\}$ and $\{r_i\}$ are given as in Theorem 1.2(i).). In Appendix 1, we proved Theorem 1.1(i). In Appendix 2, we present the generating relation of I-color and II-colors for $\rho = \lambda_k$ $(k \geq 3)$.

2 Net intervals, I-colors and II-colors

2.1 The definitions

Let $\frac{1}{2} < \rho < 1$. The similitudes $\phi_{0,\rho}, \phi_{1,\rho} : \mathbb{R} \to \mathbb{R}$ are defined by $\phi_{0,\rho}(x) = \rho x$ and $\phi_{1,\rho}(x) = \rho x + 1 - \rho$. For $\omega = (i_j)_{j=1}^m \in \{0,1\}^m$, write $\phi_{\omega,\rho} = \phi_{i_1,\rho} \circ \cdots \circ \phi_{i_m,\rho}$; the interval $\phi_{\omega,\rho}([0,1])$ is termed a m-th basic interval. Denote by $P_{m,\rho}$ the set of all endpoints of m-th basic intervals, i.e., $P_{m,\rho} = \cup_{\omega \in \{0,1\}^m} \phi_{\omega,\rho}(\{0,1\})$. It is clear that $P_{m,\rho}$ consists of all the points of the form

$$\rho^m r_m + (1-\rho)\sum_{n=0}^{m-1} \rho^n r_n \qquad (r_n = 0, \text{or } 1 \text{ for } 0 \le n \le m)$$

and $P_{m,\rho} \subset P_{m+1,\rho}$.

The points in $P_{m,\rho}$ partition $[0,1]$ into some non-overlap closed intervals, each of which is called *a m-th net interval associated with* ρ. For example, $P_{1,\rho} = \{0, 1-\rho, \rho, 1\}$ and thus the 1-th net intervals associated with any ρ are $[0, 1-\rho]$, $[1-\rho, \rho]$ and $[\rho, 1]$ respectively; Similarly the 2-th net intervals associated with $\rho = (\sqrt{5}-1)/2$ are $[0, \rho^3]$, $[\rho^3, \rho^2]$, $[\rho^2, \rho]$, $[\rho, 2\rho^2]$ and $[2\rho^2, 1]$; The 3-th net intervals associated with $\rho = (\sqrt{5}-1)/2$ are

$$[0,\rho^4], [\rho^4,\rho^3], [\rho^3,\rho^2], [\rho^2, 2\rho^3], [2\rho^3, \rho^2+\rho^4], [\rho^2+\rho^4, \rho], [\rho, 2\rho^2],$$

and $[2\rho^2, \rho^2+2\rho^3]$, $[\rho^2+2\rho^3, 1]$.

Since $P_{m,\rho} \subset P_{m+1,\rho}$, it follows that each m-th net interval is the union of some $(m+1)$-th net intervals, and each $(m+1)$-th net interval is contained in a unique m-th net interval. Denote by $\mathcal{I}_{m,\rho}$ the collection of all m-th net intervals. Now, we define a mapping $\Gamma_{m,\rho} : \mathcal{I}_{m,\rho} \to 2^{\mathbb{R}} \times \mathbb{R}$ by

$$[a,b] \mapsto \left(\left\{ \frac{\phi_{\omega,\rho}(0) - a}{\rho^m} : \ \omega \in \{0,1\}^m \text{ such that } -\rho^m < \phi_{\omega,\rho}(0) - a \le 0 \right\}, \frac{b-a}{\rho^m} \right).$$

We call $\Gamma_{m,\rho}$ to be the m-th *I-color mapping*, and call $\Gamma_{m,\rho}([a,b])$ to be the m-th *I-color* of $[a,b]$. We can see from the definition that $\Gamma_{m,\rho}([a,b])$ contains the following information about the net interval $[a,b]$: (i) the various relative distances (with a ratio ρ^{-m}) between the point a and the points of the form $(1-\rho)\sum_{n=0}^{m-1}\rho^n r_n$ ($r_n = 0$,or 1) which lie on the left side of a and have distance less than ρ^m from a; (ii) the relative length of $[a,b]$(with a ratio ρ^{-m}).

For $\omega \in \{0,1\}^m$, write $<\omega>_\rho := \{v \in \{0,1\}^m : \ \phi_{v,\rho}(0) = \phi_{\omega,\rho}(0)\}$, and use $\# <\omega>_\rho$ to denote the cardinal of $<\omega>_\rho$. Define another mapping $\Upsilon_{m,\rho} : \ \mathcal{I}_{m,\rho} \to 2^{\mathbb{R}\times\mathbb{N}} \times \mathbb{R}$ by

$$\begin{aligned} [a,b] \mapsto \quad & (\{(\frac{\phi_{\omega,\rho}(0) - a}{\rho^m}, \# <\omega>_\rho) : \omega \in \{0,1\}^m \text{ such that} \\ & -\rho^m < \phi_{\omega,\rho}(0) - a \le 0\}, \frac{b-a}{\rho^m}) \end{aligned}$$

We call $\Upsilon_{m,\rho}$ to be the m-th *II-color mapping*, and call $\Upsilon_{m,\rho}([a,b])$ to be the m-th *II-color* of $[a,b]$. Compared with the I-color of $[a,b]$, the II-color $\Upsilon_{m,\rho}([a,b])$ contains the following extra information: (iii) the multiplicity of the points of the form $(1-\rho)\sum_{n=0}^{m-1}\rho^n r_n$ ($r_n = 0$ or 1) which lie on the left side of a and have distance less than ρ^m from a.

Let us take an example. Suppose $\rho = (\sqrt{5}-1)/2$, let us consider the I-color and II-color for the 3-th net interval $[a,b] = [\rho^2, 2\rho^3]$. Since the points (with the multiplicity) of the form $(1-\rho)\sum_{n=0}^{2}\rho^n r_n$ ($r_n = 0$,or 1) can be written as:

$$0, \ \rho^4, \ \rho^3, \ \underbrace{\rho^2, \ \rho^2}_{2\text{'s}}, \ \rho^2+\rho^4, \ \rho^2+\rho^3, \ 2\rho^2.$$

Among the above points, only ρ^3(with multiplicity 1) and ρ^2 (with multiplicity 2) lie on the left side of $a = \rho^2$ and have distance less than ρ^3 from a. Thus the 3-th I-color of $[\rho^2, 2\rho^3]$ is

$$(\{\frac{\rho^3-\rho^2}{\rho^3}, \frac{\rho^2-\rho^2}{\rho^3}\}, \frac{2\rho^3-\rho^2}{\rho^3}) = (\{-\rho, 0\}, 1-\rho),$$

and the 3-th II-color of $[\rho^2, 2\rho^3]$ is

$$(\{(-\rho, 1), (0, 2)\}, 1-\rho).$$

For $x \in [0,1]$ and $m \in \mathbb{N}$, define $N_{m,\rho}(x) = \#\{\omega \in \{0,1\}^m : \quad x \in \phi_{\omega,\rho}([0,1])\}$. We call $N_{m,\rho}(x)$ the *m-th overlap times* at x. Given a m-th net interval $[a,b]$ associated with ρ, assume its II-color to be $(\{t_1, n_1\}, \cdots, \{t_r, n_r\}, \gamma)$. For convenience, we say that the integral vector $(n_1, \cdots, n_r)$ is the *II-characteristic vector* of $[a,b]$. It is an elementary fact that

$$N_{m,\rho}(x) = \sum_{i=1}^{r} n_r, \tag{2.1}$$

when $x \in (a,b)$. For this reason, we call $\sum_{i=1}^{r} n_r$ the *m-th overlap times* of $[a,b]$ and denote it by $N_{m,\rho}([a,b])$. The definitions of net interval and II-color imply the following property:

$$\begin{aligned} N_{m,\rho}([a,b]) &= \#\{\omega \in \{0,1\}^n : \ \phi_{\omega,\rho}([0,1]) \cap (a,b) \neq \emptyset\} \\ &= \#\{\omega \in \{0,1\}^n : \ \phi_{\omega,\rho}([0,1]) \supset [a,b]\} \end{aligned}$$

2.2 The properties of I-colors and II-colors

Let us first summarize some basic properties of I-colors and II-colors associated with the reciprocals of Pisot numbers. These results were first proved in [Fe], here we give a sketch of proofs.

Proposition 2.1 *Let $J = [a,b]$ be a m-th net interval associated with ρ $(1/2 < \rho < 1)$, and denote by $J_1, \cdots, J_l$ all the $(m+1)$-th net intervals which are contained in $[a,b]$. Then the $(m+1)$-th I-colors of $J_1, \cdots, J_l$ are completely determined by the m-th I-color of $[a,b]$.*

Proof. It can be deduced directly from the definition of I-color. ■

Proposition 2.2 *If $\rho(> 1/2)$ is the reciprocal of a Pisot number, then*

(i) *the number of all different I-colors associated with ρ is finite, this is, the set*

$$\mathcal{C}_\rho := \bigcup_{m \geq 1} \{\Gamma_{m,\rho}([a,b]) : \quad [a,b] \in \mathcal{I}_{m,\rho}\}$$

is finite.

(ii) *there exist two positive constant C, D (only depending on ρ) such that for each m-th net interval J with II-color $(\{(t_1,n_1),\cdots,(t_r,n_r)\},\gamma)$,*

$$C\rho^m \le |J| \le \rho^m, \tag{2.2}$$

$$\mu_\rho(J) = 2^{-m}\sum_{i=1}^r n_i\mu_\rho([-t_i,-t_i+\gamma]), \tag{2.3}$$

$$D2^{-m}N_{m,\rho}(J) \le \mu_\rho(J) \le 2^{-m}N_{m,\rho}(J) \tag{2.4}$$

where $|J|$ denotes the length of J.

(iii) *there exists a positive constant c (only depending on ρ) such that*

$$mc^{-1}\mu_\rho(J) \le \mu_\rho(I) \le mc\mu_\rho(J) \tag{2.5}$$

for any two adjacent m-th net intervals I, J associated with ρ.

(iv) *for any real number q, the L^q-spectrum $\tau_{\mu_\rho}(q)$ of μ_ρ is equal to*

$$\liminf_{m\to\infty}\frac{\log(\sum_{J\in\mathcal{I}_{m,\rho}}(\mu_\rho(J)^q)}{m\log\rho} = -\frac{q\log 2}{\log\rho} + \liminf_{m\to\infty}\frac{\log(\sum_{J\in\mathcal{I}_{m,\rho}}(N_{m,\rho}(J))^q)}{m\log\rho}$$

Proof. We first prove (i). To see this, note that when ρ^{-1} is a Pisot number, Garsia's result (Lemma 1.51 of [G1]) implies that for each positive integer d there exists a positive constants c_d, such that if each r_i $(i=1,\cdots,n)$ takes only the value $\pm d, \pm(d-1),\cdots,\pm 1$ or 0, then

$$\sum_{i=1}^n \rho^{-n}r_n = 0, \qquad \text{or} \qquad |\sum_{i=1}^n \rho^{-n}r_n| \ge c_d.$$

The above result implies that the number of different points of the form $\sum_{i=1}^m \rho^{-n}r_n$ $(r_n = \pm 1, 0)$ which lie in a given interval (a,b) is not greater than $\dfrac{b-a}{c_2}$ (noting that the distance between any different two of these points is of the form $\sum_{i=1}^m \rho^{-n}r_n$ $(r_n = \pm 2, \pm 1, 0)$ and thus not less than c_2). Therefore, the sets

$$\bigcup_{m\ge 0}\left\{\frac{\phi_{\omega,\rho}(0)-\phi_{v,\rho}(0)}{\rho^m}:\quad |\phi_{\omega,\rho}(0)-\phi_{v,\rho}(0)| \le \rho^m,\ \omega, v\in\{0,1\}^m\right\}$$

and

$$\bigcup_{m\ge 0}\left\{\frac{\phi_{\omega,\rho}(0)-\phi_{v,\rho}(1)}{\rho^m}:\quad |\phi_{\omega,\rho}(0)-\phi_{v,\rho}(1)| \le \rho^m,\ \omega, v\in\{0,1\}^m\right\}$$

contain only finite many elements. This fact and the definition of $\mathcal{C}_\rho$ yield the desired result.

The inequality (2.2) in the statement (ii) is a direct corollary of (i). To show the equality (2.3), we iterate the self-similarity relation (1.3) of μ_ρ for m times, then

$$\mu_\rho(I) = 2^{-m}\sum_{\omega\in\{0,1\}^n}\mu_\rho(\phi_{\omega,\rho}^{-1}(I))$$

for each interval I. Replace I by the m-th net interval J, we can obtain (2.3) by the definition of II-color. The inequality (2.4) follows from (2.3) by using the finiteness of $\mathcal{C}_\rho$.

The statement (iv) follows directly from the definition of L^q-spectrum and (2.2), (2.4).

The statement (iii) (it is true for all $\rho \in (1/2, 1)$) is not so easy to prove. In what follows, we will prove it by induction. One may testify (2.5) directly for the case $m = 1$ since there are just three 1-th net intervals with the overlap times $1, 2, 1$ respectively. Now assume that (2.5) holds for $m \le k$. In the following we will show that (2.5) holds for $m = k + 1$. Suppose that $I,\ J$ are two adjoint $(k+1)$-th net intervals, where I lies on the left side of J. There are two possible cases:

(A) $I,\ J$ are contained in the same one k-th net interval U.

(B) $I,\ J$ are contained in two adjoint k-th net interval I', J' respectively.

