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Exact packing measure of linear Cantor sets

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Let K be the attractor of a linear iterated function system $S_j x = \rho_j x + b_j$ $(j = 1, \ldots, m)$ on the real line satisfying the open set condition (where the open set is an interval). It is well known that the packing dimension of K is equal to α, the unique positive solution y of the equation $\sum_{j=1}^m \rho_j^y = 1$; and the α–dimensional packing measure $\mathcal{P}^\alpha(K)$ is finite and positive. Denote by μ the unique self–similar measure for the IFS $\{S_j\}_{j=1}^m$ with the probability weight $\{\rho_j^\alpha\}_{j=1}^m$. In this paper, we prove that $\mathcal{P}^\alpha(K)$ is equal to the reciprocal of the so–called "minimal centered density" of μ, and this yields an explicit formula of $\mathcal{P}^\alpha(K)$ in terms of the parameters ρ_j, b_j $(j = 1, \ldots, m)$. Our result implies that $\mathcal{P}^\alpha(K)$ depends continuously on the parameters whenever $\sum_j \rho_j < 1$.

1 Introduction

In this paper we deal with the exact computation of packing measures for a special kind of linear Cantor sets. Recall that a δ–packing of a given set $E \subset \mathbf{R}^n$ is a countable family of disjoint closed balls of radii at most δ and with centers in E. For $s \geq 0$, the s–dimensional *packing premeasure* of E is defined as

$$P^s(E) \;=\; \inf_{\delta > 0} \{P^s_\delta(E)\}\,,$$

where $P^s_\delta(E) = \sup\big\{\sum_{B_i \in \mathcal{R}} |B_i|^s : \mathcal{R} \text{ is a } \delta\text{–packing of } E\big\}$ and $|B_i|$ denotes the diameter of B_i. The s–dimensional *packing measure* of E is defined as

$$\mathcal{P}^s(E) \;=\; \inf\left\{\sum_{i=1}^\infty P^s(E_i) : \; E \subset \bigcup_{i=1}^\infty E_i\right\}.$$

The *packing dimension* of E is by definition the quantity

$$\dim_{\mathcal{P}}(E) \;:=\; \inf\big\{s \geq 0 : \; \mathcal{P}^s(E) = 0\big\} \;=\; \sup\big\{s \geq 0 : \; \mathcal{P}^s(E) = \infty\big\}\,.$$

The packing measure and packing dimension, introduced by Tricot [15], Taylor & Tricot [13, 14] and Sullivan [12], play an important role in the study of fractal geometry in a manner dual to the Hausdorff measure and Hausdorff dimension (see [9] and [4] for further properties of the above measures and dimensions). However, because of the difficulty in the definition there are few results about the explicit computation of packing measures for fractal sets. This is the motivation of this paper.

Let $S_j x = \rho_j x + b_j$ $(j = 1, \ldots, m)$ be a linear iterated function system (IFS for short) on the real line, with contraction ratios satisfying $0 < \rho_j < 1$. We assume the following form of the open set condition: there exists an open interval I such that $S_j I \subset I$ and $S_j I$ are disjoint. We remark that this open

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set condition is less general than the usual one defined by [7] (see [2] for an example). Without loss of generality, we take $I = (0, 1)$, and we assume that the images $S_j I$ are in increasing order, with $S_1(0) = 0$ and $S_m(1) = 1$. Define $l_j = S_{j+1}(0) - S_j(1)$ for $j = 1, 2, \dots, m-1$. Let K denote the attractor of the IFS (K is also called a *self–similar set*; see [7] for detailed properties). It is well known (see e. g. Theorem 2.7 of [5]) that the Hausdorff dimension and the packing dimension of K are both equal to α, where α satisfies

$$\sum_{j=1}^{m} \rho_j^\alpha \;=\; 1\,.$$

Moreover the α–dimensional Hausdorff measure $\mathcal{H}^\alpha(K)$ and α–dimensional packing measure $\mathcal{P}^\alpha(K)$ of K are both positive and finite. The problem of the exact computation of $\mathcal{H}^\alpha(K)$ has been independently studied by Marion [8], and Ayer & Strichartz [1]. In these papers, the exact value of $\mathcal{H}^\alpha(K)$ is obtained (under some additional hypothesis).

