THE VARIATIONAL PRINCIPLE FOR PRODUCTS OF NON-NEGATIVE MATRICES

DE-JUN FENG

Abstract. Let (Σ_A, σ) be a subshift of finite type and let $M(x)$ be a continuous function on Σ_A taking values in the set of non-negative matrices. We set up the variational principle between the pressure function, entropy and Lyapunov exponent for M on Σ_A. We also present some properties about equilibrium states.

1. Introduction

Let σ be the shift map on $\Sigma = \{1, 2, \cdots, m\}^{\mathbb{N}}$, $m \geq 2$. As usual Σ is endowed with the metric $d(x, y) = m^{-n}$ where $x = (x_k),\ y = (y_k)$ and n is the smallest of the k such that $x_k \neq y_k$. Given an $m \times m$ matrix A with entries 0 or 1, we consider the subshift of finite type (Σ_A, σ) (see [1]). We shall always assume that A is primitive.

Suppose M is a continuous function on Σ_A taking values in the set of all non-negative $d \times d$ matrices. Here a matrix $A = (A_{i,j})_{1\leq i,j\leq d}$ is said to be *non-negative* if $A_{i,j} \geq 0$ for all $1 \leq i, j \leq d$. Similarily we say A is *strictly positive* if $A_{i,j} > 0$ for all $1 \leq i, j \leq d$. For $q \in \mathbb{R}$, the *pressure function* $P(q)$ of M is defined by

$$P(q) := P(M, q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{J\in\Sigma_{A,n}} \sup_{x\in[J]} \|M(x)M(\sigma x)\dots M(\sigma^{n-1}x)\|^q, \tag{1.1}$$

where $\Sigma_{A,n}$ denotes the set of all admissible indices of length n over $\{1, \dots, m\}$; for $J = j_1 \cdots j_n \in \Sigma_{A,n}$, $[J]$ denotes the cylinder set $\{x = (x_i) \in \Sigma_A :\ x_i = j_i,\ 1 \leq i \leq n\}$, $\|\cdot\|$ denotes the matrix norm defined by $\|B\| := \mathbf{1}^t B\mathbf{1}$, $\mathbf{1}^t = (1, 1, \dots, 1)$. By using a sub-additive argument, it is easy to show that for $q > 0$, the limit in the above definition exists. With some additional conditions on the matrices (e.g. M is strictly positive), the limit exists for $q \in \mathbb{R}$.

The pressure function of a matrix-valued function is a natural generalization of that of the scalar case (i.e., $M(x) = e^{\phi(x)}$ where $\phi(x)$ is a real valued function called the *potential* of the subshift). The reader is referred to [1, 8, 13, 14] for the pressure

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and variational principle in the classical scalar case. In [5], Feng and Lau considered the pressure functions and Gibbs measures for the products of matrices, where the matrix function was assumed to be either strictly positive and Hölder continuous, or local non-negative constant satisfying an irreducibility assumption. For instance in the former setting, they proved

Theorem A. *Suppose that M is a Hölder continuous function on Σ_A taking values in the set of strictly positive $d \times d$ matrices. Then for any $q \in \mathbb{R}$, there is a unique σ-invariant, ergodic probability measure μ_q on Σ_A for which one can find constants $C_1 > 0$, $C_2 > 0$ such that*

$$C_1 \leq \frac{\mu_q([J])}{\exp(-nP(q)) \cdot \|M(x)M(\sigma x) \ldots M(\sigma^{n-1}x)\|^q} \leq C_2 \tag{1.2}$$

for any $n > 0$, $J \in \Sigma_{A,n}$ and $x \in [J]$.

In [3], the author used the pressure function to analyze the multifractal structure of the Lyapunov exponents for the products of matrices, and proved

Theorem B. *Suppose M is a continuous function on Σ_A taking values in the set of strictly positive $d \times d$ matrices. For any $\alpha \in \mathbb{R}$, if the set $\{x \in \Sigma_A : \ \lambda_M(x) = \alpha\}$ is not empty, then*

$$\begin{aligned} \dim_H\{x \in \Sigma_A : \ \lambda_M(x) = \alpha\} &= \frac{1}{\log m} \inf_{q \in \mathbb{R}} \{-\alpha q + P(q)\} \\ &= \frac{1}{\log m} \sup\{h(\mu) : \ \mu \in \mathcal{M}(\Sigma_A, \sigma), \ M_*(\mu) = \alpha\}, \end{aligned}$$

where $\dim_H$ denotes the Hausdorff dimension, $\lambda_M(x)$ is the upper Lyapunov exponent of M at x defined by

$$\lambda_M(x) = \lim_{n \to \infty} \frac{1}{n} \log \|M(x)M(\sigma x) \ldots M(\sigma^{n-1}x)\| \tag{1.3}$$

when the limit exists, $\mathcal{M}(\Sigma_A, \sigma)$ denotes the collection of all σ-invariant Borel probability measures on Σ_A, and

$$M_*(\mu) = \lim_{n \to \infty} \frac{1}{n} \int \log \|M(y)M(\sigma y) \ldots M(\sigma^{n-1}y)\| d\mu(y). \tag{1.4}$$

Theorem B was also proved in [5] under an additional condition that M is Hölder continuous. For $\mu \in \mathcal{M}(\Sigma_A, \sigma)$, $M_*(\mu)$ is often called the *upper Lyapunov exponent* of μ associated with M. It was first proved by Furstenberg and Kesten [7] that $\lambda_M(x)$ exists for μ almost all x and $\int \lambda_M(x) d\mu(x) = M_*(\mu)$.