(Let us recall the property of overlap times for net interval: if Q is a n-th net interval, then

$$\begin{aligned} N_{n,\rho}(Q) &= \#\{\omega \in \{0,1\}^n :\ \phi_{\omega,\rho}([0,1]) \cap int(Q) \neq \emptyset\} \\ &= \#\{\omega \in \{0,1\}^n :\ \phi_{\omega,\rho}([0,1]) \supset Q\} \end{aligned} \tag{2.6}$$

) In the case (A), it is clear that

$$N_{k,\rho}(U) \le N_{k+1,\rho}(I) \le 2N_{k,\rho}(U),\ \ N_{k,\rho}(U) \le N_{k+1,\rho}(J) \le 2N_{k,\rho}(U),$$

and therefore

$$\frac{1}{2}N_{k+1,\rho}(J) \le N_{k+1,\rho}(I) \le 2N_{k+1,\rho}(J).$$

In the case (B), let us define

$$\begin{aligned} &\mathcal{A}_1 = \{\omega \in \{0,1\}^k :\ \phi_{\omega,\rho}([0,1]) \supset I' \text{ and they share the same right end-point}\} \\ &\mathcal{A}_2 = \{\omega \in \{0,1\}^k \backslash \mathcal{A}_1 :\ \phi_{\omega,\rho}([0,1]) \supset I'\}, \\ &\mathcal{A}_3 = \{\omega \in \{0,1\}^k :\ \phi_{\omega,\rho}([0,1]) \supset J' \text{ and they share the same left end-point}\}, \\ &\mathcal{A}_4 = \{\omega \in \{0,1\}^k \backslash \mathcal{A}_3 :\ \phi_{\omega,\rho}([0,1]) \supset J'\}. \end{aligned}$$

From the definition of net interval and the property (2.6), we have

$$\begin{aligned} &\mathcal{A}_2 = \mathcal{A}_4 \\ &N_{k,\rho}(I') = \#\mathcal{A}_1 + \#\mathcal{A}_2, \\ &N_{k,\rho}(J') = \#\mathcal{A}_3 + \#\mathcal{A}_4, \\ &\#\mathcal{A}_1 + \#\mathcal{A}_2 \le N_{k+1,\rho}(I) \le \#\mathcal{A}_1 + 2\#\mathcal{A}_2, \\ &\#\mathcal{A}_3 + \#\mathcal{A}_4 \le N_{k+1,\rho}(J) \le \#\mathcal{A}_3 + 2\#\mathcal{A}_4. \end{aligned}$$

According to the above relation, we can deduce

$$\frac{1}{k+2}N_{k+1,\rho}(J) \le N_{k+1,\rho}(I) \le (k+2)N_{k+1,\rho}(J)$$

from the assumption $\frac{1}{k+1}N_{k,\rho}(J') \le N_{k,\rho}(I') \le (k+1)N_{k,\rho}(J')$. $\blacksquare$

2.3 The case $\rho = \lambda_2$

In this subsection, we always assume $\rho = \lambda := \dfrac{\sqrt{5}-1}{2}$.

We first consider the I-colors associated with λ. Let J be any m-th net interval, and $J_1, \cdots, J_l$ be the adjoint (from left to right) $(m+1)$-th net subintervals of J. Denote by U, $U_i (1 \le i \le l)$ the I-colors of J, $J_i (1 \le i \le l)$ respectively, then we would like to express their relation by

$$U \longrightarrow U_1 + \cdots + U_l,$$

and say that U generates out U_i, $1 \le i \le l$.

Under this expression, by direct calculation, we have

$$\begin{array}{lll} (\{0\}, \lambda) & \longrightarrow & (\{0\}, \lambda) + (\{-\lambda, 0\}, 1-\lambda) \\ (\{-\lambda, 0\}, 1-\lambda) & \longrightarrow & (\{\lambda - 1, 0\}, \lambda) \\ (\{\lambda - 1\}, \lambda) & \longrightarrow & (\{-\lambda, 0\}, 1-\lambda) + (\{\lambda-1\}, \lambda) \\ (\{\lambda - 1, 0\}, \lambda) & \longrightarrow & (\{-\lambda, 0\}, 1-\lambda) + (\{\lambda-1\}, 2\lambda-1) + (\{-\lambda, 0\}, \lambda) \\ (\{\lambda - 1\}, 2\lambda-1) & \longrightarrow & (\{-\lambda, 0\}, 1-\lambda) \end{array} \tag{2.7}$$

As we have seen, there are only five elements in the set $\mathcal{C}_\lambda$. In the following process, we will label the net intervals according to the above generating relations.

Let $\Xi = \{a, b, c, d, e, f, \overline{f}\}$ be an alphabet set. For any $m \in \mathbb{N}$, we will label every m-th net interval uniquely by a letter string of length m in the following way. Let J be a m-th net interval, for convenience, we denote it also by $J^{(m)}$. For each $1 \le i \le m-1$, there is only one i-th net interval that contains J, which we denote by $J^{(i)}$. Then J is labelled as $(x_i)_{i=1}^m \in \Xi^m$, where

$$x_i = \begin{cases} a & \text{if } \Gamma_{i,\lambda}(J^{(i)}) = (\{0\}, \lambda) \\ b & \text{if } \Gamma_{i,\lambda}(J^{(i)}) = (\{-\lambda, 0\}, 1-\lambda), \text{ and} \\ & \text{either} i = 1, \text{ or } i > 1 \text{ with } \Gamma_{i-1,\lambda}(J^{(i-1)}) = (\{\lambda-1\}, 2\lambda-1) \\ c & \text{if } \Gamma_{i,\lambda}(J^{(i)}) = (\{\lambda-1\}, \lambda) \\ d & \text{if } \Gamma_{i,\lambda}(J^{(i)}) = (\{\lambda-1, 0\}, \lambda) \\ e & \text{if } \Gamma_{i,\lambda}(J^{(i)}) = (\{\lambda-1\}, 2\lambda-1) \\ f & \text{if } \Gamma_{i,\lambda}(J^{(i)}) = (\{-\lambda, 0\}, 1-\lambda), \ i > 1, \\ & \quad \Gamma_{i-1,\lambda}(J^{(i-1)}) = (\{\lambda-1, 0\}, \lambda), \\ & \quad \text{and } J^{(i)} \text{ has the same left endpoint as } J^{(i-1)} \\ \overline{f} & \text{if } \Gamma_{i,\lambda}(J^{(i)}) = (\{-\lambda, 0\}, 1-\lambda), \ i > 1, \\ & \quad \Gamma_{i-1,\lambda}(J^{(i-1)}) = (\{\lambda-1, 0\}, \lambda), \\ & \quad \text{and } J^{(i)} \text{ has the same right endpoint as } J^{(i-1)} \end{cases} \tag{2.8}$$

For example, let us consider the Markov code for the 3-th net intervals $J = [1-\lambda, 2\lambda^3]$ and $J' = [\lambda^2 + \lambda^4, \lambda]$. By direct check, $[1-\lambda, \lambda]$ is the unique 1-th net interval (and also the 2-th net interval) which contains J (and also J'); the 1-th I-color for $[1-\lambda, \lambda]$, 2-th I-color for $[1-\lambda,\lambda]$ and 3-th I-color for J (or J') are

$$(\{-\lambda,0\},1-\lambda),\ (\{\lambda-1,0\},\lambda),\ (\{-\lambda,0\},1-\lambda)$$

respectively. By our labelling principle, the Markov codes for J , J' are bdf, $bd\overline{f}$ respectively.

By the above labelling principle, any two different m-th net intervals correspond to different relative Markov codes. A formal expression of the generating relation (2.7) can be given below:

$$\begin{cases} a & \longrightarrow & a+b \\ b & \longrightarrow & d \\ c & \longrightarrow & b+c \\ d & \longrightarrow & f+e+\overline{f} \\ e & \longrightarrow & b \\ f & \longrightarrow & d \\ \overline{f} & \longrightarrow & d \end{cases} \tag{2.9}$$

We will say that i generates out j if there is an arrow from i to j. The above relation determine a 0-1 matrix $H=(H_{i,j})_{i,j\in\Xi}$ by $H_{i,j}=1$ if i generates out j. That is

$$H = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ \overline{f} \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & \overline{f} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{2.10}$$

For $m\geq 2$, it follows from (2.8) and (2.9) that each m-th net interval can be coded as an element in

$$S^m := \{(x_i)_{i=1}^m \in \Xi^m : \quad H_{x_i,x_{i+1}} = 1, 1\leq i\leq m-1, \quad x_1 = a, b \text{ or } c\}, \tag{2.11}$$

and each element of the above set corresponds to unique one m-th net interval. For any $\omega\in S^m$, we will use V_ω to denote the m-th net interval corresponding to ω.

We would like to know more about the possible forms of the elements in S^m. For this purpose, we write $X_0=f$ and $X_1=\overline{f}$, and define $\mathcal{B}_\lambda=\mathcal{B}$ to be a collection of letter strings as follows

$$\mathcal{B} := \{bde\}\bigcup\{bdX_{i_1}d\cdots X_{i_k}de : k\in\mathbb{N}, i_1,\cdots,i_k = 0 \text{ or } 1\}. \tag{2.12}$$

Then by the generating relation (2.9), each element in S^m is the prefix of a letter string of the form of the following three cases:

$$
\begin{aligned}
&\omega_1 \circ \omega_2 \cdots \circ \omega_n \circ \cdots, \\
&\underbrace{a \cdots a}_{r\ a\text{'s}} \circ \omega_1 \circ \omega_2 \cdots \circ \omega_n \circ \cdots, \\
&\underbrace{c \cdots c}_{r\ c\text{'s}} \circ \omega_1 \circ \omega_2 \cdots \circ \omega_n \circ \cdots,
\end{aligned}
\tag{2.13}
$$

where $r \in \mathbb{N}$ and $\omega_i \in \mathcal{B}$, $i \in \mathbb{N}$.

In what follows we consider the II-colors associated with λ.

Let J be any m-th net interval, suppose that $J_1, \cdots, J_l$ are the $(m+1)$-th net intervals (from left to right) which contained in J. Let Θ, $\Theta_i(1 \le i \le l)$ be the II-colors of J, $J_i(1 \le i \le l)$ respectively. We express this generating relation by

$$\Theta \Longrightarrow \Theta_1 + \cdots + \Theta_l.$$

Under this notion, we have

$$
\begin{aligned}
&(\{(0,r)\},\lambda) &&\Longrightarrow (\{(0,r)\},\lambda) + (\{(-\lambda,r),(0,r)\},1-\lambda) \\
&(\{(-\lambda,p),(0,q)\},1-\lambda) &&\Longrightarrow (\{(\lambda-1,p),(0,q)\},\lambda) \\
&(\{(\lambda-1,r)\},\lambda) &&\Longrightarrow (\{(-\lambda,r),(0,r)\},1-\lambda) + (\{(\lambda-1,r)\},\lambda) \\
&(\{(\lambda-1,p),(0,q)\},\lambda) &&\Longrightarrow (\{(-\lambda,p),(0,p+q)\},1-\lambda) + (\{(\lambda-1,p+q)\},2\lambda-1) \\
& && \qquad + (\{(-\lambda,p+q),(0,q)\},1-\lambda) \\
&(\{(\lambda-1,r)\},2\lambda-1) &&\Longrightarrow (\{(-\lambda,r)\},\{(0,r)\},1-\lambda)
\end{aligned}
$$

where $p, q, r \in \mathbb{N}$.

Denote by

$$
\begin{aligned}
A^{(r)} &:= (\{(0,r)\},\lambda) \\
B^{(p,q)} &:= (\{(-\lambda,p),(0,q)\},1-\lambda) \\
C^{(r)} &:= (\{(\lambda-1,r)\},\lambda) \\
D^{(p,q)} &:= (\{(\lambda-1,p),(0,q)\},\lambda) \\
E^{(r)} &:= (\{(\lambda-1,r)\},2\lambda-1) \\
F^{(p,q)} &:= (\{(-\lambda,p),(0,q)\},1-\lambda) \\
\overline{F}^{(p,q)} &:= (\{(-\lambda,p),(0,q)\},1-\lambda)
\end{aligned}
$$

then the generating relations of II-colors can be written as

$$
\begin{cases}
A^{(r)} & \Longrightarrow\ A^{(r)} + B^{(r,r)} \\
B^{(p,q)} & \Longrightarrow\ D^{(p,q)} \\
C^{(r)} & \Longrightarrow\ B^{(r,r)} + C^{(r)} \\
D^{(p,q)} & \Longrightarrow\ F^{(p,p+q)} + E^{(p+q)} + \overline{F}^{(p+q,q)} \\
E^{(r)} & \Longrightarrow\ B^{(r,r)} \\
F^{(p,q)} & \Longrightarrow\ D^{(p,q)} \\
\overline{F}^{(p,q)} & \Longrightarrow\ D^{(p,q)}
\end{cases}
\tag{2.14}
$$

Now according to the above generating relations, we define a family of matrixes $T_{i,j}$ for each pair $(i,j) \in \Xi \times \Xi$ with $H_{i,j} = 1$:

$$\begin{cases} T_{a,a} = 1, & T_{a,b} = (1,1), \\ T_{b,d} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \\ T_{c,b} = (1,1), & T_{c,c} = 1, \\ T_{d,f} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & T_{d,e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T_{d,\overline{f}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\ T_{e,b} = (1,1), & \\ T_{f,d} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \\ T_{\overline{f},d} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. & \end{cases} \tag{2.15}$$