This paper is devoted to the exact computation of $\mathcal{P}^\alpha(K)$ (we always assume that $m \geq 2$ and $\alpha < 1$).

Denote by μ the unique probability measure satisfying the self–similar relation

$$\mu \;=\; \sum_{j=1}^{m} \rho_j^\alpha\, \mu \circ S_j^{-1}\,. \tag{1.1}$$

Then by the scaling property of $\mathcal{P}^\alpha$, $\mu = c\,\mathcal{P}^\alpha|_K$ for $c = 1/\mathcal{P}^\alpha(K)$. In this paper, we introduce the notion of *minimal centered density* of μ. For any closed interval J, let $d(J) = \mu(J)/|J|^\alpha$ be the (α, J)*–density* of μ. Then the minimal centered density of μ, denoted by $d_{\min}$, is defined by

$$d_{\min} \;=\; \inf\{d(J) :\ J \text{ a closed interval centered in } K \text{ with } J \subset [0,1]\}\,.$$

Using the density theorem of packing measure proved by Saint Raymond and Tricot (Corollary 7.2 of [10]; see also [9], p. 95), we show that $\mathcal{P}^\alpha(K)$ is equal to the reciprocal of $d_{\min}$. Hence, our main purpose is to determine the constant $d_{\min}$. For any $x \in \mathbf{R}$, let $\operatorname{dist}(x, K)$ denote the distance between x and K, that is

$$\operatorname{dist}(x, K) \;=\; \inf\{|x - y| :\ y \in K\}\,.$$

Now we can formulate the main result of this paper as follows.

Theorem 1.1 *With the above setting, we have* $\mathcal{P}^\alpha(K) = d_{\min}^{-1}$, *where*

$$d_{\min} \;=\; \begin{cases} \min\{2^{-\alpha}R_0, 2^{-\alpha}R_1\} & \textit{if} \quad m \;=\; 2\,,\\ \min\{2^{-\alpha}R_0, 2^{-\alpha}R_1, R_2\} & \textit{if} \quad m \;\geq\; 3\,, \end{cases}$$

and the constant R_0, R_1 *and* R_2 *are respectively defined by*

$$R_0 \;=\; \min_{2 \leq j \leq m} \frac{\sum_{k=1}^{j-1} \rho_k^\alpha}{|S_j(0)|^\alpha}\,;$$

$$R_1 \;=\; \min_{1 \leq j \leq m-1} \frac{\sum_{k=j+1}^{m} \rho_k^\alpha}{|1 - S_j(1)|^\alpha}\,;$$

$$R_2 \;=\; \min_{1 \leq j_1 < j_2 < m} \frac{\sum_{k=j_1+1}^{j_2} \rho_k^\alpha}{\left(S_{j_2+1}(0) - S_{j_1}(1) - 2\operatorname{dist}\left(\frac{S_{j_2+1}(0) + S_{j_1}(1)}{2}, K\right)\right)^\alpha}\,.$$

Let us give a simple example of the application of Theorem 1.1. For $0 < \beta < 1$, denote by C_β the attractor of the IFS $\left\{\frac{1-\beta}{2}\,x,\ \frac{1-\beta}{2}\,x + \frac{1+\beta}{2}\right\}$. The set C_β is called the β–center Cantor set, of packing dimension $\alpha(\beta) = \frac{\log 2}{-\log(1-\beta)/2}$. It is well–known that the $\alpha(\beta)$–dimensional Hausdorff measure of C_β is equal to 1 (cf. Theorem 7.1 of [8], or Theorem 4.2 of [1]). By Theorem 1.1, one gets that $\mathcal{P}^{\alpha(\beta)}\big(C_\beta\big) = \big(\frac{2+2\beta}{1-\beta}\big)^{\alpha(\beta)}$.

It was proved in [1] that the α–dimensional Hausdorff measure $\mathcal{H}^\alpha(K)$ does not depend continuously on the parameters ρ_j and b_j $(j = 1, \ldots, m)$. Notice that Theorem 1.1 implies that $\mathcal{P}^\alpha(K)$ depends continuously on these parameters.