The main purpose of this paper is to set up the variational principle for the non-negative matrix-valued functions. We prove the following general theorem, which does not need any additional smoothness condition or the strict positivity of M.

Theorem 1.1. *Suppose that M is a continuous function on Σ_A taking values in the set of non-negative $d \times d$ matrices. Then for any $q > 0$, we have*

$$P(q) = \sup\left\{h_\mu(\sigma) + qM_*(\mu) : \ \mu \in \mathcal{M}(\Sigma_A, \sigma)\right\}, \tag{1.5}$$

and this supremum is attained.

If furthermore M is strictly positive, then (1.5) holds for any $q \in \mathbb{R}$, and the corresponding supremum is attained.

When $d = 1$, $M(x) = e^{\phi(x)}$ becomes a scarlar function; and in this case (1.5) is just the classical variational principal formula for the potential $q\phi(x)$ (see, e.g., [8, Theorem 4.4.11], where ϕ may takes the value $-\infty$).

A member μ of $\mathcal{M}(\Sigma_A, \sigma)$ is called an *equilibrium state for M with respect to q* if

$$P(q) = h_\mu(\sigma) + qM_*(\mu).$$

Let $\mathcal{I}(M, q)$ denotes the collection of all equilibrium states of M with respect to q. It is interesting to consider under what condition $\mathcal{I}(M, q)$ contains only one element (in this case we say that M has a unique equilibrium state with respect to q). The following theorem establishes the derivative formula of the pressure function which is an extension of the classical Ruelle formula to matrix-valued function (for the classical Ruelle formula, see [13, Ex. 5, p. 99], [11, Lemma 4] and [8, Theorem 4.3.5])

Theorem 1.2. *Suppose that M is a continuous function on Σ_A taking values in the set of non-negative $d \times d$ matrices with $P(q) \neq -\infty$ for all $q > 0$. Then*

$$P'(q+) := \lim_{\epsilon \downarrow 0} \frac{P(q+\epsilon) - P(q)}{\epsilon} = \sup\left\{M_*(\mu) : \ \mu \in \mathcal{I}(M, q)\right\} \tag{1.6}$$

$$P'(q-) := \lim_{\epsilon \downarrow 0} \frac{P(q-\epsilon) - P(q)}{-\epsilon} = \inf\left\{M_*(\mu) : \ \mu \in \mathcal{I}(M, q)\right\} \tag{1.7}$$

for any $q > 0$.

If furthermore M is strictly positive, then (1.6) and (1.7) hold for any $q \in \mathbb{R}$.

We remark that there are examples of $M \not\equiv \mathbf{0}$ satisfying $P(M, q) \equiv -\infty$. For instance, take $\Sigma = \{1, 2\}^{\mathbb{N}}$ and define $f \in C(\Sigma)$ by

$$f(x) = \begin{cases} 0 & \text{if } x_1x_2 = 00 \text{ or } 11, \\ 2^{-n} & \text{if } x_1 \ldots x_{2n+1} = (01)^n 1 \text{ or } (10)^n 0, \end{cases}$$

where $x = (x_i)_{i=1}^{\infty} \in \Sigma$. Take $M(x) = f(x)I_d$, where I_d denotes the $d \times d$ identity matrix. Then $P(M, q) \equiv -\infty$. We point out that the condition $P(q) \neq -\infty$ for all $q > 0$ is equivalent to $P(q) \neq -\infty$ for some $q > 0$. A sufficient condition insuring $P(q) \neq -\infty$ is that there exists $x \in \Sigma_A$ such that

$$\overline{\lambda}_M(x) := \limsup_{n \to \infty} \frac{1}{n} \log \|M(x)M(\sigma x) \dots M(\sigma^{n-1}x)\| \neq -\infty.$$

We also remark that for any fixed M and q, the pressure $P(e^{\phi(x)}M, q)$ is a convex function of $\phi \in C(\Sigma_A)$. It can be derived directly by Theorem 1.1 and the fact $\left(e^{\phi(x)}M\right)^* (\mu) = \int \phi d\mu + M^*(\mu)$. Anyway we don't know whether there is any kind of convexity of $P(M, q)$ on M.

As a direct corollary of Theorem 1.2, we have

Corollary 1.3. *Let M be a continuous function on Σ_A taking values in the set of non-negative (strictly positive resp.) $d \times d$ matrices. A necessary condition for M having a unique equilibrium state with respect to some $q > 0$ ($q \in \mathbb{R}$ resp.), is that $P(q)$ is differentiable at q.*

Under some additional assumptions, we can show the existence of unique equilibrium state for M (see Theorem 3.1, Corollary 3.2).