With the above definition, the generating relation (2.14) can be re-written as:

$$\begin{cases} A^{(r)} & \Longrightarrow & A^{(r)T_{a,a}} + B^{(r)T_{a,b}} \\ B^{(p,q)} & \Longrightarrow & D^{(p,q)T_{b,d}} \\ C^{(r)} & \Longrightarrow & B^{(r)T_{c,b}} + C^{(r)T_{c,c}} \\ D^{(p,q)} & \Longrightarrow & F^{(p,q)T_{d,f}} + E^{(p,q)T_{d,e}} + \overline{F}^{(p,q)T_{d,\overline{f}}} \\ E^{(r)} & \Longrightarrow & B^{(r)T_{e,b}} \\ F^{(p,q)} & \Longrightarrow & D^{(p,q)T_{f,d}} \\ \overline{F}^{(p,q)} & \Longrightarrow & D^{(p,q)T_{\overline{f},d}} \end{cases}$$

This is, if $i \longrightarrow i_1 + \cdots + i_l$, then we have

$$I^{(n_1,\cdots,n_r)} \Longrightarrow I_1^{(n_1,\cdots,n_r)\cdot T_{i,i_1}} + \cdots I_l^{(n_1,\cdots,n_r)\cdot T_{i,i_l}}. \tag{2.16}$$

For any matrix M, denote by $||M||$ the absolute value sum of all the entries of M. For convenience, we write

$$T_{x_1 x_2 \cdots x_m} := T_{x_1,x_2} \cdots T_{x_{m-1},x_m}. \tag{2.17}$$

Then according to the formula (2.16), we obtain the following lemma at once:

Lemma 2.3 *Let J be a m-th net interval ($m \geq 2$) coded as $\omega = (x_i)_{i=1}^m \in S^m$, suppose its II-color is $(\{(t_1,n_1),\cdots,(t_r,n_r)\},\gamma)$, then*

$$(n_1,\cdots,n_r) = \begin{cases} T_{x_1x_2\cdots x_m} & \textit{if } x_1 = a \textit{ or } c \\ (1,1)\cdot T_{x_1x_2\cdots x_m} & \textit{if } x_1 = b \end{cases}$$

and

$$N_{m,\lambda}(J) := \sum_{i=1}^r n_i = ||T_{x_1x_2\cdots x_m}||. \tag{2.18}$$

Furthermore, suppose that $\omega \in S^m$ can be written as the concatenation $\omega_1 \circ \omega_2$, where the end-letter of ω_1 is e. Then

$$N_{m,\lambda}(V_\omega) = ||T_\omega|| = ||T_{\omega_1}|| \times ||T_{\omega_2}||, \tag{2.19}$$

where V_ω is the m-th net interval corresponding to ω.

3 μ_λ is a locally infinitely-generated self-similar measure without overlap

As we have mentioned in Section 2.3, for any positive integer m there is a one-to-one correspondence between the collection of all m-th net intervals associated with λ and the string set S^m which is defined by

$$S^m := \{(x_i)_{i=1}^m \in \Xi^m : \quad H_{x_i,x_{i+1}} = 1, 1 \leq i \leq m-1, \quad x_1 = a, b \text{ or } c\},$$

where the 0-1 matrix H is defined by (2.10).

For any $\omega \in \cup_{m\geq 1} S^m$, we use V_ω to denote the net interval corresponding to ω.

Define

$$\mathcal{D} = \{\omega = (x_i)_{i=1}^n \in \cup_{m\geq 1} S^m : \ x_n = e, x_i \neq e \text{ for } 1 \leq i < n\}.$$

We will show that μ_λ is supported on $\cup_{\omega\in\mathcal{D}} V_\omega$, and for each $\omega \in \mathcal{D}$, the restriction of μ_λ on V_ω is an infinite self-similar measure without overlap.

Let us recall the definition of the string set $\mathcal{B}$ (see (2.12)), it is easy to see that

$$\mathcal{B} = \{\omega = (x_i)_{i=1}^n \in \mathcal{D} : \ x_1 = b\}.$$

For a fixed $\omega \in \mathcal{D}$, it is clear $\omega \circ \mathbf{i} \in \cup_{m\geq 1} S^m$ for each $\mathbf{i} \in \mathcal{B}$; let us denote by $g_{\omega,\mathbf{i}}$ the similitude (preserving-orientation) so that $g_{\omega,\mathbf{i}}(V_\omega) = V_{\omega\circ\mathbf{i}}$, obviously, $g_{\omega,\mathbf{i}}$ is determined uniquely and has the contraction ratio $\lambda^{|\mathbf{i}|}$, where $|\mathbf{i}|$ denotes the length of the string $\mathbf{i}$.

Theorem 3.1 *μ_λ is supported on $\cup_{\omega\in\mathcal{D}} V_\omega$. For each $\omega \in \mathcal{D}$,*

$$\mu_\lambda^{(\omega)} = \sum_{\mathbf{i}\in\mathcal{B}} 2^{-|\mathbf{i}|} \cdot ||T_{\mathbf{i}}|| \cdot \mu_\lambda^{(\omega)} \circ g_{\omega,\mathbf{i}}^{-1}, \tag{3.1}$$

where $\mu_\lambda^{(\omega)}$ denotes the restriction of μ_λ on V_ω, that is, $\mu_\lambda^{(\omega)}(A) = \mu_\lambda(A \cap V_\omega)$ for any $A \subset \mathbb{R}$.

Before giving the proof of above result, we will prove the following three statements at first:

(a1) $\mu_\lambda[1-\lambda, \lambda]) = \frac{1}{3}$,

(a2) $\sum_{\mathbf{i}\in\mathcal{B}} \lambda^{|\mathbf{i}|} = 1$, (a3) $\sum_{\mathbf{i}\in\mathcal{B}} 2^{-|\mathbf{i}|}\cdot ||T_{\mathbf{i}}|| = 1$.

We know that $V_a=[0,1-\lambda]$ and $V_c=[\lambda,1]$. Using the result of Proposition 2.2(ii) to V_a and V_c, we obtain that

$$\begin{aligned}\mu_\lambda([0,1-\lambda]) &= \tfrac{1}{2}(\mu_\lambda([0,1-\lambda])+\mu_\lambda([1-\lambda,\lambda])),\\ \mu_\lambda([\lambda,1]) &= \tfrac{1}{2}(\mu_\lambda([1-\lambda,\lambda])+\mu_\lambda([\lambda,1])),\end{aligned}$$

which implies that $\mu_\lambda([0,1-\lambda])=\mu_\lambda([1-\lambda,\lambda])=\mu_\lambda([\lambda,1])=\frac{1}{3}$.

To prove the statement (a2), we observe that each string in $\mathcal{B}$ is of length $3+2n$ ($n\geq 0$), and for each $n\geq 0$ there are just 2^n different strings in $\mathcal{B}$ which are of length $3+2n$. Therefore

$$\begin{aligned}\sum_{\mathbf{i}\in\mathcal{B}}\lambda^{|\mathbf{i}|} &= \sum_{n\geq 0} 2^n\lambda^{3+2n} = \lambda^3\sum_{n\geq 0}(2\lambda^2)^n\\ &= \frac{\lambda^3}{1-2\lambda^2} = 1.\end{aligned}$$

Now, let us prove the statement (a3).

$$\begin{aligned}\sum_{\mathbf{i}\in\mathcal{B}} 2^{-|\mathbf{i}|}\cdot||T_{\mathbf{i}}|| &= 2^{-3}\cdot 2+\sum_{n\geq 1}\sum_{i_1,\cdots,i_n\in\{0,1\}} 2^{-3-2n}(1,1)M_{i_1}\cdots M_{i_n}(1,1)'\\ &= 2^{-2}+\sum_{n\geq 1}2^{-3-2n}(1,1)(M_0+M_1)^n(1,1)'\\ &= 2^{-2}+\sum_{n\geq 1}2^{-3-2n}(1,1)\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}^n(1,1)'\\ &= 2^{-2}+\sum_{n\geq 1}2^{-3-2n}\cdot 2\cdot 3^n = 2^{-2}+2^{-2}\sum_{n\geq 1}(\tfrac{3}{4})^n\\ &= 1.\end{aligned}$$

The proof of Theorem 3.1. We first show that the intervals V_ω ($\omega\in\mathcal{D}$) are disjoint. To see this, pick any two different elements ω and ω' from $\mathcal{D}$. There are two possible cases: (i) $|\omega|=|\omega'|$; (ii) $|\omega|\neq|\omega'|$, in this case we assume $|\omega|>|\omega'|$. In the first case, we write $\omega=\nu\circ e$ and $\omega'=\nu'\circ e$. The net interval V_ν and $V_{\nu'}$ have no overlap because they are two different $|\nu|$-th net intervals. Since $V_\nu=V_{\nu\circ f}\cup V_{\nu\circ e}\cup V_{\nu\circ\overline{f}}$ (the end letter of ν is "d"), it follows that $V_\omega=V_{\nu\circ e}$ has no common endpoint with V_ν. Therefore, V_ω and $V_{\omega'}$ are disjoint. In the second case, we write $\omega=\omega_1\circ\omega_2$ where $|\omega_1|=|\omega'|$. Since the end letter of ω_1 is not "e", the net interval V_{ω_1} and $V_{\omega'}$ have no overlap, and therefore V_ω and $V_{\omega'}$ are disjoint (noting that V_ω has no common endpoint with V_{ω_1}).

Now we show that μ_λ is supported on $\cup_{\omega\in\mathcal{D}}V_\omega$. For this purpose, we give below the precise values for $\mu_\lambda(V_\omega),\omega\in\mathcal{D}$. Notice that $\mathcal{D}=\mathcal{B}\cup\{a^n\circ\mathbf{i}:n\in\mathbb{N},\mathbf{i}\in\mathcal{B}\}\cup\{c^n\circ\mathbf{i}:n\in\mathbb{N},\mathbf{i}\in\mathcal{B}\}$. For each $\mathbf{i}\in\mathcal{B}$, using the result of Proposition 2.2(ii) we obtain that

$$\begin{aligned}\mu_\lambda(V_{\mathbf{i}}) &= 2^{-|\mathbf{i}|}\cdot||T_{\mathbf{i}}||\cdot\mu_\lambda([1-\lambda,\lambda]),\\ \mu_\lambda(V_{a^n\circ\mathbf{i}}) &= 2^{-|\mathbf{i}|-n}\cdot||T_{\mathbf{i}}||\cdot\mu_\lambda([1-\lambda,\lambda]),\\ \mu_\lambda(V_{c^n\circ\mathbf{i}}) &= 2^{-|\mathbf{i}|-n}\cdot||T_{\mathbf{i}}||\cdot\mu_\lambda([1-\lambda,\lambda]),\end{aligned}$$

therefore

$$\begin{aligned}\mu_\lambda(\cup_{\omega\in\mathcal{D}}V_\omega) &= \sum_{\omega\in\mathcal{D}}\mu_\lambda(V_\omega)\\ &= \sum_{\mathbf{i}\in\mathcal{B}}2^{-|\mathbf{i}|}\cdot||T_{\mathbf{i}}||\cdot\mu_\lambda([1-\lambda,\lambda])+\\ &\quad 2\sum_{\mathbf{i}\in\mathcal{B}}\sum_{n\geq 1}2^{-|\mathbf{i}|-n}\cdot||T_{\mathbf{i}}||\cdot\mu_\lambda([1-\lambda,\lambda])\\ &= 3\sum_{\mathbf{i}\in\mathcal{B}}2^{-|\mathbf{i}|}\cdot||T_{\mathbf{i}}||\cdot\mu_\lambda([1-\lambda,\lambda])=1,\end{aligned}$$

where the last equality follows from the statement (a1) and (a3). Similarly, we can show that

$$\mu_\lambda(\cup_{\mathbf{i}_1,\cdots,\mathbf{i}_n\in\mathcal{B}} V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}) = \mu_\lambda(V_\omega),\ \forall n\in\mathbb{N}, \omega\in\mathcal{D}.$$

For a fixed $\omega\in\mathcal{D}$, we begin to prove (3.1). It suffices to show that (3.1) holds for each net interval $V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}$ where $\mathbf{i}_1,\cdots,\mathbf{i}_n\in\mathcal{B}$ (since if so, a standard argument can show that (3.1) holds for any Borel subset of $\mathbb{R}$).

For any $\mathbf{j},\mathbf{i}_1,\cdots,\mathbf{i}_n\in\mathcal{B}$, we claim that $g_{\omega,\mathbf{j}}(V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}) = V_{\omega\circ\mathbf{j}\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}$. To see this, given two intervals $[a,b]\supset[c,d]$ we say that the ratio $\frac{c-a}{b-a}$ is *the relative place* of $[c,d]$ in $[a,b]$. It is clear that the relative place is invariant under any linear preserving orientation mapping, hence, the relative place of $g_{\omega,\mathbf{j}}(V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n})$ in $g_{\omega,\mathbf{j}}(V_\omega)=V_{\omega,\mathbf{j}}$ is the same as that of $V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}$ in V_ω. To prove our claim, it suffices to show that the relative place of $V_{\omega\circ\mathbf{j}\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}$ in $V_{\omega,\mathbf{j}}$ is the same as that of $V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}$ in V_ω, and the lengths of intervals $g_{\omega,\mathbf{j}}(V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}), V_{\omega\circ\mathbf{j}\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}$ are equal. However, these two facts are easy to check according to the generating relationship of I-colors.