If we admit the parameters ρ_j to be negative, it seems hard to get a uniform formula for $\mathcal{P}^\alpha(K)$. However, in some special cases (for example, $\rho_1\rho_m > 0$ and $l_j > 0,\ l_{j+1} > 0$ for some $1 \le j \le m-2$), it can be proved by using a similar method that the result of Theorem 1.1 still remains true $\big($in which $S_j(0),\ S_j(1)$ are replaced by $u_j,\ v_j$ respectively for each j, with $S_j(I) = [u_j, v_j]\big)$.

This paper is organized as follows: in Section 2 we consider the pointwise lower α–density of μ. We prove that it is equal to $d_{\min}$ for μ almost all $x \in \mathbf{R}$, which implies $\mathcal{P}^\alpha(K) = d_{\min}^{-1}$. In Section 3 we give the explicit computation of $d_{\min}$, which yields the proof of Theorem 1.1.

2 The pointwise lower α–densities of μ

In this section we will consider the pointwise lower α–densities of μ. For a given measure ν on $\mathbf{R}$ and $x \in \mathbf{R}$, the lower α–density of ν at x is defined by

$$\Theta_*^\alpha(\nu, x) \ := \ \liminf_{r\to 0} \nu([x-r, x+r])/(2r)^\alpha\,.$$

The upper α–density $\Theta^{*\alpha}(\nu, x)$ is defined similarly by taking the upper limit. We have the following result:

Theorem 2.1 *For μ almost all $x \in \mathbf{R}$, $\Theta_*^\alpha(\mu, x) = d_{\min}$, where $d_{\min}$ is the minimal centered density of μ.*

The analogue fact that the upper α–densities are μ–almost all equal to a constant was first proved by Salli [11] in a more general setting. And the fact for the lower α–densities can now also be derived from the so–called tangential measure (see [6]).

For the convenience of the readers, we would like to give a direct and elementary proof of Thereom 2.1, based on the following lemma which follows immediately from (1.1):

Lemma 2.2 *Let $J \subset [0,1]$ be a closed interval, then for any $1 \le j \le m$, the interval $S_j(J)$ has the same density as J, that is $d\big(S_j(J)\big) = d(J)$. In other words, if J' is a subinterval of $S_j([0,1])$ for some $1 \le j \le m$, then $d(J') = d\big(S_j^{-1}(J')\big)$.*

P r o o f of Theorem 2.1. By the definition of $d_{\min}$, we have $\Theta_*^\alpha(\mu, x) \ge d_{\min}$ for all $x \in K$. Hence $\Theta_*^\alpha(\mu, x) \ge d_{\min}$ for μ almost all $x \in \mathbf{R}$ since μ is supported on K.

In what follows we prove the reverse inequality. It suffices to prove for any fixed $\epsilon > 0$,

$$\Theta_*^\alpha(\mu, x) \ \le \ d_{\min} + \epsilon \quad \text{for} \ \ \mu \ \ \text{a.a.} \ \ x\,.$$

By the definition of $d_{\min}$, there exist $x_0 \in K$ and $0 < r_0 < 1/2$ such that $[x_0 - r_0, x_0 + r_0] \subset (0,1)$ and $d([x_0 - r_0, x_0 + r_0]) < d_{\min} + \epsilon/2$. Thus there exist $n_0 \in N$ and $t_1, t_2, \ldots, t_{n_0} \in \{1, \ldots, m\}$ such that

$$[y - r_0, y + r_0] \subset (0,1)\,, \quad d([y - r_0, y + r_0]) \ \le \ d_{\min} + \epsilon \tag{2.1}$$

for any $y \in S_T([0,1]) := S_{t_1}S_{t_2}\ldots S_{t_{n_0}}([0,1])$. We define for any $p \in N$,

$$A_p \ = \ \bigcup_{n=p}^{\infty} \ \bigcup_{j_1,\ldots,j_n \in\{1,\ldots,m\}} S_{j_1}\ldots S_{j_n}(S_T([0,1]))\,. \tag{2.2}$$

Then we have the following statements:

(i) For each $x \in A_p$, there exists r (depending on x) such that $0 < r \le \rho_{\max}^p$ and $d([x-r, x+r]) \le d_{\min} + \epsilon$, where $\rho_{\max} = \max\{\rho_j :\ 1 \le j \le m\}$;

(ii) $\mu\big(A_p\big) = 1$ for all $p \in N$.