As we have seen from Theorem A and Theorem B, the pressure function $P(q)$ is an important term in studying the Gibbs measures of $M(x)$ and the Hausdorff dimension of level sets of $\lambda_M(x)$. We should point out that $P(q)$ has also appeared naturally in the study of multifractal phenomena about measures. In [4] the author studied the multifractal structure of a class of self-similar measures with overlaps (namely, self-similar measures satisfying the finite type condition). He proved that these measures locally can be expressed as the product of a finite family of non-negative matrices, and their L^q-spectra $\tau(q)$ (one of the basic ingredients in the study of multifractal phenomena, see [2, 12]) differ from $P(q)$ only by a factor (see [4, Lemma 4.1 and Theorem 5.2]). The readers are referred to [4, 6, 9, 10] and the references therein for the multifractal theory for self-similar measures with overlaps.

A first thought of proving Theorem 1.1 is to re-express the pressure function $P(q)$ of M as the classical pressure P_f for some scalar function f. However this thought seems only possible for the case that $q = 1$ and M is strictly positive. In this case we may enlarge the symbolic set $\{1, 2, \dots, m\}$ to $S = \{(i, j) : i = 1, \dots, d, \ j = 1, \dots, m\}$ and define the 0-1 matrix $\widehat{A} = \widehat{A}_{S \times S}$ by

$$\widehat{A}_{(i,j),(i',j')} = \begin{cases} 1, & \text{if } A_{j,j'} = 1 \\ 0, & \text{otherwise.} \end{cases}$$

One may check by definition that $P(1)$ equals P_f for a scalar function f on the subshift space $S_{\widehat{A}}^{\mathbb{N}}$ defined by

$$f\left(((i_1, j_1), (i_2, j_2), \ldots,)\right) = \log M_{i_1, i_2}(j_1 j_2 \cdots).$$

Even in this case we still have some difficulty to pull back the variational result from $S_{\widehat{A}}^{\mathbb{N}}$ to Σ_A.

Our proof of Theorem 1.1 is essentially based on the existence of "Gibbs" measures (see Theorem A). In fact, by Theorem A and a standard argument we prove Theorem 1.1 immediately in the special case where M is Hölder continuous and takes values in the set of strictly positive $d \times d$ matrices. The original part of our proof is the generalization of this result to functions which are continuous and with values in $d \times d$ non-negative matrices. We do this with two approximation steps: of continuous maps by Hölder continuous ones and of non-negative matrices by strictly positive ones.

We organize the paper as follows. In Section 2, we prove Theorem 1.1 (see Propositions 2.6-2.8). In section 3, we consider the equilibrium states of M, and give a proof of Theorem 1.2.

2. The proof of Theorem 1.1

For convenience, we use Γ_+ (Γ resp.) to denote the collection of all continuous functions on Σ_A taking values in the set of all strictly positive (non-negative resp.) $d \times d$ matrices. For $M \in \Gamma$, we write $\pi_n M(x)$ for the product $M(x)M(\sigma x) \cdots M(\sigma^{n-1} x)$.

Lemma 2.1. *Let $M \in \Gamma$. We have*

$$\|\pi_{n+\ell} M(x)\| \leq \|\pi_n M(x)\| \|\pi_\ell M(\sigma^n x)\|, \qquad \forall\ n, \ell \in \mathbb{N},\ x \in \Sigma_A.$$

Moreover if $M \in \Gamma_+$, then there exists a constant $C > 0$ (depending on M) such that

$$\|\pi_{n+\ell} M(x)\| \geq C \|\pi_n M(x)\| \|\pi_\ell M(\sigma^n x)\|, \qquad \forall\ n, \ell \in \mathbb{N},\ x \in \Sigma_A.$$

Proof. The first inequality is trivial. The second one was proved in [5]. However for the reader's convenience, we include the detailed proof. Since $M \in \Gamma_+$, there is a constant $C > 0$ such that

$$\frac{\min_{i,j} M_{i,j}(x)}{\max_{i,j} M_{i,j}(x)} \geq dC, \qquad \forall\ x \in \Sigma_A,$$

which implies that $M(x) \geq CEM(x)$ (here and afterwards we write $B^{(1)} \geq B^{(2)}$ for two matrices $B^{(1)}$, $B^{(1)}$ if $B^{(1)}_{i,j} \geq B^{(2)}_{i,j}$ for each index (i, j)), here $E = (E_{i,j})_{1 \leq i,j \leq d}$

is the matrix whose entries are all equal to 1. Let $\mathbf{1}$ be the d-dimensional column vector each coordinate of which is 1. Then using $M(\sigma^n x) \geq CEM(\sigma^n x)$, we have

$$\begin{aligned} \|\pi_{n+\ell}M(x)\| &\geq \|\,(\pi_n M(x))\,CE\,(\pi_\ell M(\sigma^n x))\,\| \\ &= C\|\,(\pi_n M(x))\,\mathbf{1}^t\mathbf{1}\,(\pi_\ell M(\sigma^n x))\,\| \\ &= C\|\pi_n M(x)\|\|\pi_\ell M(\sigma^n x)\|. \end{aligned}$$

□

Lemma 2.2. (i) *If $M \in \Gamma$, then for $q > 0$ the limit*

$$P(q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{J\in\Sigma_{A,n}} \sup_{x\in[J]} \|\pi_n M(x)\|^q \tag{2.1}$$

exists and equals $\inf_n \frac{1}{n}\log \sum_{J\in\Sigma_{A,n}} \sup_{x\in[J]} \|\pi_n M(x)\|^q$.