For $n\in\mathbb{N}$ and $\mathbf{i}_1,\cdots,\mathbf{i}_n\in\mathcal{B}$, by the above analysis, we have $g_{\omega,\mathbf{i}_1}(V_{\omega\circ\mathbf{i}_2\circ\cdots\circ\mathbf{i}_n}) = V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}$. For $\mathbf{j}\in\mathcal{B}$ and $\mathbf{j}\neq\mathbf{i}_1$, it is clear that $g_{\omega,\mathbf{j}}(V_\omega)\cap V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}=\emptyset$ since $g_{\omega,\mathbf{j}}(V_\omega)=V_{\omega,\mathbf{j}}$. Therefore

$$\begin{aligned}\textstyle\sum_{\mathbf{i}\in\mathcal{B}} 2^{-|\mathbf{i}|}\cdot||T_\mathbf{i}||\cdot\mu_\lambda^{(\omega)}\circ g_{\omega,\mathbf{i}}^{-1}(V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}) &= 2^{-|\mathbf{i}_1|}\cdot||T_{\mathbf{i}_1}||\cdot\mu_\lambda^{(\omega)}\circ g_{\omega,\mathbf{i}_1}^{-1}(V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n})\\ &= 2^{-|\mathbf{i}_1|}\cdot||T_{\mathbf{i}_1}||\cdot\mu_\lambda^{(\omega)}(V_{\omega\circ\mathbf{i}_2\circ\cdots\circ\mathbf{i}_n})\\ &= 2^{-|\mathbf{i}_1|}\cdot||T_{\mathbf{i}_1}||\cdot\mu_\lambda^{(\omega)}(V_{\omega\circ\mathbf{i}_2\circ\cdots\circ\mathbf{i}_n})\\ &= \mu_\lambda^{(\omega)}(V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}),\end{aligned}$$

which shows that (3.1) holds for $V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n}$. ■

Remark 1 For each $\omega\in\mathcal{D}$, $\mu_\lambda^{(\omega)}$ is equivalent to the image of one Bernoulli shift measure. To see this, consider the one-side shift space $(\mathcal{B}^\mathbb{N},\sigma)$ endowed with the product measure η, where the factor measure on each $\mathbf{i}$ is equal to $2^{-|\mathbf{i}|}\cdot||T_\mathbf{i}||$. Define the projection $\Pi^{(\omega)}$ from $\mathcal{B}^\mathbb{N}$ to V_ω by

$$\Pi^{(\omega)}((\mathbf{i}_n)_{n=1}^\infty) = \cap_{m\geq 1} V_{\omega\circ\mathbf{i}_1\circ\cdots\circ\mathbf{i}_n},$$

then Theorem 3.1 implies that

$$\eta = \frac{1}{\mu_\lambda(V_\omega)}\mu_\lambda^{(\omega)}\circ\Pi^{(\omega)},$$

or equivalently,

$$\mu_\lambda^{(\omega)} = \mu_\lambda(V_\omega)\cdot\eta\circ(\Pi^{(\omega)})^{-1}$$

4 The multifractal analysis of μ_λ

Suppose ν is a Borel measure on $\mathbb{R}$, define $\mathcal{R}(\nu)=\{\alpha\geq 0:\ \lim_{r\downarrow 0}\log\nu([x-r,x+r])/\log r = \alpha\ \text{ for some } x\in\mathbb{R}\}$.

Proposition 4.1 *Let ρ ($\rho \geq \frac{1}{2}$) be the reciprocal of a Pisot number. Then for each α in $\mathcal{R}(\mu_\rho)$,*

$$\dim_{\mathcal{P}}\{x \in [0,1] : \lim_{r\downarrow 0} \log \mu_\rho([x-r, x+r])/\log r = \alpha\} \leq \inf_{q\in\mathbb{R}}\{-\tau_{\mu_\rho}(q) + \alpha q\}, \tag{4.1}$$

where $\dim_{\mathcal{P}}$ denotes the packing dimension.

Proof. For $\alpha \geq 0$, denote by K_α the set $\{x \in [0,1] : \lim_{r\downarrow 0} \log \mu_\rho([x-r, x+r])/\log r = \alpha\}$. It is clear $K_\alpha \neq \emptyset$ if and only if $\alpha \in \mathcal{R}(\mu_\rho)$. For each point x in $[0,1]$ and integer $m > 0$, denote by $J_{m,\rho}(x)$ the m-th net interval associated with ρ which contains x (if x ($\neq 0,1$) is the endpoint of a m-th net interval, then there are two m-th net intervals which contain x; in this case, we select the left one of these two net intervals as $J_{m,\rho}(x)$).

By (ii) and (iii) of Proposition 2.2,

$$K_\alpha = \{x \in [0,1] : \lim_{m\to\infty} \frac{\log \mu_\rho(J_{m,\rho}(x))}{\log \rho^m} = \alpha\}. \tag{4.2}$$

In the following we prove (4.1) by considering $q \geq 0$ and $q < 0$ respectively.

First take $q \geq 0$. For any integer $n > 0$ and real number $\epsilon > 0$, denote

$$F_{\alpha,n,\epsilon} = \{x \in [0,1] : \mu_\rho(J_{m,\rho}(x)) \geq \rho^{m(\alpha+\epsilon)} \text{ for any } m \geq n\}. \tag{4.3}$$

By (4.2) and (4.3), it is clear that

$$K_\alpha \subset \cup_{n=1}^\infty F_{\alpha,n,\epsilon} \tag{4.4}$$

for any $\epsilon > 0$.

Let us estimate the upper box-counting dimension of $F_{\alpha,n,\epsilon}$ for fixed n and ϵ. To do this, for each integer $m > 0$ denote

$$\begin{aligned} \Omega_{\alpha,m,\epsilon} &= \{J \in \mathcal{I}_{m,\rho} : \mu_\rho(J) \geq \rho^{m(\alpha+\epsilon)}\}, \\ t_{\alpha,m,\epsilon} &= \#\Omega_{\alpha,m,\epsilon}. \end{aligned}$$

By (4.3) and Proposition 2.2(ii), $\Omega_{\alpha,m,\epsilon}$ is a ρ^m-cover of $F_{\alpha,n,\epsilon}$ for each $m \geq n$, therefore by the definition of the upper box-counting dimension,

$$\overline{\dim}_B F_{\alpha,n,\epsilon} \leq \limsup_{m\to\infty} \frac{\log t_{\alpha,m,\epsilon}}{\log \rho^{-m}}. \tag{4.5}$$

Note that

$$\sum_{J\in\mathcal{I}_{m,\rho}} (\mu_\rho(J))^q \geq \sum_{J\in\Omega_{\alpha,m,\epsilon}} (\mu_\rho(J))^q \geq t_{\alpha,m,\epsilon}\rho^{m(\alpha+\epsilon)q},$$

thus by (4.5) and Proposition 2.2(iv),

$$\begin{aligned} \overline{\dim}_B F_{\alpha,n,\epsilon} &\leq \limsup_{m\to\infty} \frac{\log(\sum_{J\in\mathcal{I}_{m,\rho}} (\mu_\rho(J))^q/\rho^{m(\alpha+\epsilon)q})}{\log \rho^{-m}} \\ &= -\liminf_{m\to\infty} \frac{\log(\sum_{J\in\mathcal{I}_{m,\rho}} (\mu_\rho(J))^q)}{\log \rho^m} + (\alpha+\epsilon)q \\ &= -\tau_{\mu_\rho}(q) + (\alpha+\epsilon)q. \end{aligned} \tag{4.6}$$

Using the fact that the packing dimension of a given set is always less than or equal to its upper box-counting dimension, and the fact that the packing dimension is countably stable, by (4.4) and (4.6) we obtain that

$$\dim_{\mathcal{P}} K_\alpha \le -\tau_{\mu_\rho}(q) + (\alpha+\epsilon)q.$$

Letting $\epsilon \downarrow 0$ we obtain

$$\dim_{\mathcal{P}} K_\alpha \le -\tau_{\mu_\rho}(q) + \alpha q. \tag{4.7}$$

Now take $q < 0$. A parallel argument shows that (4.7) still holds. To see this, one may consider the sets

$$\widetilde{F}_{\alpha,n,\epsilon} = \{x \in [0,1] : \mu_\rho(J_{m,\rho}(x)) \le \rho^{m(\alpha-\epsilon)} \text{ for any } m \ge n\}$$

and prove similarly that

$$\overline{\dim}_B \widetilde{F}_{\alpha,n,\epsilon} \le -\tau_{\mu_\rho}(q) + (\alpha+\epsilon)q.$$

■

Lemma 4.2 ([Y]) *Let ξ be a finite Borel measure on the line $\mathbb{R}$. If there exists a nonnegative real number s such that*

$$\lim_{r\downarrow 0} \log \xi[x-r, x+r]/\log r = s$$

for ξ almost all $x \in \mathbb{R}$, then

$$\dim_H \xi = s.$$

Proposition 4.3 *Let q_0 be defined as in Theorem 1.1(i). Then for each $q \ne q_0$,*

$$\dim_H\{x \in [0,1] : \lim_{r\downarrow 0} \log \mu_\lambda[x-r, x+r]/\log r = \alpha(q)\} \ge -\tau(\alpha(q)) + \alpha(q)\cdot q, \tag{4.8}$$

where $\tau(q) := \tau_{\mu_\lambda}(q)$ denotes the L^q-spectrum of μ_λ and $\alpha(q) = \tau'(q)$.

Proof. For $\alpha \ge 0$, denote by K_α the set $\{x \in [0,1] : \lim_{r\downarrow 0} \log \mu_\lambda[x-r,x+r]/\log r = \alpha\}$.

Suppose $q < q_0$. According to Theorem 1.1(i) we have $\alpha(q) = -\frac{\log 2}{\log \rho}$ and $-\tau_{\mu_\lambda}(\alpha(q)) + \alpha(q)\cdot q = 0$. One may check that $K_{\alpha(q)} \ne \emptyset$ by showing $K_{\alpha(q)}$ contains the point 0. Therefore (4.8) holds for this case.

From now on, we assume $q > q_0$. To prove (4.8) we concentrate a measure ν_q on $K_{\alpha(q)}$ and examine the power law behavior of $\nu_q[x-r, x+r]$ as $r \downarrow 0$, so that we can use lemma 4.2 to give a lower bound of $\dim_H K_{\alpha(q)}$ by finding the dimension of ν_q.

First we define a probability product measure $\widehat{\nu}_q$ on the shift space $(\mathcal{B}^{\mathbb{N}}, \sigma)$ with the weights $\widetilde{p}_{\mathbf{i}} = p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)}$ for each $\mathbf{i} \in \mathcal{B}$, where $p_{\mathbf{i}} = 2^{-|\mathbf{i}|}||T_{\mathbf{i}}||$ and $r_{\mathbf{i}} = \lambda^{|\mathbf{i}|}$(Theorem 1.1(i) implies that $\sum_{\mathbf{i}\in\mathcal{B}} \widetilde{p}_{\mathbf{i}} = 1$). It is well known that $\widehat{\nu}_q$ is a σ-invariant ergodic measure. Consider the projection Π from $\mathcal{B}^{\mathbb{N}}$ to the net interval $V_{\mathbf{i}_0}$ with $\mathbf{i}_0 = bde \in \mathcal{B}$, which is defined by

$$\Pi((\mathbf{i}_n)_{n=1}^\infty) = \cap_{n\ge 1} V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n}.$$

It is clear that Π is a continuous injection. Define ν_q to be the image measure of $\widehat{\nu}_q$ under the projection Π, that is, $\nu_q = \widehat{\nu}_q \circ \Pi^{-1}$. Then

$$\nu_q(V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n}) = \widetilde{p}_{\mathbf{i}_1}\cdots\widetilde{p}_{\mathbf{i}_n}. \tag{4.9}$$

Note that

$$\mu_\lambda(V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n}) = \mu_\lambda(V_{\mathbf{i}_0})\cdot p_{\mathbf{i}_1}\cdots p_{\mathbf{i}_n}, \tag{4.10}$$

$$|V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n}| = |V_{\mathbf{i}_0}|\cdot r_{\mathbf{i}_1}\cdots r_{\mathbf{i}_n}. \tag{4.11}$$

Since $\widehat{\nu}_q$ is a σ-invariant ergodic, by the Birkhoff Ergodic Theorem, there exists a Borel measurable set $G_q \subset \mathcal{B}^{\mathbb{N}}$ with $\widehat{\nu}_q(G_q) = 1$ such that for each $\omega = (\mathbf{i}_n)_{n=1}^\infty \in G_q$,

$$\begin{aligned}\lim_{n\to+\infty}\frac{\log\nu_q(V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n})}{n} &= \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\widetilde{p}_{\sigma^n\omega|1}\\ &= \sum_{\mathbf{i}\in\mathcal{B}}\widetilde{p}_{\mathbf{i}}\log\widetilde{p}_{\mathbf{i}}\\ &= \sum_{\mathbf{i}\in\mathcal{B}}p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)}\log(p_{\mathbf{i}}r_{\mathbf{i}}^{-\tau(q)})\end{aligned} \tag{4.12}$$

$$\begin{aligned}\lim_{n\to+\infty}\frac{\log\mu_\lambda(V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n})}{n} &= \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log p_{\sigma^n\omega|1}\\ &= \sum_{\mathbf{i}\in\mathcal{B}}\widetilde{p}_{\mathbf{i}}\log p_{\mathbf{i}}\\ &= \sum_{\mathbf{i}\in\mathcal{B}}p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)}\log p_{\mathbf{i}},\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\lim_{n\to+\infty}\frac{\log|V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n}|}{n} &= \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log r_{\sigma^n\omega|1}\\ &= \sum_{\mathbf{i}\in\mathcal{B}}\widetilde{p}_{\mathbf{i}}\log r_{\mathbf{i}}\\ &= \sum_{\mathbf{i}\in\mathcal{B}}p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)}\log r_{\mathbf{i}}.\end{aligned} \tag{4.14}$$