To prove (i), take any $x \in A_p$. Then $x \in S_{j_1} \dots S_{j_n}(S_T([0,1]))$ for some $n \geq p$ and $j_1, \dots, j_n \in \{1, \dots, m\}$. Take $y = \big(S_{j_1} \dots S_{j_n}\big)^{-1}(x)$. Then $y \in S_T([0,1])$. By Lemma 2.2 and (2.1),

$$\begin{aligned} d\big([x - \rho_{j_1 \dots} \rho_{j_n} r_0, x + \rho_{j_1 \dots} \rho_{j_n} r_0]\big) &= d\big(S_{j_1} \dots S_{j_n}\big([y - r_0, y + r_0]\big)\big) = d\big([y - r_0, y + r_0]\big) \\ &\leq d_{\min} + \epsilon\,, \end{aligned}$$

which proves (i). Now let us turn to the proof of (ii). By (2.2) we have

$$A_p = \bigcup_{j_1, \dots, j_p \in \{1, \dots, m\}} S_{j_1} \dots S_{j_p}(A)\,,$$

where

$$A = \bigcup_{n=0}^{\infty} \bigcup_{j_1, \dots, j_n \in \{1, \dots, m\}} S_{j_1} \dots S_{j_n}(S_T([0,1]))\,.$$

By (1.1), we have

$$\mu(A_p) = \sum_{j_1, \dots, j_p \in \{1, \dots, m\}} \rho_{j_1}^{\alpha} \dots \rho_{j_p}^{\alpha} \mu(A) = \mu(A)\,.$$

Thus we only need to prove $\mu(A) = 1$. Define $B_0 = S_T([0,1])$. For any integer $k \geq 1$, we define

$$B_k = \bigcup_{j_1 \dots j_{kn_0} \in \mathcal{F}_k} S_{j_1} \dots S_{j_{kn_0}}(S_T([0,1]))\,,$$

where $\mathcal{F}_k := \Big\{ j_1 \dots j_{kn_0} \in \{1, \dots, m\}^{kn_0} : \; j_{sn_0+1} \dots j_{(s+1)n_0} \neq t_1 t_2 \dots t_{n_0} \text{ for } 0 \leq s \leq k-1 \Big\}$. This definition implies that the sets B_k $(k \geq 0)$ have no overlap (more precisely, $B_k \cap B_{k'}$ consists of at most finitely many points for $k \neq k'$), and $\mu(B_k) = \big(1 - \rho_{t_1}^{\alpha} \dots \rho_{t_{n_0}}^{\alpha}\big)^k \rho_{t_1}^{\alpha} \dots \rho_{t_{n_0}}^{\alpha}$. Hence

$$\mu(A) \geq \mu\left(\bigcup_{k \geq 0} B_k\right) = \sum_{k \geq 0} \mu(B_k) = \sum_{k \geq 0} \big(1 - \rho_{t_1}^{\alpha} \dots \rho_{t_{n_0}}^{\alpha}\big)^k \rho_{t_1}^{\alpha} \dots \rho_{t_{n_0}}^{\alpha} = 1\,,$$

and thus $\mu(A) = 1$, which implies (ii).

Now take $E = \bigcap_{p=1}^{\infty} A_p$. By (i), we have $\Theta_*^{\alpha}(\mu, x) \leq d_{\min} + \epsilon$ for any $x \in E$. By (ii) we have $\mu(E) = 1$. This proves the proposition. □

Denote by $\mathcal{P}^{\alpha}|_K$ the restriction of the α–dimensional packing measure on K, that is, $\mathcal{P}^{\alpha}|_K(A) = \mathcal{P}^{\alpha}(A \cap K)$ for any Borel set $A \subset \mathbf{R}$. Since $\mu = c\,\mathcal{P}^{\alpha}|_K$ with $c = 1/\mathcal{P}^{\alpha}(k)$, we have

$$\Theta_*^{\alpha}(\mu, x) = \frac{1}{\mathcal{P}^{\alpha}(K)} \Theta_*^{\alpha}(\mathcal{P}^{\alpha}|_K, x)\,, \quad \text{for all} \quad x \in \mathbf{R}\,. \tag{2.3}$$

As a corollary of Theorem 2.1, we have

Corollary 2.3 $\mathcal{P}^{\alpha}(K) = d_{\min}^{-1}$.