(ii) *If $M \in \Gamma$, then for any $\mu \in \mathcal{M}(\Sigma_A, \sigma)$,*

$$M_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \int \log \|\pi_n M(x)\| d\mu(x) = \inf_n \frac{1}{n} \int \log \|\pi_n M(x)\| d\mu(x).$$

(iii) *If $M \in \Gamma_+$, then for any $q \in \mathbb{R}$ the limit (2.1) exists. Moreover let C be the constant as in Lemma 2.1, then*

$$P(q) = \begin{cases} \inf_n \frac{1}{n}\log \sum_{J\in\Sigma_{A,n}} \sup_{x\in[J]} \|\pi_n M(x)\|^q, & \text{if } q \geq 0, \\ \inf_n \frac{1}{n}\log \sum_{J\in\Sigma_{A,n}} C^q \sup_{x\in[J]} \|\pi_n M(x)\|^q, & \text{if } q < 0. \end{cases}$$

Proof. Suppose $I \in \Sigma_{A,n}$, $J \in \Sigma_{A,\ell}$ with $IJ \in \Sigma_{A,n+\ell}$. By Lemma 2.1, we have for $M \in \Gamma$ and $q \geq 0$,

$$\begin{aligned} \sup_{x\in[IJ]} \|\pi_{n+\ell}M(x)\|^q &\leq \sup_{x\in[IJ]} \left(\|\pi_n M(x)\|^q \|\pi_\ell M(\sigma^n x)\|^q\right) \\ &\leq \sup_{x\in[I]} \|\pi_n M(x)\|^q \cdot \sup_{y\in[J]} \|\pi_\ell M(y)\|^q. \end{aligned}$$

While for $M \in \Gamma_+$ and $q < 0$,

$$C^q \sup_{x\in[IJ]} \|\pi_{n+\ell}M(x)\|^q \leq \left(C^q \sup_{x\in[I]} \|\pi_n M(x)\|^q\right)\left(C^q \sup_{y\in[J]} \|\pi_\ell M(y)\|^q\right).$$

Using a sub-additive argument, we obtain (i) and (iii). The statement (ii) is obtained similarly by using the fact $\|\pi_{n+\ell}M(x)\| \leq \|\pi_n M(x)\|\|\pi_\ell M(\sigma^n x)\|$. □

Lemma 2.3. *Let $M \in \Gamma_+$. For any $\mu \in \mathcal{M}(\Sigma_A, \sigma)$ and $n \in \mathbb{N}$, we have*

$$\int \frac{\log \|\pi_n M(x)\|}{n} d\mu(x) + \frac{\log C}{n} \leq M_*(\mu) \leq \int \frac{\log \|\pi_n M(x)\|}{n} d\mu(x),$$

where C is the constant in Lemma 2.1.

Proof. Take any $\mu \in \mathcal{M}(\Sigma_A, \sigma)$. By Lemma 2.1 and the invariance of μ, we have for any $n, m \in \mathbb{N}$,

$$\begin{aligned}\int \log \|\pi_{n+\ell} M(x)\| d\mu(x) &\leq \int \log \|\pi_n M(x)\| d\mu(x) + \int \log \|\pi_\ell M(\sigma^n x)\| d\mu(x) \\ &= \int \log \|\pi_n M(x)\| d\mu(x) + \int \log \|\pi_\ell M(x)\| d\mu(x).\end{aligned}$$

A sub-additive argument yields the second desired inequality. Similarly we have

$$\begin{aligned}\int \log \left(C\|\pi_{n+\ell} M(x)\|\right) d\mu(x) &\geq \int \log \left(C\|\pi_n M(x)\|\right) d\mu(x) \\ &\quad + \int \log \left(C\|\pi_\ell M(x)\|\right) d\mu(x),\end{aligned}$$

which proves the first inequality by a super-additive argument. □

As a corollary, we have

Corollary 2.4. *Suppose $\mu_k \in \mathcal{M}(\Sigma_A, \sigma)$ converges to μ in the weak-star topology. Then for any $M \in \Gamma_+$,*

$$\lim_{k\to\infty} M_*(\mu_k) = M_*(\mu). \tag{2.2}$$

Proof. By Lemma 2.3, for any $n \in \mathbb{N}$ we have

$$\left| M_*(\mu) - \frac{1}{n} \int \log \|\pi_n M(x)\| d\mu(x) \right| \leq \frac{|\log C|}{n}$$

and

$$\left| M_*(\mu_k) - \frac{1}{n} \int \log \|\pi_n M(x)\| d\mu_k(x) \right| \leq \frac{|\log C|}{n}.$$