(The integrality of functions $\omega\mapsto\widetilde{p}_{\omega|1}$, $\omega\mapsto p_{\omega|1}$ and $\omega\mapsto r_{\omega|1}$, or equivalently, the finiteness of $\sum_{\mathbf{i}\in\mathcal{B}}p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)}\log(p_{\mathbf{i}}r_{\mathbf{i}}^{-\tau(q)})$, $\sum_{\mathbf{i}\in\mathcal{B}}p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)}\log p_{\mathbf{i}}$ and $\sum_{\mathbf{i}\in\mathcal{B}}p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)}\log r_{\mathbf{i}}$ come from Corollary 5.11)

Now fix $\omega = (\mathbf{i}_n)_{n=1}^\infty \in G_q$. By (4.11) it is clear

$$V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n} \subset [\Pi(\omega) - |V_{\mathbf{i}_0}|\cdot r_{\mathbf{i}_1}\cdots r_{\mathbf{i}_n},\ \Pi(\omega) + |V_{\mathbf{i}_0}|\cdot r_{\mathbf{i}_1}\cdots r_{\mathbf{i}_n}],\ \forall n. \tag{4.15}$$

On the other hand, we know that the end letter of the string $\mathbf{i}_{n+1}$ is e, changing from $\mathbf{i}_{n+1}$ this letter to f and $\overline{f}$, we get two letter string $\mathbf{j}_{n+1}$ and $\mathbf{j}'_{n+1}$ respectively. By the generating relation (2.9), i.e., "$d \to f + e + \overline{f}$ ", we know that $V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n} \supset V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n\mathbf{j}_{n+1}} \cup V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n\mathbf{i}_{n+1}} \cup V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n\mathbf{j}'_{n+1}}$ where these three intervals does not overlap, the first and the third intervals have length larger than the second. Thus,

$$\begin{aligned} V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n} &\supset [\Pi(\omega) - |V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n\mathbf{i}_{n+1}}|,\ \Pi(\omega) + |V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_n\mathbf{i}_{n+1}}|] \\ &= [\Pi(\omega) - |V_{\mathbf{i}_0}| \cdot r_{\mathbf{i}_1} \cdots r_{\mathbf{i}_n} r_{\mathbf{i}_{n+1}},\ \Pi(\omega) + |V_{\mathbf{i}_0}| \cdot r_{\mathbf{i}_1} \cdots r_{\mathbf{i}_n} r_{\mathbf{i}_{n+1}}],\ \forall n. \end{aligned} \tag{4.16}$$

Now for any small number $r > 0$, select n so that $|V_{\mathbf{i}_0}| \cdot r_{\mathbf{i}_1} \cdots r_{\mathbf{i}_n} r_{\mathbf{i}_{n+1}} \le r < |V_{\mathbf{i}_0}| \cdot r_{\mathbf{i}_1} \cdots r_{\mathbf{i}_n}$. Then by (4.15) and (4.16), we obtain

$$\begin{aligned} [\Pi(\omega) - r, \Pi(\omega) + r] &\subset [\Pi(\omega) - |V_{\mathbf{i}_0}| \cdot r_{\mathbf{i}_1} \cdots r_{\mathbf{i}_n},\ \Pi(\omega) + |V_{\mathbf{i}_0}| \cdot r_{\mathbf{i}_1} \cdots r_{\mathbf{i}_n}] \\ &\subset V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_{n-1}}, \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} [\Pi(\omega) - r, \Pi(\omega) + r] &\supset [\Pi(\omega) - |V_{\mathbf{i}_0}| \cdot r_{\mathbf{i}_1} \cdots r_{\mathbf{i}_{n+1}},\ \Pi(\omega) + |V_{\mathbf{i}_0}| \cdot r_{\mathbf{i}_1} \cdots r_{\mathbf{i}_{n+1}}] \\ &\supset V_{\mathbf{i}_0\mathbf{i}_1\cdots\mathbf{i}_{n+1}}. \end{aligned} \tag{4.18}$$

The above two relations, formula (4.12)-(4.14) and Theorem 1.1 imply that

$$\begin{aligned} \lim_{n\to+\infty} \frac{\log \nu_q[\Pi(\omega) - r, \Pi(\omega) + r]}{\log r} &= \frac{\sum_{\mathbf{i}\in\mathcal{B}} p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)} \log(p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)})}{\sum_{\mathbf{i}\in\mathcal{B}} p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)} \log r_{\mathbf{i}}} \\ &= -\tau(q) + \alpha(q) \cdot q, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} \lim_{n\to+\infty} \frac{\log \mu_\lambda[\Pi(\omega) - r, \Pi(\omega) + r]}{\log r} &= \frac{\sum_{\mathbf{i}\in\mathcal{B}} p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)} \log p_{\mathbf{i}}}{\sum_{\mathbf{i}\in\mathcal{B}} p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)} \log r_{\mathbf{i}}} \\ &= \alpha(q). \end{aligned} \tag{4.20}$$

(4.20) implies $\Pi(G_q) \subset K_{\alpha(q)}$. Note that $\nu_q(\Pi(G_q)) = \widehat{\nu}_q(G_q) = 1$. Using Lemma 4.2 and (4.19) we obtain

$$\dim_H \nu_q = -\tau(q) + \alpha(q) \cdot q, \tag{4.21}$$

therefore

$$\dim_H K_{\alpha(q)} \ge \dim_H \Pi(G_q) \ge \dim_H \nu_q = -\tau(q) + \alpha(q) \cdot q.$$

■

Remark 2 The proof of Proposition 4.3 contains a direct way to obtain the formula for the Hausdorff dimension of μ_λ and the logarithm density of μ_λ at almost all (with respect to Lebesgue measure) $x \in [0,1]$. To see this, define ν_q the same as in the The proof of Proposition 4.3. By the definition of ν_q, one can see that ν_1 is just equivalent to the restriction μ_λ (with a constant ratio) on the interval $V_{\mathbf{i}_0}$ (denoting it by $\mu_\lambda^{(\mathbf{i}_0)}$). By (4.19),

$$\dim_H \nu_1 = \frac{\sum_{\mathbf{i}\in\mathcal{B}} p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)} \log(p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)})}{\sum_{\mathbf{i}\in\mathcal{B}} p_{\mathbf{i}}^q r_{\mathbf{i}}^{-\tau(q)} \log r_{\mathbf{i}}} = \frac{\sum_{\mathbf{i}\in\mathcal{B}} p_{\mathbf{i}} \log p_{\mathbf{i}}}{\sum_{\mathbf{i}\in\mathcal{B}} p_{\mathbf{i}} \log r_{\mathbf{i}}}$$

(it is easy to check that $\tau(1) = 0$ by the definition of L^q-spectrum). Since $\dim_H \mu_\lambda^{(\mathbf{i}_0)}$ $= \dim_H \nu_1$ and $\mathbf{i}_0$ can be replaced by any element of $\mathcal{B}$, it follows that $\dim_H \mu_\lambda = \dim_H \nu_1$.

Similarly the measure ν_0 is the restriction of the Lebesgue measure on $V_{\mathbf{i}_0}$ (with a ratio). The formula (4.20) implies that

$$\lim_{r\downarrow 0} \log \mu_\lambda([x-r, x+r])/\log r = \frac{\sum_{\mathbf{i}\in\mathcal{B}} r_{\mathbf{i}} \log p_{\mathbf{i}}}{\sum_{\mathbf{i}\in\mathcal{B}} r_{\mathbf{i}} \log r_{\mathbf{i}}}$$

for almost all $x \in [0,1]$ with respect to the Lebesgue measure.

Combine Proposition 4.1 and Proposition 4.3 to obtain the following theorem.

Theorem 4.4 *Let q_0 defined as in Theorem 1.1(i). Then for each $q \in \mathbb{R}\backslash\{q_0\}$, the multifractal formalism (1.1) of μ_λ holds for $\alpha = \alpha(q) := \tau'_{\mu_\lambda}(q)$, where $\tau_{\mu_\lambda}(q)$ is the L^q-spectrum of μ_λ; moreover the Hausdorff dimension and the packing dimension of the set*

$$\{x \in [0,1] : \lim_{r\downarrow 0} \log \mu_\lambda[x-r, x+r]/\log r = \alpha(q)\}$$

coincide, the common value is equal to $-\tau_{\mu_\lambda}(\alpha(q)) + \alpha(q) \cdot q$.

5 Appendix

5.1 The proof of Theorem 1.1(i)

To make this paper self-contained, in this part we will prove (i) of Theorem 1.1. By Proposition 2.2(iv) and Lemma 2.3, we only need to consider about the limit $\lim_{m\to\infty}(\sum_{\omega\in S^m} ||T_\omega||^q)^{\frac{1}{m}}$ for any real number q, where S^m is defined as in (2.11) and T_ω's are defined by (2.15),(2.17).

Let the matrixes M_0, M_1 be defined as in (1.2). For $\mathbf{j} = j_1 \cdots j_n \in \{0,1\}^n$, denote $M_{\mathbf{j}} = M_{j_1} \circ \cdots \circ M_{j_n}$. For any $q \in \mathbb{R}$, define

$$u_{0,q} = 2^q, \quad u_{n,q} = \sum_{\mathbf{j}\in\{0,1\}^n} ||M_{\mathbf{j}}||^q \quad (n \geq 1). \tag{5.1}$$

We will prove the following theorem.

Theorem 5.1 *For any real number q, the limit $\lim_{m\to\infty}(\sum_{\omega\in S^m}||T_\omega||^q)^{\frac{1}{m}}$ exists and is equal to $\mathbf{x}(q)^{-1}$, where $\mathbf{x}(q)$ is defined by*

$$\mathbf{x}(q) := \sup\{x \geq 0 : \sum_{n\geq 0} u_{n,q}x^{2n+3} \leq 1\}. \tag{5.2}$$

Moreover, let $q = q_0$ be the real root of $\sum_{n\geq 0} u_{n,q} = 1$. then $q_0 \in (-\infty, -2)$. And when $q > q_0$, $\mathbf{x}(q)$ is the root of $\sum_{n=0}^{\infty}(\sum_{|J|=n}||M_J||^q)x^{2n+3} = 1$, and it is infinitely differentiable on $(q_0, +\infty)$; When $q \leq q_0$, $\mathbf{x}(q) = 1$. Furthermore, $\mathbf{x}(q)$ is not differentiable at $q = q_0$,

$$x'(q_0-) = 0,\ x'(q_0+) = -\frac{\sum_{n\geq 0}(\sum_{|J|=n}||M_J||^{q_0}\log||M_J||)}{\sum_{n\geq 0} u_{n,q_0}\cdot(2n+3)} \in (-\infty, 0).$$

We will prove the above theorem by a series of lemmas. At first, we define

$$S_b^m = \{(x_i)_{i=1}^m \in S^m : \quad x_1 = b\},$$

$$v_{m,q} = \sum_{\omega\in S_b^m}||T_\omega||^q,$$

for any positive integer m and real number q.

Lemma 5.2 $v_{1,q} = 2^q = u_{0,q}, v_{2,q} = 2^q = u_{0,q}, v_{3,q} = u_{0,q} + u_{1,q}$,
and for $k \geq 2$

$$\begin{aligned} v_{2k,q} &= (\textstyle\sum_{i=0}^{k-2} u_{i,q}v_{2k-(2i+3),q}) + u_{k-1,q} \\ v_{2k+1,q} &= (\textstyle\sum_{i=0}^{k-2} u_{i,q}v_{2k+1-(2i+3),q}) + u_{k-1,q} + u_{k,q} \end{aligned}$$

Proof. Since $S_b^1 = \{b\}$, $S_b^2 = \{bd\}$, $S_b^3 = \{bde, bdf, bd\overline{f}\}$, we can calculate $v_{1,q}, v_{2,q}$ and $v_{3,q}$ directly. Denote $X_0 = f$ and $X_1 = \overline{f}$. For any $k \geq 2$, by (2.13) each element $\omega \in S_b^{2k}$ can be written as one of the following two cases:

(i) $\omega = bdX_{i_1}d\cdots X_{i_{k-1}}d$, $i_1, \cdots, i_{k-1} \in \{0,1\}$.

(ii) $\omega = bdX_{i_1}d\cdots X_{i_l}de\circ\omega_2$, $0 \leq l \leq k-2$, $i_1, \cdots, i_l \in \{0,1\}$ and $\omega_2 \in S_b^{2k-(3+2l)}$.

For the case (i), $||T_\omega|| = ||M_{i_1}\cdots M_{i_{k-1}}||$. For the case (ii), by the formula (2.19), $||T_\omega|| = ||M_{i_1}\cdots M_{i_l}||\cdot||T_{\omega_2}||$. Thus

$$\begin{aligned} v_{2k,q} &= \sum_{\omega\in S_b^{2k}}||T_\omega||^q \\ &= \sum_{i_1,\cdots,i_{k-1}\in\{0,1\}}||M_{i_1\cdots i_{k-1}}||^q + \sum_{0\leq l\leq k-2}(\sum_{i_1,\cdots,i_l\in\{0,1\}}||M_{i_1\cdots i_l}||^q\cdot\sum_{\omega\in S_b^{2k-(3+2l)}}||T_\omega||^q) \\ &= u_{k-1,q} + (\sum_{l=0}^{k-2}u_{l,q}\cdot v_{2k-(2l+3),q}) \end{aligned}$$

In the other hand, by (2.13) each element $\omega \in S_b^{2k+1}$ can be written as one of the following three cases:

(iii) $\omega = bdX_{i_1}d\cdots X_{i_{k-1}}dX_{i_k}$, $i_1, \cdots, i_k \in \{0,1\}$.