Proof. By (2.3), $\Theta_*^{\alpha}(\mathcal{P}^{\alpha}|_K, x) = \mathcal{P}^{\alpha}(K)\Theta_*^{\alpha}(\mu, x)$ for any $x \in \mathbf{R}$. Thus by Theorem 2.1, $\Theta_*^{\alpha}(\mathcal{P}^{\alpha}|_K, x) = \mathcal{P}^{\alpha}(K) d_{\min}$ for $\mathcal{P}^{\alpha}|_K$ almost all $x \in \mathbf{R}$. However, by using the lower density theorem proved by Saint Raymond and Tricot (Corollary 7.2 of [10]; see also [9], p. 95), we have $\Theta_*^{\alpha}(\mathcal{P}^{\alpha}|_K, x) = 1$ for $\mathcal{P}^{\alpha}|_K$ almost all $x \in \mathbf{R}$, which implies the desired result. □

3 The exact computation of $d_{\min}$

At first, we consider the minimal value of (α, J)–densities of μ when J is of the form $[0,x]$ or $[y,1]$. For this purpose, define

$$D_0 \;=\; \inf\{d([0,x]) :\; 0 < x \le 1\}\,, \quad D_1 \;=\; \inf\{d([y,1]) :\; 0 \le y < 1\}\,. \tag{3.1}$$

We would like to characterize D_0, D_1 by the parameters ρ_j and l_j $(j = 1, \ldots, m)$.

By Lemma 2.2, $d([0,x]) = d\big([0, \rho_1^{-1}x]\big)$ for $0 < x \le \rho_1$. Thus to determine the exact value of D_0, we only need to consider the interval $[0,x]$ with $\rho_1 < x \le 1$. Since $d([0,x])$ is a continuous function of x, $d([0,x])$ attains its minimum D_0 at some $x_0 \in [\rho_1, 1]$; furthermore we can assume $x_0 > \rho_1$ (noting $d([0,1]) = D_0$ whenever $d([0,\rho_1]) = D_0$). Then we have the following result about x_0:

Lemma 3.1 $x_0 \in \big\{S_j(0) : 2 \le j \le m\big\}$.

Proof. It is clear that x_0 is not contained in the interior of the set $[0,1]\setminus\bigcup_{j=1}^m S_j([0,1])$. Therefore $x_0 \in S_j([0,1])$ for some $j \ge 2$. Fix this j and take $u = x_0 - S_j(0)$. It suffices to show $u = 0$. Assume that $u > 0$. Then

$$\begin{aligned} d([0,x_0]) &= \frac{\mu([0,S_j(0)]) + \mu([S_j(0), S_j(0)+u])}{(S_j(0)+u)^\alpha} \\ &> \frac{\mu([0,S_j(0)]) + \mu([S_j(0), S_j(0)+u])}{(S_j(0))^\alpha + u^\alpha} \\ &\ge \min\left\{\frac{\mu([0,S_j(0)])}{(S_j(0))^\alpha}, \frac{\mu([S_j(0), S_j(0)+u])}{u^\alpha}\right\} \\ &= \min\{d([0,S_j(0)]),\, d([S_j(0), S_j(0)+u])\} \\ &= \min\big\{d([0,S_j(0)]),\, d\big([0, S_j^{-1}(u)]\big)\big\}\,, \end{aligned}$$

which contradicts the minimality of $d([0,x_0])$. This completes the proof. □

As a corollary, we have

Corollary 3.2 *Let D_0 be defined as in* (3.1)*, then*

$$D_0 \;=\; \min_{2\le j\le m} d\big([0,S_j(0)]\big) \;=\; \min_{2\le j\le m} \frac{\sum_{k=1}^{j-1}\rho_k^\alpha}{|S_j(0)|^\alpha}\,.$$

Considering D_1 dually, we have

Corollary 3.3 *Let D_1 be defined as in* (3.1)*, then*

$$D_1 \;=\; \min_{1\le j\le m-1} d\big([S_j(1),1]\big) \;=\; \min_{1\le j\le m-1} \frac{\sum_{k=j+1}^{m}\rho_k^\alpha}{|1-S_j(1)|^\alpha}\,.$$

Lemma 3.4 $d_{\min} \le \min\big\{2^{-\alpha}D_0,\, 2^{-\alpha}D_1\big\}$.