Since $\lim_{k\to\infty} \int \log \|\pi_n M(x)\| d\mu_k(x) = \int \log \|\pi_n M(x)\| d\mu(x)$, we have

$$\limsup_{k\to\infty} |M_*(\mu) - M_*(\mu_k)| \leq \frac{2|\log C|}{n}.$$

Letting $n \to \infty$, we obtain the desired result. □

Lemma 2.5. *(cf.* [14, Lemma 9.9]*) Let $a_1, \cdots, a_k$ be given real numbers. If $p_i \geq 0$ and $\sum_{i=1}^k p_i = 1$ then*

$$\sum_{i=1}^k p_i(a_i - \log p_i) \leq \log \left(\sum_{i=1}^k e^{a_i} \right).$$

Proposition 2.6. *For any $M \in \Gamma_+$ and $q \in \mathbb{R}$ (resp., for any $M \in \Gamma$ and $q > 0$),*

$$P(q) \geq \sup \left\{ h_\mu(\sigma) + qM_*(\mu) : \ \mu \in \mathcal{M}(\Sigma_A, \sigma) \right\}.$$

Proof. The following argument is classical. Let $\mu \in \mathcal{M}(\Sigma_A, \sigma)$. By Lemma 2.5, for any $n \in \mathbb{N}$,

$$\begin{aligned} & \log \sum_{I \in \Sigma_{A,n}} \sup_{x \in [I]} \|\pi_n M(x)\|^q \\ \geq & \sum_{I \in \Sigma_{A,n}} \left[-\mu([I]) \log \mu([I]) + \mu([I]) \log \sup_{x \in [I]} \|\pi_n M(x)\|^q \right] \\ \geq & \sum_{I \in \Sigma_{A,n}} (-\mu([I]) \log \mu([I])) + \int \log \|\pi_n M(x)\|^q d\mu(x) \\ = & \sum_{I \in \Sigma_{A,n}} (-\mu([I]) \log \mu([I])) + q \int \log \|\pi_n M(x)\| d\mu(x). \end{aligned}$$

Dividing the both sides by n, and letting $n \to \infty$, we have

$$P(q) \geq h_\mu(\sigma) + qM_*(\mu).$$

□

Proposition 2.7. *For any $M \in \Gamma_+$ and $q \in \mathbb{R}$, there exists $\mu \in \mathcal{M}(\Sigma_A, \sigma)$ such that*

$$P(q) = h_\mu(\sigma) + qM_*(\mu).$$

Proof. Fix $q \in \mathbb{R}$, we divide the proof into two steps.

Step 1. Assume M is Hölder continuous. In this case let $\mu = \mu_q$ be the Gibbs measure in Theorem A. Then for each $n \in \mathbb{N}$, $I \in \Sigma_{A,n}$ and $x \in [I]$,

$$\log C_1 \leq nP(q) + \log \mu([I]) - q \log \|\pi_n M(x)\| \leq \log C_2.$$

Integrating by μ, and dividing both sides by n, we have

$$\frac{\log C_1}{n} \leq P(q) + \frac{1}{n} \sum_{I \in \Sigma_{A,n}} \mu([I]) \log \mu([I]) - q \int \frac{\log \|\pi_n M(x)\|}{n} d\mu(x) \leq \frac{\log C_2}{n}.$$

Letting $n \to \infty$ we have $P(q) = h_\mu(\sigma) + M_*(\mu)$.

Step 2. Now let us consider M without the Hölder continuity assumption. For each $k \in \mathbb{N}$, define a matrix-valued function $M^{(k)}$ on Σ_A by

$$M^{(k)}_{i,j}(x) = \sup_{y \in I_k(x)} M_{i,j}(y), \qquad 1 \leq i, j \leq d,$$

where $I_k(x) = [x_1 x_2 \cdots x_k]$ for $x = (x_i)$.

By the definition $M^{(k)}$ depends only on the first k coordinates of x, and thus it is Hölder continuous. As we proved in Step 1, there exists $\mu_k \in \mathcal{M}(\Sigma_A, \sigma)$ such that

$$P(M^{(k)}, q) = h_{\mu_k}(\sigma) + q\left(M^{(k)}\right)_*(\mu_k).$$

Since M is strictly positive and continuous, there exists a sequence of positive numbers ϵ_k such that $\lim_k \epsilon_k = 0$ and

$$M(x) \leq M^{(k)}(x) \leq (1+\epsilon_k)M(x), \qquad \forall x \in \Sigma_A,$$

From which we deduce that

$$|P(q) - P(M^{(k)}, q)| \leq |q| \log(1+\epsilon_k)$$

and

$$\left|\frac{\log \|\pi_n M(x)\|}{n} - \frac{\log \|\pi_n M^{(k)}(x)\|}{n}\right| \leq \log(1+\epsilon_k), \quad \forall x \in \Sigma_A.$$

By the above two inequalities, we have

$$\begin{aligned} P(q) &= \lim_{k\to\infty} P(M^{(k)}, q) = \lim_{k\to\infty} \left[h_{\mu_k}(\sigma) + q\left(M^{(k)}\right)_*(\mu_k)\right] \\ &= \lim_{k\to\infty} \left[h_{\mu_k}(\sigma) + qM_*(\mu_k)\right]. \end{aligned} \tag{2.3}$$

Since $\mathcal{M}(\Sigma_A, \sigma)$ is compact in the weak-star topology, there exists a subsequence $\{\mu_{k_i}\}$ of $\{\mu_k\}$ such that μ_{k_i} converges to some $\mu \in \mathcal{M}(\Sigma_A, \sigma)$.