(iv) $\omega = bdX_{i_1}d\cdots X_{i_{k-1}}de$, $i_1, \cdots, i_{k-1} \in \{0,1\}$.

(v) $\omega = bdX_{i_1}d\cdots X_{i_l}de\circ\omega_2$, $0\le l\le k-2$, $i_1,\cdots,i_l\in\{0,1\}$ and $\omega_2\in S_b^{2k+1-(3+2l)}$.

For the case (iii), $||T_\omega|| = ||M_{i_1}\cdots M_{i_k}||$. For the case (iv), $||T_\omega|| = ||M_{i_1}\cdots M_{i_{k-1}}||$. And for the case (v), by the formula (2.19), $||T_\omega|| = ||M_{i_1}\cdots M_{i_l}||\cdot||T_{\omega_2}||$. Thus by a discussion similar to that for $v_{2k,q}$,we have

$$v_{2k+1,q} = (\sum_{l=0}^{k-2} u_{l,q}v_{2k+1-(2l+3),q}) + u_{k-1,q} + u_{k,q}.$$

■

Lemma 5.3 $\lim_{m\to\infty}(\sum_{\omega\in S_b^m}||T_\omega||^q)^{\frac{1}{m}} = \mathbf{x}(q)^{-1}$*, where* $\mathbf{x}(q)$ *is given by (5.2).*

Proof. We will prove the statement in two steps.

(i) $\overline{\lim}_{m\to\infty}(v_{m,q})^{\frac{1}{m}} \le \mathbf{x}(q)^{-1}$

Since $\sum\limits_{n\ge0} u_{n,q}\mathbf{x}(q)^{3+2n}\le 1$ it follows that

$$\begin{cases} \mathbf{x}(q)^{-2k} & \ge \sum\limits_{i=0}^{k} u_{i,q}\mathbf{x}(q)^{(3+2i)-2k} \ge \sum\limits_{i=0}^{k-2} u_{i,q}\mathbf{x}(q)^{(3+2i)-2k} + u_{k-1,q}\mathbf{x}(q) \\ \mathbf{x}(q)^{-2k-1} & \ge \sum\limits_{i=0}^{k} u_{i,q}\mathbf{x}(q)^{(3+2i)-2k-1} \ge \sum\limits_{i=0}^{k-2} u_{i,q}\mathbf{x}(q)^{(3+2i)-2k-1} + u_{k-1,q} + u_{k,q}\mathbf{x}(q)^2 \end{cases} \tag{5.3}$$

Select a positive number $C > \max\{1,\mathbf{x}(q)^{-2},\mathbf{x}(q)^{-1}\}$ such that

$$v_{i,q} < C\cdot\mathbf{x}(q)^{-i}, \quad i = 1,2,3$$

Now we will prove by induction that

$$v_{i,q} < C\cdot\mathbf{x}(q)^{-i} \tag{5.4}$$

for all $i\in\mathbb{N}$. Suppose that this inequality holds for any $i<2k$, then by Lemma 5.2 and Inequality (5.3), we have

$$\begin{aligned} v_{2k,q} &= (\textstyle\sum_{i=0}^{k-2} u_{i,q}v_{2k-(2i+3),q}) + u_{k-1,q} \\ &\le C(\textstyle\sum_{i=0}^{k-2} u_{i,q}\mathbf{x}(q)^{(2i+3)-2k}) + u_{k-1,q} \\ &\le C(\textstyle\sum_{i=0}^{k-2} u_{i,q}\mathbf{x}(q)^{(2i+3)-2k}) + C\mathbf{x}(q)u_{k-1,q} \\ &\le C\mathbf{x}(q)^{-2k}, \\ v_{2k+1,q} &= (\textstyle\sum_{i=0}^{k-2} u_{i,q}v_{2k+1-(2i+3),q}) + u_{k-1,q} + u_{k,q} \\ &\le C(\textstyle\sum_{i=0}^{k-2} u_{i,q}\mathbf{x}(q)^{(2i+3)-2k-1}) + u_{k-1,q} + u_{k,q} \\ &\le C(\textstyle\sum_{i=0}^{k-2} u_{i,q}\mathbf{x}(q)^{(2i+3)-2k-1}) + Cu_{k-1,q} + C\mathbf{x}(q)^2u_{k,q} \\ &\le C\mathbf{x}(q)^{-2k-1}. \end{aligned}$$

Thus the inequality (5.4) holds also for $i = 2k, 2k+1$. By induction, Inequality (5.4) holds for all $i\in\mathbb{N}$, which proves the statement (i).

(ii) $\underline{\lim}_{m\to\infty}(v_{m,q})^{\frac{1}{m}} \geq \mathbf{x}(q)^{-1}$

Given any $0 < y < \mathbf{x}(q)^{-1}$, then there exists positive integer N such that

$$1 < \sum_{i=0}^{N-2} u_{i,q} y^{-3-2i}.$$

Thus when $k \geq N$, we have

$$\begin{cases} y^{2k} & \leq \sum_{i=0}^{k-2} u_{i,q} y^{2k-(3+2i)}, \\ y^{2k+1} & \leq \sum_{i=0}^{k-2} u_{i,q} y^{2k+1-(3+2i)}, \end{cases} \tag{5.5}$$

Select a positive number $D < \min\{1, \mathbf{x}(q)^{-1}, \mathbf{x}(q)^{-2}\}$ such that

$$v_{i,q} > Dy^i, \quad i = 1, \cdots, 2N-1.$$

Then by Lemma (5.2), Formula (5.5) and a discussion similar to that in the part (i), we have

$$v_{i,q} > Dy^i, \quad \forall i \in \mathbb{N},$$

which yields $\lim\limits_{m\to\infty}(v_{m,q})^{\frac{1}{m}} \geq y$ $(0 < y < \mathbf{x}(q)^{-1})$. Thus $\lim\limits_{m\to\infty}(v_{m,q})^{\frac{1}{m}} \geq \mathbf{x}(q)^{-1}$. ■

Lemma 5.4 $\lim\limits_{m\to\infty}(\sum\limits_{\omega\in S^m} ||T_\omega||^q)^{\frac{1}{m}} = \mathbf{x}(q)^{-1}$, *where* $\mathbf{x}(q)$ *is given by (5.2).*

Proof. By (2.13) each element in S^m can be written as $\underbrace{a\cdots a}_{m_1}\circ\omega$, or $\underbrace{c\cdots c}_{m_1}\circ\omega$, where $0 \leq m_1 \leq m$ and $\omega \in S_b^{m-m_1}$, thus we have

$$\sum_{\omega\in S^m} ||T_\omega||^q = \sum_{\omega\in S_b^m} ||T_\omega||^q + 2\sum_{j=1}^{m-1}\sum_{\omega\in S_b^j} ||T_\omega||^q + 2. \tag{5.6}$$

Since $u_{n,q} > ||M_0^n||^q = (n+2)^q$, it follows that the series $\sum_{n\geq 0} u_{n,q}x^{2n+3}$ diverges for $x > 1$. By the definition of $\mathbf{x}(q)$, we have $\mathbf{x}(q) \leq 1$ and thus $\mathbf{x}(q)^{-1} \geq 1$. By (5.6) and Lemma 5.3, we have

$$\lim_{m\to\infty}(\sum_{\omega\in S^m} ||T_\omega||^q)^{\frac{1}{m}} = \mathbf{x}(q)^{-1}.$$

■

The following lemmas consider about the differentiability of $\mathbf{x}(q)$.

Lemma 5.5 *(i) If* $q \geq 0$, *then for any* $m, n \in \mathbb{N}$,

$$u_{m,q}u_{n,q} \geq u_{m+n,q}.$$

(ii) If $q < 0$, *then for any* $m, n \in \mathbb{N}$,

$$u_{m,q}u_{n,q} \leq u_{m+n,q}.$$

Proof. The above statement follows immediately from the observation that for any integer $m, n \geq 1$,

$$\begin{aligned}
u_{m,q}u_{n,q} &= \sum_{\mathbf{i}\in\{0,1\}^m}\left((1,1)M_{\mathbf{i}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q \sum_{\mathbf{j}\in\{0,1\}^n}\left((1,1)M_{\mathbf{j}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q \\
&= \sum_{\mathbf{i}\in\{0,1\}^m}\sum_{\mathbf{j}\in\{0,1\}^n}\left((1,1)M_{\mathbf{i}}\begin{bmatrix}1\\1\end{bmatrix}(1,1)M_{\mathbf{j}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q, \\
&= \sum_{\mathbf{i}\in\{0,1\}^m}\sum_{\mathbf{j}\in\{0,1\}^n}\left((1,1)M_{\mathbf{i}}\begin{bmatrix}1&1\\1&1\end{bmatrix}M_{\mathbf{j}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q, \\
u_{m+n,q} &= \sum_{\mathbf{i}\in\{0,1\}^m}\sum_{\mathbf{j}\in\{0,1\}^n}\left((1,1)M_{\mathbf{i}}M_{\mathbf{j}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q, \\
&= \sum_{\mathbf{i}\in\{0,1\}^m}\sum_{\mathbf{j}\in\{0,1\}^n}\left((1,1)M_{\mathbf{i}}\begin{bmatrix}1&0\\0&1\end{bmatrix}M_{\mathbf{j}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q,
\end{aligned}$$

■

Lemma 5.6 *Let θ_0 be the positive root of $x^2+2x-\frac{9}{8}=0$, i.e., $\theta_0 \approx 0.45774$. And let ζ be the Riemann-Zeta function, that is $\zeta(x)=\sum_{n\geq 1} n^{-x}$ $(x>1)$. Then for any $q \in (-\zeta^{-1}(\frac{16}{11}), -\zeta^{-1}(1+\theta_0)) \approx (-2.2599, -2.2543)$, we have*

$$1 < \sum_{n\geq 0} u_{n,q} < +\infty.$$

Proof. Denote $U=(-\zeta^{-1}(\frac{16}{11}), -\zeta^{-1}(1+\theta_0))$. By direct check, we have $\zeta(3)\approx 1.2021 < \frac{16}{11}$ and $\zeta(2)\approx 1.6449 > 1+\theta_0$, therefore $(\zeta(3),\zeta(2)) \supset (\frac{16}{11}, 1+\theta_0)$, it follows $U \subset (-3,-2)$. Furthermore by computation, $U \approx (-2.2599, -2.2543)$.

Since that any element in $\{0,1\}^n$ can be written as $0^{n_1}1^{n_2}\cdots$, or $1^{n_1}0^{n_2}\cdots$, it follows that

$$\begin{aligned}
\textstyle\sum_{n\geq 0} u_{n,q} &= \textstyle\sum_{n\geq 0}\sum_{|J|=n} ||M_J||^q \\
&= 2^q + 2\textstyle\sum_{n\geq 1}||M_0^n||^q + 2\sum_{l\geq 1}\sum_{n_1,\cdots,n_{2l}\geq 1}||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2l-1}}M_1^{n_{2l}}||^q \\
&\quad +2\textstyle\sum_{l\geq 1}\sum_{n_1,\cdots,n_{2l+1}\geq 1}||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2l-1}}M_1^{n_{2l}}M_0^{n_{2l+1}}||^q.
\end{aligned} \tag{5.7}$$

Since

$$\begin{cases}
||(M_0^{n_1}M_1^{n_2})\cdots(M_0^{n_{2l-1}}M_1^{n_{2l}})|| & = ||\begin{pmatrix}1+n_1n_2 & n_1\\ n_2 & 1\end{pmatrix}\cdots\begin{pmatrix}1+n_{2l-1}n_{2l} & n_{2l-1}\\ n_{2l} & 1\end{pmatrix}|| \\
& \geq (1+n_1n_2)\cdots(1+n_{2l-1}n_{2l}) \\
||M_0^{n_1}M_1^{n_2}\cdots M_1^{n_{2l}}M_0^{n_{2l+1}}|| & \geq ||\begin{pmatrix}(1+n_1n_2)\cdots(1+n_{2l-1}n_{2l}) & *\\ * & *\end{pmatrix}\begin{pmatrix}1 & n_{2l+1}\\ 0 & 1\end{pmatrix}|| \\
& \geq (1+n_1n_2)\cdots(1+n_{2l-1}n_{2l})(1+n_{2l+1})
\end{cases} \tag{5.8}$$

and

$$\begin{cases}
||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2l-1}}M_1^{n_{2l}}|| & \leq (1+n_1)(1+n_2)\cdots(1+n_{2l-1})(2+n_{2l}) \\
||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2l}}M_1^{n_{2l+1}}|| & \leq (1+n_1)(1+n_2)\cdots(1+n_{2l})(2+n_{2l+1})
\end{cases}$$

(which follows from that $(1,1)M_i^n \leq (n+1,n+1) = (n+1)(1,1)$ for any $i \in \{0,1\},\ n \geq 0.$), by(5.7), when $q < 0$ we have

$$\sum_{n\geq 0} u_{n,q} \leq 2^q + 2\sum_{n\geq 1}(2+n)^q + 2\cdot(1+\sum_{n\geq 1} n^q)\cdot(\sum_{l\geq 1}(\sum_{n_1,n_2\geq 1}(1+n_1n_2)^q)^l) \tag{5.9}$$

and

$$\sum_{n\geq 0} u_{n,q} \geq 2^q + 2\sum_{n\geq 1}(2+n)^q\cdot(1+\sum_{l\geq 1}(\sum_{n\geq 1}(1+n)^q)^l) \tag{5.10}$$

From now on, we assume that $q \in U$. As we have proved, $-3 < q < -2$.