Proof. It suffices to show that there exist intervals I_0, I_1 centered in K with $I_j \subset [0,1]$ $(j = 0,1)$ such that $d(I_0) = 2^{-\alpha}D_0$ and $d(I_1) = 2^{-\alpha}D_1$. For simplicity, we only prove the first equality. By Corollary 3.2, there exists $x_0 \in \{S_j(0) : 2 \le j \le m\}$ such that $d([0,x_0]) = D_0$. Since $\alpha < 1$, there exists $1 \le i \le m-1$ with $l_i > 0$, where $l_i = S_{i+1}(0) - S_i(1)$. Fix this i and take a positive integer k large enough so that $\rho_1^k x_0 < l_i$. Then the interval $\big[S_{i+1}(0) - \rho_{i+1}\rho_1^k x_0, S_{i+1}(0)\big]$ is contained in $[S_i(1), S_{i+1}(0)]$ and thus $\mu\big(\big[S_{i+1}(0) - \rho_{i+1}\rho_1^k x_0, S_{i+1}(0)\big]\big) = 0$. Define $I_0 := \big[S_{i+1}(0) - \rho_{i+1}\ \rho_1^k x_0, S_{i+1}(0) + \rho_{i+1}\rho_1^k x_0\big]$. We have

$$\begin{aligned} d(I_0) &= \frac{\mu\big(\big[S_{i+1}(0), S_{i+1}(0) + \rho_{i+1}\rho_1^k x_0\big]\big)}{2^\alpha\big(\rho_{i+1}\rho_1^k x_0\big)^\alpha} = \frac{\mu\big(S_{i+1}\circ S_1^k([0,x_0])\big)}{2^\alpha\big|S_{i+1}\circ S_1^k([0,x_0])\big|^\alpha} \\ &= 2^{-\alpha} d\big(S_{i+1}\circ S_1^k([0,x_0])\big) \;=\; 2^{-\alpha} d([0,x_0]) \;=\; 2^{-\alpha} D_0\,, \end{aligned}$$

which concludes the proof. □

Define $\mathcal{E} = \big\{$closed interval $J \subset [0,1]$ centered in K: $J \not\subset S_j([0,1])$, $\; j = 1,\ldots,m\big\}$. By Lemma 2.2, we have

$$d_{\min} \;=\; \inf\{d(J) : \; J \in \mathcal{E}\}\,. \tag{3.2}$$

Lemma 3.5 *If $J \in \mathcal{E}$ satisfies $d(J) < \min\{2^{-\alpha}D_0, 2^{-\alpha}D_1\}$, then*

$$|J| \;>\; \min\{\rho_{\min}, l_{\min}\}\,,$$

where $\rho_{\min} = \min\{\rho_j : \; 1 \le j \le m\}$, $\; l_{\min} = \min\big\{l_j : \; 1 \le j \le m-1, \; l_j \ne 0\big\}$ and $|J|$ denotes the length of J.

P r o o f. It suffices to show $d(J) \ge \min\{2^{-\alpha}D_0,\, 2^{-\alpha}D_1\}$ under the assumption that $J \in \mathcal{F}_1$ and $|J| \le \min\{\rho_{\min},\, l_{\min}\}$. It is clear that under this assumption there are only three possible cases for J :

(i) There exists j such that $S_j([0,1])$ and $S_{j+1}([0,1])$ are touching, and $J = J_1 \cup J_2$, where $J_1 \subset S_j([0,1])$ and $J_2 \subset S_{j+1}([0,1])$.

(ii) There exists j such that $S_j([0,1])$ and $S_{j+1}([0,1])$ are separate, and $J = J_1 \cup J_2$, where $J_1 \subset S_j([0,1])$ and $J_2 \subset [S_j(1), S_{j+1}(0)]$.

(iii) There exists j such that $S_j([0,1])$ and $S_{j+1}([0,1])$ are separate, and $J = J_1 \cup J_2$, where $J_1 \subset \big[S_j(1), S_{j+1}(0)\big]$ and $J_2 \subset S_{j+1}([0,1])$.