By the upper semi-continuity of the measure-theoretic entropy on $M(\Sigma_A, \sigma)$ (cf. [14, Theorem 8.2]), we have

$$\limsup_{i\to\infty} h_{\mu_{k_i}}(\sigma) \leq h_\mu(\sigma). \tag{2.4}$$

On the other hand, by Corollary 2.4 we have

$$\lim_{i\to\infty} M_*(\mu_{k_i}) = M_*(\mu). \tag{2.5}$$

Combining (2.3)-(2.5) yields

$$P(q) \leq h_\mu(\sigma) + qM_*(\mu),$$

and thus $P(q) = h_\mu(\sigma) + qM_*(\mu)$ by Proposition 2.6. □

By Propositions 2.6-2.7, to finish the proof of Theorem 1.1, we only need to prove

Proposition 2.8. *For any $M \in \Gamma$ and $q > 0$, there exists $\mu \in \mathcal{M}(\Sigma_A, \sigma)$ such that*

$$P(q) = h_\mu(\sigma) + qM_*(\mu).$$

Now fix $M \in \Gamma$. For any $\epsilon > 0$, define a matrix-valued function M_ϵ on Σ_A by

$$M_\epsilon(x) = M(x) + \epsilon E,$$

where E is the $d \times d$ matrix of which each entry equals 1. It is clear that M_ϵ is continuous and strictly positive. Note that since $\|\pi_n M_\epsilon(x)\|$ is a polynomial of ϵ with continuous coefficients, we have

Lemma 2.9. *For a fixed $n \in \mathbb{N}$, there exist $a > 0$ and $\epsilon_0 > 0$ such that*

$$\|\pi_n M_\epsilon(x)\| \leq \|\pi_n M(x)\| + a\epsilon, \qquad \forall x \in \Sigma_A,\ \epsilon < \epsilon_0. \tag{2.6}$$

To prove Proposition 2.8, we still need the following simple lemma.

Lemma 2.10. *For any $q > 0$, $P(q) = \lim_{\epsilon \to 0} P(M_\epsilon, q)$.*

Proof. Fix $q > 0$. It is clear that $P(M_\epsilon, q) \geq P(q)$ for any $\epsilon > 0$. Let $\delta > 0$. By Lemma 2.2, there exists $n_0 \in \mathbb{N}$ such that

$$P(q) \geq \frac{1}{n_0} \log \sum_{J \in \Sigma_{A,n_0}} \sup_{x \in [J]} \|\pi_{n_0} M(x)\|^q - \delta.$$

Since

$$\lim_{\epsilon \to 0} \frac{1}{n_0} \log \sum_{J \in \Sigma_{A,n_0}} \sup_{x \in [J]} \|\pi_{n_0} M_\epsilon(x)\|^q = \frac{1}{n_0} \log \sum_{J \in \Sigma_{A,n_0}} \sup_{x \in [J]} \|\pi_{n_0} M(x)\|^q,$$

it follows from Lemma 2.2 that

$$\limsup_{\epsilon \to 0} P(M_\epsilon, q) \leq \lim_{\epsilon \to 0} \frac{1}{n_0} \log \sum_{J \in \Sigma_{A,n_0}} \sup_{x \in [J]} \|\pi_{n_0} M_\epsilon(x)\|^q \leq P(q) + \delta,$$

which implies the desired result. □

Proof of Proposition 2.8. Fix $q > 0$. For any $k \in \mathbb{N}$, the matrix-valued function $M_{1/k}$ is continuous and strictly positive. Therefore by Proposition 2.7, there exists $\mu_k \in \mathcal{M}(\Sigma_A, \sigma)$ such that

$$P(M_{1/k}, q) = h_{\mu_k}(\sigma) + q \left(M_{1/k}\right)_* (\mu_k). \tag{2.7}$$

Let $\{\mu_{k_i}\}$ be a weak-star convergent subsequence of $\{\mu_k\}$ and μ the limit point. We show below that $P(q) = h_\mu(\sigma) + qM_*(\mu)$. To see this we first show that

$$\limsup_{i \to \infty} \left(M_{1/k_i}\right)_* (\mu_{k_i}) \leq M_*(\mu). \tag{2.8}$$