At first, we have

$$\begin{aligned}\textstyle\sum_{n_1,n_2\geq 1}(1+n_1n_2)^q &= 2\textstyle\sum_{n\geq 1}(1+n)^q - 2^q + \sum_{n_1,n_2\geq 2}(1+n_1n_2)^q \\ &< 2\textstyle\sum_{n\geq 1}(1+n)^q - 2^q + (\sum_{n\geq 2} n^q)^2 \\ &= 2(\zeta(-q)-1) - 2^q + (\zeta(-q)-1)^2 \\ &< 2\theta_0 - \tfrac{1}{8} + \theta_0^2 = 1,\end{aligned} \tag{5.11}$$

by Inequality (5.9) we have $\sum_{n\geq 0} u_{n,q} < +\infty$.

On the other hand,

$$\begin{aligned}&2^q + 2\textstyle\sum_{n\geq 1}(2+n)^q\cdot(1+\sum_{l\geq 1}(\sum_{n\geq 1}(1+n)^q)^l) \\ &= 2^q + 2\cdot\dfrac{\zeta(-q)-1-2^q}{2-\zeta(-q)} = 1 + \dfrac{(3-2^q)\zeta(-q)-4}{2-\zeta(-q)} \\ &> 1 + \dfrac{(3-2^{-2})\zeta(-q)-4}{2-\zeta(-q)} > 1 + \dfrac{(3-2^{-2})\cdot\frac{16}{11}-4}{2-\zeta(-q)} = 1,\end{aligned}$$

by Inequality (5.10) we have $\sum_{n\geq 0} u_{n,q} > 1$. ■

Corollary 5.7 *(i) $\sum_{n\geq 0} u_{n,q}$ tends to 0 when q tends to $-\infty$.*

(ii) There exists unique $q_0 < -2.25$ such that $\sum_{n\geq 0} u_{n,q_0} = 1$.

Proof. By Lemma 5.6, there exists real number $q_1 < -2.25$ such that $1 < \sum_{n\geq 0} u_{n,q_1} < +\infty$. Thus from the definition of $u_{n,q}$, the sum $\sum_{n\geq 0} u_{n,q}$ (as a function of q) is increasing and continuous on $(-\infty, q_1)$. On the other hand, note that

$$\frac{u_{n,q}}{u_{n,q'}} \leq \max_{|J|=n} ||M_J||^{q-q'} \leq 2^{q-q'}$$

for any integer $n > 0$ and real numbers $q < q' \leq q_1$, therefore

$$\frac{\sum_{n\geq 0} u_{n,q}}{\sum_{n\geq 0} u_{n,q'}} < 2^{q-q'}$$

holds for any $q < q' \leq q_1$, which implies (i). The statement (ii) follows from the continuity of $\sum_{n\geq 0} u_{n,q}$ on $(-\infty, q_1)$. ■

Lemma 5.8 *Let θ_0 be the positive root of $x^2 + 2x - \frac{9}{8} = 0$, then $\sum_{n\geq 1} n\cdot u_{n,q} < +\infty$ if $q < -\zeta^{-1}(1+\theta_0) \approx -2.2544$, where ζ is the Riemann-Zeta function.*

Proof. By Inequality (5.8), for $q < 0$ we have

$$\begin{aligned}
\sum_{n\geq 0} n\cdot u_{n,q} &= 2\sum_{n\geq 1} n||M_0^n||^q + 2\sum_{k\geq 1}\sum_{n_1,\cdots n_{2k}\geq 1}(n_1+\cdots+n_{2k})||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2k-1}}M_1^{n_{2k}}||^q \\
&+2\sum_{k\geq 1}\sum_{n_1,\cdots n_{2k+1}\geq 1}(n_1+\cdots+n_{2k+1})||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2k-1}}M_1^{n_{2k}}M_0^{n_{2k+1}}||^q \\
&\leq 2\sum_{n\geq 1} n(2+n)^q + 2\sum_{k\geq 1}\sum_{n_1,n_2,\cdots,n_{2k}\geq 1}(n_1+\cdots+n_{2k})(1+n_1n_2)^q\cdots(1+n_{2k-1}n_{2k})^q \\
&+2\sum_{k\geq 1}\sum_{n_1,n_2,\cdots,n_{2k},n_{2k+1}\geq 1}(n_1+\cdots+n_{2k+1})(1+n_1n_2)^q\cdots(1+n_{2k-1}n_{2k})^q n_{2k+1}^q \\
&= 2\sum_{n\geq 1} n(2+n)^q + 2\sum_{k\geq 1}\sum_{n_1,n_2,\cdots,n_{2k}\geq 1} 2kn_1(1+n_1n_2)^q\cdots(1+n_{2k-1}n_{2k})^q \\
&+2\sum_{k\geq 1}\sum_{n_1,n_2,\cdots,n_{2k},n_{2k+1}\geq 1} 2kn_1(1+n_1n_2)^q\cdots(1+n_{2k-1}n_{2k})^q n_{2k+1}^q \\
&+2\sum_{k\geq 1}\sum_{n_1,n_2,\cdots,n_{2k},n_{2k+1}\geq 1} n_{2k+1}(1+n_1n_2)^q\cdots(1+n_{2k-1}n_{2k})^q n_{2k+1}^q \\
&= 2\sum_{n\geq 1} n(2+n)^q + \sum_{n_1,n_2\geq 1} n_1(1+n_1n_2)^q \times \sum_{k\geq 1} 4k(\sum_{m_1,m_2\geq 1}(1+m_1m_2)^q)^{k-1} \\
&+ \sum_{n_1,n_2\geq 1} n_1(1+n_1n_2)^q \times \sum_{n\geq 1} n^q \times \sum_{k\geq 1} 4k(\sum_{m_1,m_2\geq 1}(1+m_1m_2)^q)^{k-1} \\
&+2\sum_{n\geq 1} n^{q+1} \times \sum_{k\geq 1}(\sum_{m_1,m_2\geq 1}(1+m_1m_2)^q)^k
\end{aligned} \tag{5.12}$$

Now suppose $q < -\zeta^{-1}(1+\theta_0)$. By Inequality (5.11) , we have

$$\sum_{n_1,n_2\geq 1}(1+n_1n_2)^q < 1.$$

On the other hand, since $q < -2$, it follows that the series $\sum_{n\geq 1} n^{q+1}$ and $\sum_{n_1,n_2\geq 1} n_1(1+n_1n_2)^q$ converge. Therefore by Inequality (5.12), $\sum_{n\geq 1} n\cdot u_{n,q} < +\infty$. ■

Lemma 5.9 *Suppose that $q \in \mathbb{R}$ satisfies $\sum_{n\geq 0} u_{n,q} = +\infty$, then for any integer L there exists $0 < y < 1$ such that*

$$L < \sum_{n\geq 0} u_{n,q} y^n < +\infty.$$

Proof. Case 1: $q > 0$.

In this case, $u_{n,q} > 1$ for $n \geq 0$, therefore $\sum_{n\geq 0} u_{n,q} = +\infty$. By Lemma 5.5, $\{u_{n,q}\}_n$ is submultiplicative, therefore

$$\lim_{n\to+\infty} u_{n,q}^{1/n} = \inf_{n\geq 1} u_{n,q}^{1/n}.$$

Denote by r_q the value of above limit, then $1 \leq r_q < \infty$ and $u_{n,q} \geq r_q^n$ for $n \geq 1$. Hence $\lim_{x\to r_q^{-1}} \sum_{n\geq 0} u_{n,q}x^n = +\infty$, which implies the desired result since the series $\sum_{n\geq 0} u_{n,q}y^n$ converges on $(0, r_q^{-1})$.

Case 2: $q < 0$, and $\sum_{n\geq 0} u_{n,q} = \infty$.

For any integer $l \geq 0$ and positive integers $n_1, n_2, \cdots, n_l$, define

$$\begin{aligned} a(n_1, n_2, \cdots, n_l) &= (1,0) M_0^{n_1} M_1^{n_2} \cdots M_{l (\bmod 2)}^{n_l} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ b(n_1, n_2, \cdots, n_l) &= (1,1) M_0^{n_1} M_1^{n_2} \cdots M_{l (\bmod 2)}^{n_l} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

It is clear that

$$a(n_1, n_2, \cdots, n_l) \leq b(n_1, n_2, \cdots, n_l)$$

and

$$a(n_1, n_2, \cdots, n_l) a(m_1, m_2, \cdots, m_s) \leq a(n_1, n_2, \cdots, n_l, m_1, m_2, \cdots, m_s), \tag{5.13}$$

where $m_1, m_2, \cdots, m_s$ are positive integers. It is not hard to show that

$$a(n_1, n_2, \cdots, n_l) \geq \frac{1}{4} b(n_1, n_2, \cdots, n_l), \text{ if } l \text{ is even.} \tag{5.14}$$

(To see this, denote

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = (M_0^{n_1} M_1^{n_2}) \cdots (M_0^{n_{l-1}} M_1^{n_l})$$

for even integer l. Then by induction on l, one can verify that among the x_i's, x_1 is the greatest and x_4 the smallest .)

For any integer $L \geq 1$, take an integer $y(L) \geq L \cdot 4^{-q}$, and define $p = 2^{y(L)}$. Now for any $0 < x < 1$,

$$\begin{aligned} \sum_{n\geq 0} u_{n,q} x^n &= 2^q + 2 \cdot \sum_{j=1}^{2p-1} \sum_{n_1,\cdots,n_j \geq 1} b(n_1, n_2, \cdots, n_j)^q \cdot x^{n_1+\cdots+n_j} \\ &\quad + 2 \cdot \sum_{j=0}^{2p-1} \sum_{k=1}^{+\infty} \sum_{n_1,\cdots,n_{2kp+j} \geq 1} b(n_1, \cdots, n_{2kp+j})^q \cdot x^{n_1+\cdots+n_{2kp+j}} \\ &\leq 2^q + 2 \cdot \sum_{j=1}^{2p-1} \sum_{n_1,\cdots,n_j \geq 1} b(n_1, n_2, \cdots, n_j)^q \cdot x^{n_1+\cdots+n_j} \\ &\quad + 2 \cdot \sum_{j=0}^{2p-1} \sum_{k=1}^{+\infty} \sum_{n_1,\cdots,n_{2kp+j} \geq 1} a(n_1, \cdots, n_{2kp+j})^q \cdot x^{n_1+\cdots+n_{2kp+j}} \\ &\leq 2^q + 2 \cdot \sum_{j=1}^{2p-1} \sum_{n_1,\cdots,n_j \geq 1} b(n_1, n_2, \cdots, n_j)^q \cdot x^{n_1+\cdots+n_j} \\ &\quad + 2 \cdot \sum_{j=0}^{2p-1} \sum_{k=1}^{+\infty} \sum_{n_1,\cdots,n_{2kp+j} \geq 1} a(n_1, \cdots, n_{2kp})^q a(n_{2kp+1}, \cdots, n_{2kp+j})^q x^{n_1+\cdots+n_{2kp+j}} \\ &\leq 2^q + 2 \cdot \sum_{j=1}^{2p-1} \sum_{n_1,\cdots,n_j \geq 1} b(n_1, n_2, \cdots, n_j)^q \cdot x^{n_1+\cdots+n_j} \end{aligned}$$

$$+2\cdot(\sum_{j=0}^{2p-1}\sum_{n_1,\cdots,n_j\geq 1}a(n_1,\cdots,n_j)^q\ x^{n_1+\cdots+n_j})$$

$$(\sum_{k=1}^{+\infty}(\sum_{n_1,\cdots,n_{2p}\geq 1}a(n_1,\cdots,n_{2p})^q\ x^{n_1+\cdots+n_{2p}})^k). \tag{5.15}$$

Since $a(n_1,n_2,\cdots,n_l), b(n_1,n_2,\cdots,n_l)$ are polynomials about $n_1,n_2,\cdots,n_l$ and $0<x<1$, it follows

$$\sum_{n_1,\cdots,n_l\geq 1}a(n_1,\cdots,n_l)^q\ x^{n_1+\cdots+n_l}<\infty$$

$$\sum_{n_1,\cdots,n_l\geq 1}b(n_1,\cdots,n_l)^q\ x^{n_1+\cdots+n_l}<\infty$$

for any positive integer l. Thus by (5.15), $\sum_{n\geq 0}u_{n,q}x^n<\infty$ provided that $\sum_{n_1,\cdots,n_{2p}\geq 1}a(n_1,\cdots,n_{2p})^q\ x^{n_1+\cdots+n_{2p}}<1$.