Fix $n \in \mathbb{N}$. For any integer $N > 0$, define $g_N(x) = \max\{-N, \frac{1}{n} \log \|\pi_n M(x)\|\}$. By Lemma 2.9, for any $\delta > 0$, there exists i_0 (depending on N) such that

$$\frac{1}{n} \log \|\pi_n M_{1/k_i}(x)\| \leq g_N(x) + \delta, \qquad \forall x \in \Sigma_A,\ i \geq i_0.$$

Therefore

$$\left(M_{1/k_i}\right)_* (\mu_{k_i}) \le \int \frac{1}{n}\log\|\pi_n M_{1/k_i}(x)\| d\mu_{k_i}(x) \le \int g_N(x) d\mu_{k_i}(x) + \delta, \qquad \forall i \ge i_0.$$

Letting $i \to \infty$ and then $\delta \to 0$, we have

$$\limsup_{i\to\infty} \left(M_{1/k_i}\right)_* (\mu_{k_i}) \le \int g_N(x) d\mu(x), \ \forall N \in \mathbb{N}. \tag{2.9}$$

Note that $\{g_N(x)\}_{N\ge 1}$ is a sequence of continuous functions on Σ_A having a uniform upper bound. By Fatou theorem,

$$\limsup_{N\to\infty} \int g_N(x) d\mu(x) \le \int \limsup_{N\to\infty} g_N(x) d\mu(x) = \int \frac{1}{n} \log\|\pi_n M(x)\| d\mu(x).$$

This combining (2.9) yields

$$\limsup_{i\to\infty} \left(M_{1/k_i}\right)_* (\mu_{k_i}) \le \int \frac{1}{n}\log\|\pi_n M(x)\| d\mu(x), \qquad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, we obtain (2.8). By the upper semi-continuity of the measure-theoretic entropy on $M(\Sigma_A, \sigma)$, we have

$$\limsup_{i\to\infty} h_{\mu_{k_i}}(\sigma) \le h_\mu(\sigma). \tag{2.10}$$

Combining (2.10), (2.8) and (2.7) yields

$$P(q) = \lim_{k\to\infty} P(M_{1/k}, q) \le h_\mu(\sigma) + qM_*(\mu),$$

and thus $P(q) = h_\mu(\sigma) + qM_*(\mu)$ by Proposition 2.6. □

3. The proof of Theorem 1.2

In this section, we first give a proof of Theorem 1.2, then we give the existence result for the unique equilibrium state in some cases.

Proof of Theorem 1.2. First assume that M is a continuous function on Σ_A satisfying $P(q) \ne -\infty$ for all $q > 0$. Then $P(q)$ is a convex continuous function. Therefore $P'(q+)$ and $P'(q-)$ exist for any $q > 0$.

Fix $q > 0$. By Theorem 1.1, $\mathcal{I}(M, q) \ne \emptyset$. For any $\mu \in \mathcal{I}(M, q)$ and $\epsilon > 0$, we have

$$P(q+\epsilon) \ge h_\mu(\sigma) + (q+\epsilon)M_*(\mu), \quad P(q) = h_\mu(\sigma) + qM_*(\mu).$$

It follows that $P'(q+) \ge M_*(\mu)$ and thus

$$P'(q+) \ge \sup\{M_*(\mu) : \ \mu \in \mathcal{I}(M, q)\}. \tag{3.1}$$

Similarly we have

$$P'(q-) \leq \inf\{M_*(\mu): \ \mu \in \mathcal{I}(M,q)\}. \tag{3.2}$$

By (3.1) and (3.2), we know

$$M_*(\mu) = P'(q), \qquad \forall \mu \in \mathcal{I}(M,q) \tag{3.3}$$

if $P'(q)$ exists.

Since $P(\cdot)$ is convex, there exists a sequence of real numbers $q_k \downarrow q$ such that $P'(q_k)$ exist and $P'(q+) = \lim_{k\to\infty} P'(q_k)$. Take $\mu_k \in \mathcal{I}(M,q_k)$. Without loss of generality, we assume $\mu_k \to \mu$ in the weak-star topology. We claim that

$$\mu \in \mathcal{I}(M,q) \quad \text{and} \quad M_*(\mu) = \limsup_{k\to\infty} M_*(\mu_k). \tag{3.4}$$

To prove the claim, note that

$$\limsup_{k\to\infty} M_*(\mu_k) \leq \lim_{k\to\infty} \int \frac{1}{n} \log \|\pi_n M(x)\| d\mu_k(x) = \int \frac{1}{n} \log \|\pi_n M(x)\| d\mu(x)$$

for any $n \in \mathbb{N}$. Thus we have

$$\limsup_{k\to\infty} M_*(\mu_k) \leq M_*(\mu). \tag{3.5}$$

This combining $\limsup_{k\to\infty} h_{\mu_k}(\sigma) \leq h_\mu(\sigma)$ yields

$$P(q) = \lim_{k\to\infty} P(q_k) = \lim_{k\to\infty} (h_{\mu_k}(\sigma) + q_k M_*(\mu_k)) \leq h_\mu(\sigma) + qM_*(\mu),$$

which implies (3.4) (here we have used the positivity of q). By (3.3) and (3.4) we have

$$P'(q+) = \lim_{k\to\infty} P'(q_k) = \lim_{k\to\infty} M_*(\mu_k) = M_*(\mu),$$

which combining (3.1) yields (1.6). An analogous argument proves (1.7).