Since $\sum_{n\geq 0}u_{n,q}=\infty$, it follows from (5.15) that $\sum_{n_1,\cdots,n_{2p}\geq 1}a(n_1,\cdots,n_{2p})^q\geq 1$ (or $=+\infty$). Therefore there exists $0<z\leq 1$ such that $\sum_{n_1,\cdots,n_{2p}\geq 1}a(n_1,\cdots,n_{2p})^q z^{n_1+\cdots+n_{2p}}=1$. Moreover,

$$\sum_{n\geq 0}u_{n,q}x^n<\infty\quad\text{for } x\in(0,z). \tag{5.16}$$

For $l=2,2^2,\cdots,p$, by Inequality (5.13), we obtain that

$$\sum_{n_1,\cdots,n_{2p}\geq 1}a(n_1,\cdots,n_{2p})^q z^{n_1+\cdots+n_{2p}}\leq(\sum_{n_1,\cdots,n_l\geq 1}a(n_1,\cdots,n_l)^q z^{n_1+\cdots+n_l})^{2p/l},$$

which implies that $\sum_{n_1,\cdots,n_l\geq 1}a(n_1,\cdots,n_l)^q z^{n_1+\cdots+n_l}\geq 1$. Thus by (5.14), we have

$$\sum_{n_1,\cdots,n_l\geq 1}b(n_1,\cdots,n_l)^q z^{n_1+\cdots+n_l}\geq 4^q,\quad l=2,2^2,\cdots,p.$$

Therefore

$$\begin{aligned}\lim_{x\to z-}\sum_{n\geq 0}u_{n,q}x^n &\geq 2^q+2\cdot\sum_{j=1}^{2p-1}\sum_{n_1,\cdots,n_j\geq 1}b(n_1,\cdots,n_j)^q\cdot z^{n_1+\cdots+n_j}\\ &\geq 2^q+2\cdot y(L)\cdot 4^q\\ &\geq 2^q+2L,\end{aligned}$$

this and (5.16) yield the desired result. ■

Proposition 5.10 *Let $\mathbf{x}(q)$ be defined by (5.2) and q_0 be given as in Corollary 5.7 (ii), then*

(i) $\mathbf{x}(q)=1$ for $q\leq q_0$;

(ii) if $q>q_0$, then $\mathbf{x}(q)$ is the positive root of $\sum_{n\geq 0}u_{n,q}x^{2n+3}=1$, and it is infinitely differentiable on $(q_0,+\infty)$, and

$$x'(q)=-\frac{\sum_{n\geq 0}(\sum_{|J|=n}||M_J||^q\log||M_J||)\cdot\mathbf{x}(q)^{2n+3}}{\sum_{n\geq 0}u_{n,q}\cdot(2n+3)\cdot\mathbf{x}(q)^{2n+2}},$$

(iii) $\mathbf{x}(q)$ is not differentiable at $q=q_0$, moreover,

$$x'(q_0-)=0,\ x'(q_0+)=-\frac{\sum_{n\geq 0}(\sum_{|J|=n}||M_J||^{q_0}\log||M_J||)}{\sum_{n\geq 0}u_{n,q_0}\cdot(2n+3)}<0.$$

Proof. Fix $q \le q_0$. Since $\sum_{n\ge 0} u_{n,q} \le 1$, it follows $\mathbf{x}(q) \ge 1$ by the definition (5.2). On the other hand, $u_{n,q} > ||M_0^n||^q = (n+1)^q$, therefore $\sum_{n\ge 0} u_{n,q} x^{2n+3} = \infty$ if $x > 1$, thus $\mathbf{x}(q) \le 1$ by (5.2). The statement (i) follows.

To show (ii), let $q > q_0$. We have either $1 < \sum_{n\ge 0} u_{n,q} < \infty$ or $\sum_{n\ge 0} u_{n,q} = \infty$. In the former case, $\sum_{n\ge 0} u_{n,q} x^{2n+3}$ is continuous on $(0,1)$ and thus there exists x_0 satisfying $\sum_{n\ge 0} u_{n,q} x_0^{2n+3} = 1$. By (5.2) $\mathbf{x}(q) = x_0$. Now we assume $\sum_{n\ge 0} u_{n,q} = \infty$. By Lemma 5.9, there exists $0 < t_1 < t_2 < 1$ such that $1 < \sum_{n\ge 0} u_{n,q} t_1^{2n} < +\infty$ and $t_1^{-3} < \sum_{n\ge 0} u_{n,q} t_2^{2n} < \infty$. Thus $1 < \sum_{n\ge 0} u_{n,q} t_2^{2n+3} < \infty$, similarly we can show that $\mathbf{x}(q)$ satisfies $\sum_{n\ge 0} u_{n,q} \mathbf{x}(q)^{2n+3} = 1$. Now we show below that $\mathbf{x}(q)$ is infinitely differentiable on $(q_0, +\infty)$. Define

$$G(q,x) = \sum_{n\ge 0} u_{n,q} x^{2n+3}.$$

Fix $q_1 \in (q_0, +\infty)$. As we have shown, there exists real number $y > \mathbf{x}(q_1)$ such that $1 < G(q_1, y) < +\infty$. Take a real number z so that $\mathbf{x}(q_1) < z < y$, and take q_2 such that

$$q_2 > q_1, \quad 4^{q_2 - q_1} < \frac{y}{z}.$$

Note that for any integer $n \ge 0$,

$$\frac{u_{n,q_2}}{u_{n,q_1}} \le \max_{|J|=n} ||M_J||^{q_2-q_1} \le 4^{n(q_2-q_1)}.$$

Therefore for any $q < q_2$ and $0 < x < z$, we have

$$\begin{aligned} G(q,x) &\le \sum_{n\ge 0} u_{n,q_2} z^{2n+3} \\ &= \sum_{n\ge 0} u_{n,q_1} y^{2n+3} 4^{n(q_2-q_1)} (\frac{z}{y})^{2n+3} < +\infty, \end{aligned}$$

$$\begin{aligned} \sum_{n\ge 0} \frac{du_{n,q}}{dq} x^{2n+3} &= \sum_{n\ge 0} \sum_{|J|=n} ||M_J||^q \log ||M_J|| x^{2n+3} \le \sum_{n\ge 0} u_{n,q} (\log 4^n) x^{2n+3} \\ &\le \sum_{n\ge 0} u_{n,q_1} y^{2n+3} (\log 4^n) 4^{n(q_2-q_1)} (\frac{z}{y})^{2n+3} < +\infty, \end{aligned}$$

and

$$\begin{aligned} \sum_{n\ge 0} u_{n,q} (3+2n) x^{2n+2} &< \sum_{n\ge 0} u_{n,q_2} (3+2n) z^{2n+3} \\ &= \sum_{n\ge 0} u_{n,q_1} y^{2n+3} (3+2n) 4^{n(q_2-q_1)} (\frac{z}{y})^{2n+3} < +\infty. \end{aligned} \tag{5.17}$$

The above three inequality imply that $G(q,x)$ is well defined and differentiable on $(-\infty, q_2) \times (0, z)$. A similar more discussion shows that $G(q,x)$ is infinitely differentiable on $(-\infty, q_2) \times (0, z)$. Thus by the Implicit Function Theorem, $\mathbf{x}(q)$ is infinitely differentiable on a neighborhood of q_1. Since q_1 is taken arbitrarily on $(q_0, +\infty)$, $\mathbf{x}(q)$ is infinitely differentiable on $(q_0, +\infty)$ and (ii) follows.

To show the statement (iii), we only need to calculate $x'(q_0+)$. For $q > q_0$, starting from the fact that

$$\sum_{n\geq 0} u_{n,q}\mathbf{x}(q)^{2n+3} - \sum_{n\geq 0} u_{n,q_0}\mathbf{x}(q_0)^{2n+3} = 0,$$

we have

$$\begin{aligned}
\frac{\mathbf{x}(q)-\mathbf{x}(q_0)}{q-q_0} &= -\frac{\displaystyle\sum_{n\geq 0}\frac{u_{n,q}-u_{n,q_0}}{q-q_0}\cdot \mathbf{x}(q_0)^{2n+3}}{\sum_{n\geq 0} u_{n,q}(\mathbf{x}(q)^{2n+2}+\mathbf{x}(q)^{2n+1}\mathbf{x}(q_0)+\cdots+\mathbf{x}(q_0)^{2n+2})} \\
&= -\frac{\displaystyle\sum_{n\geq 0}\frac{u_{n,q}-u_{n,q_0}}{q-q_0}}{\sum_{n\geq 0} u_{n,q}(\mathbf{x}(q)^{2n+2}+\mathbf{x}(q)^{2n+2}+\cdots+\mathbf{x}(q)+1)}.
\end{aligned}$$

Since $\sum_{n\geq 0} u_{n,q}(2n+3) < +\infty$ on a neighborhood of q_0 (by Lemma 5.8 and 5.6), taking $q \downarrow q_0$ we get the desired result. ■

Corollary 5.11 *Let* $\mathbf{x}(q)$ *be defined by (5.2) and* q_0 *be given as in Corollary 5.7 (ii), then*

$$\sum_{n\geq 0} n u_{n,q}\mathbf{x}(q)^{2n+3} < +\infty$$

for any $q > q_0$.

Proof. It follows immediately from the inequality (5.17). ■

Proof of Theorem 5.1. It follows immediately from Lemma 5.4, Corollary 5.7, Proposition 5.10.

5.2 The generating relations of I-colors and II-colors for $\rho = \lambda_k$ $(k \geq 3)$

Fix the integer $k \geq 3$. The generating relation of I-colors for $\rho = \lambda_k$ can be expressed by

$$\begin{cases}
a & \longrightarrow\ a+b+h_1 \\
b & \longrightarrow\ d_1 \\
c & \longrightarrow\ g_1+b+c \\
d_m\ (1\leq m\leq k-2) & \longrightarrow\ d_{m+1} \\
d_{k-1} & \longrightarrow\ f+e+\overline{f} \\
e & \longrightarrow\ g_1+b+h_1 \\
f & \longrightarrow\ d_1 \\
\overline{f} & \longrightarrow\ d_1 \\
g_m\ (1\leq m\leq k-2) & \longrightarrow\ g_{m+1}+b+h_1 \\
g_{k-1} & \longrightarrow\ h_1 \\
h_m\ (1\leq m\leq k-2) & \longrightarrow\ g_1+b+h_{m+1} \\
h_{k-1} & \longrightarrow\ g_1
\end{cases} \tag{5.18}$$

where

$$\begin{cases} a & := (\{0\}, 1-\rho^k) \\ b & := (\{\rho^k - 1, 0\}, \rho^k) \\ c & := (\{-\rho^k\}, 1-\rho^k) \\ d_m \ (1 \le m \le k-1) & := (\{\rho^{k-m} - 1, 0\}, \rho^{k-m}) \\ e & := (\{-\rho^k\}, 1-2\rho^k) \\ f & := (\{\rho^k - 1, 0\}, \rho^k) \\ \overline{f} & := (\{\rho^k - 1, 0\}, \rho^k) \\ g_m \ (1 \le m \le k-1) & := (\{-\rho^{k-m}\}, 1-\rho^k - \rho^{k-m}) \\ h_m \ (1 \le m \le k-1) & := (\{-\rho^k\}, 1-\rho^k - \rho^{k-m}) \end{cases}$$

Define the alphabet set $\Xi_k = \{a, b, c, d_1, \cdots, d_{k-1}, e, f, \overline{f}, g_1, \cdots, g_{k-1}, h_1, \cdots, h_{k-1}\}$. The generating relation (5.18) determine a 0-1 matrix $Q = (Q_{i,j})_{i,j \in \Xi_k}$, so that $Q_{i,j} = 1$ if i generates out j.

For $m \ge 2$, set

$$S_k^m := \{(x_i)_{i=1}^m \in (\Xi_k)^k : Q_{x_i, x_{i+1}} = 1, 1 \le i \le m-1, \quad x_1 = a, b \text{ or } c\}, \tag{5.19}$$

then there is a one-to-one correspondence between S_k^m and the collection of all m-th net intervals associated with $\rho = \lambda_k$.

On the other hand the generating relation of II-colors for $\rho = \lambda_k$ can be expressed as

$$\begin{cases} A^{(1)} & \Longrightarrow \quad A^{(1)} + B^{(1,1)} + H_1^{(1)} \\ B^{(p,q)} & \Longrightarrow \quad D_1^{(p,q)} \\ C^{(1)} & \Longrightarrow \quad G_1^{(1)} + B^{(1,1)} + C^{(1)} \\ D_m^{(p,q)} \ (1 \le m \le k-2) & \Longrightarrow \quad D_{m+1}^{(p,q)} \\ D_{k-1}^{(p,q)} & \Longrightarrow \quad F^{(p,p+q)} + E^{(p+q)} + \overline{F}^{(p+q,q)} \\ E^{(r)} & \Longrightarrow \quad G_1^{(r)} + B^{(r,r)} + H_1^{(r)} \\ F^{(p,q)} & \Longrightarrow \quad D_1^{(p,q)} \\ \overline{F}^{(p,q)} & \Longrightarrow \quad D_1^{(p,q)} \\ G_m^{(r)} \ (1 \le m \le k-2) & \Longrightarrow \quad G_{m+1}^{(r)} + B^{(r,r)} + H_1^{(r)} \\ G_{k-1}^{(r)} & \Longrightarrow \quad H_1^{(r)} \\ H_m^{(r)} \ (1 \le m \le k-2) & \Longrightarrow \quad G_1^{(r)} + B^{(r,r)} + H_{m+1}^{(r)} \\ H_{k-1}^{(r)} & \Longrightarrow \quad G_1^{(1)} \end{cases}$$

where

$$
\begin{cases}
A^{(1)} & := (\{(0,1)\}, 1-\rho^k) \\
B^{(p,q)} & := (\{(\rho^k-1,p),(0,q)\}, \rho^k) \\
C^{(1)} & := (\{(-\rho^k,1)\}, 1-\rho^k) \\
D_m^{(p,q)} \ (1 \le m \le k-1) & := (\{(\rho^{k-m}-1,p),(0,q)\}, \rho^{k-m}) \\
E^{(r)} & := (\{(-\rho^k,r)\}, 1-2\rho^k) \\
F^{(p,q)} & := (\{(\rho^k-1,p),(0,q)\}, \rho^k) \\
\overline{F}^{(p,q)} & := (\{(\rho^k-1,p),(0,q)\}, \rho^k) \\
G_m^{(r)} \ (1 \le m \le k-1) & := (\{(-\rho^{k-m},r)\}, 1-\rho^k-\rho^{k-m}) \\
H_m^{(r)} \ (1 \le m \le k-1) & := (\{(-\rho^k,r)\}, 1-\rho^k-\rho^{k-m})
\end{cases}
$$

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