Now assume M is strictly positive. The above argument (in which we use (2.2) to replace (3.5)) can prove (1.6) and (1.7) for $q \leq 0$. □

In the following we give two cases for which M has a unique equilibrium state with respect to q. Recall we have mentioned in Theorem A the existence and uniqueness of Gibbs measures under the assumption that M is strictly positive and Hölder continuous. In [5], Feng and Lau also proved the existence and uniqueness of Gibbs measures when M is a function taking values in the set of non-negative $d\times d$ matrices satisfying the following assumptions:

(H1) $M(x) = M_i$ if $x \in [i]$, $i = 1, \cdots, m$;

(H2) M is irreducible in the following sense: there exists $r > 0$ such that for any $i, j \in \{1, 2, \ldots, m\}$,

$$\sum_{k=1}^{r} \sum_{K \in \Sigma_{A,k;i,j}} M_K > \mathbf{0} \tag{3.6}$$

where $\Sigma_{A,k;i,j}$ denotes the set of all $K \in \Sigma_{A,k}$ such that $iKj \in \Sigma_{A,k+2}$. And $M_K = M_{u_1} M_{u_2} \cdots M_{u_k}$ for $K = u_1 u_2 \cdots u_k$.

More precisely they proved

Theorem C. *Suppose M is a function on Σ_A taking values in the set of all $d \times d$ non-negative matrices and satisfies (H1) and (H2). Then for any $q > 0$, there is a unique Gibbs measure μ_q on Σ_A as in Theorem A.*

Now we can formulate our result about the existence of the unique equilibrium state:

Theorem 3.1. (i). *Suppose M satisfies the condition of Theorem A, then $\mathcal{I}(M, q)$ contains only one element for any $q \in \mathbb{R}$;*
(ii). *Suppose M satisfies the condition of Theorem C, then $\mathcal{I}(M, q)$ contains only one element for any $q > 0$.*

We remark that Theorem 3.1 follows from the uniqueness of Gibbs measures, by a proof much similar to that given in [1, Theorem 1.22] for showing the uniqueness of the equilibrium states in the Hölder continuous real-valued functions case. The only one slight modification is to replace $S_n\phi(x)$ therein by $\log \|\pi_n M(x)\|$.

Combining Corollary 1.3 and Theorem 3.1, we have

Corollary 3.2. (i) *Suppose M satisfies the condition of Theorem A, then $P'(q)$ exists for any $q \in \mathbb{R}$;*
(ii) *Suppose M satisfies the condition of Theorem C, then $P'(q)$ exists for any $q > 0$.*

We remark Corollary 3.2 (not including the existence of $P'(0)$ in (i)) was also proved in [5] by a different method.

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References

[1] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, *Lecture notes in Math.*, No. **470**, Springer-Verlag, 1975.

[2] K. J. Falconer, Fractal geometry. Mathematical foundations and applications. John Wiley & Sons, Chichester, 1990.

[3] D.-J. Feng, Lyapunov exponent for products of matrices and multifractal analysis. Part I: Positive matrices. *Israel J. Math.* (to appear).

[4] D.-J. Feng, Smoothness of the L^q-spectrum of self-similar measures with overlaps. *J. London Math. Soc.* **68** (2003), 102–118.

[5] D.-J. Feng and K.-S. Lau, The pressure function for products of non-negative matrices, *Math. Res. Lett.* **9** (2002), 363-378.

[6] D.-J. Feng and E. Olivier, Multifractal analysis of the weak Gibbs measures and phase transition– Application to some Bernoulli convolutions. *Ergodic Theory & Dynamical System* (to appear).

[7] H. Furstenberg and H. Kesten, Products of Random matrices, *Ann. Math. Stat.,* **31** (1960), 457-469.

[8] G. Keller, Equilibrium states in ergodic theory, Cambridge University Press, 1998.

[9] K.-S. Lau, Multifractal structure and product of matrices. Preprint.

[10] K.-S. Lau and S.-M. Ngai, Multifractal measures and a weak separation condition. *Adv. Math.* **141** (1999), 45–96.

[11] E. Olivier, Multifractal analysis in symbolic dynamics and distribution of pointwise dimension for g-measures. *Nonlinearity* **12** (1999), 1571–1585.

[12] Ya. B. Pesin, Dimension theory in dynamical systems. Contemporary views and applications. University of Chicago Press, 1997.

[13] D. Ruelle, Thermodynamic formalism. The mathematical structures of classical equilibrium statistical mechanics. Encyclopedia of Mathematics and its Applications, 5. Addison-Wesley Publishing Co., Reading, Mass., 1978.

[14] P. Walters, An introduction to ergodic theory, Springer-Verlag, 1982.

De-Jun Feng
Department of Mathematical Sciences
Tsinghua University
Beijing, 100084, P. R. China
Email: dfeng@math.tsinghua.edu.cn

Current Address:
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA
Email: djfeng@math.gatech.edu