In case (i), we have

$$\begin{aligned} d(J) \;&=\; \frac{\mu(J_1)+\mu(J_2)}{(|J_1|+|J_2|)^\alpha} \;>\; \frac{\mu(J_1)+\mu(J_2)}{|J_1|^\alpha+|J_2|^\alpha} \;\ge\; \min\{d(J_1),\, d(J_2)\} \\ &=\; \min\big\{d\big(S_j^{-1}(J_1)\big), d\big(S_{j+1}^{-1}(J_2)\big)\big\}\,. \end{aligned}$$

Since $d\big(S_j^{-1}(J_1)\big)$, $\; d\big(S_{j+1}^{-1}(J_2)\big)$ are of the forms $[\,y,1]$ and $[0,x]$ respectively, it follows that $d(J) > \min\{D_0, D_1\}$.

In case (ii), we have $|J_1| \ge |J_2|$ since J is centered in K. Thus

$$d(J) \;=\; \frac{\mu(J_1)}{(|J_1|+|J_2|)^\alpha} \;\ge\; \frac{\mu(J_1)}{(2|J_1|)^\alpha} \;=\; 2^{-\alpha}d(J_1) \;=\; 2^{-\alpha}d\big(S_j^{-1}(J_1)\big)\,,$$

which implies $d(J) \ge 2^{-\alpha}D_1$ since $d\big(S_j^{-1}(J_1)\big)$ is of the forms $[\,y,1]$.

In case (iii), we have $d(J) \ge 2^{-\alpha}D_0$ by a discussion similar to (ii).

Combining the above discussions completes the proof. □

Proposition 3.6 *Assume $d_{\min} < \min\big\{2^{-\alpha}D_0,\, 2^{-\alpha}D_1\big\}$. Then $m \ge 3$ and*

$$d_{\min} \;=\; \min_{1 \le l_1 < l_2 < m} \frac{\sum_{k=l_1+1}^{l_2} \rho_k^\alpha}{\Big(S_{l_2+1}(0) - S_{l_1}(1) - 2\,\mathrm{dist}\Big(\frac{S_{l_2+1}(0)+S_{l_1}(1)}{2}, K\Big)\Big)^\alpha}\,.$$

P r o o f. Since $d_{\min} < \min\{2^{-\alpha}D_0,\, 2^{-\alpha}D_1\}$, by (3.2) and Lemma 3.5 we have

$$d_{\min} \;=\; \inf\{d(J) : \; J \in \mathcal{E},\; |J| > \min\{\rho_{\min}, l_{\min}\}\}\;.$$

By the compactness of K, there exists $J_0 \in \mathcal{F}$ with $|J_0| \ge \min\{\rho_{\min},\, l_{\min}\}$ such that

$$d_{\min} \;=\; d(J_0)\,.$$

By Lemma 2.2, we may assume $J_0 = [a_0, b_0] \in \mathcal{E}$. Denote $c_0 = (a_0+b_0)/2$, then $c_0 \in K$. For convenience, we call each interval $\big[S_j(1), S_{j+1}(0)\big]$ $(1 \le j \le m-1)$ a *lake*. First we prove the following statements:

(i) either $a_0 \in \{S_j(1) : 1 \le j \le m-1\}$ or a_0 is contained in the interior of one lake;

(ii) either $b_0 \in \{S_j(0) : 2 \le j \le m\}$ or b_0 is contained in the interior of one lake;

For simplicity we only prove (i). The statement (ii) follows by a similar argument. Assume that (i) is not true, then $a_0 \in \big[S_j(0), S_j(1)\big)$ for some $1 \le j \le m-1$. Fix this j. In the following we will lead to a contradiction. We first claim that $c_0 \notin S_j([0,1])$, i.e., $c_0 > S_j(1)$. If $c_0 \in S_j([0,1])$, then

$$\begin{aligned} d([a_0,b_0]) &= \frac{\mu([a_0,b_0])}{(b_0-a_0)^\alpha} \ge \frac{\mu([a_0,S_j(1)])}{(b_0-a_0)^\alpha} \ge \frac{\mu([a_0,S_j(1)])}{2^\alpha(S_j(1)-a_0)^\alpha} = 2^{-\alpha} d\big(S_j^{-1}([a_0,S_j(1)])\big) \\ &\ge 2^{-\alpha} D_1\,, \end{aligned}$$

which contradicts the assumption $d_{\min} < \min\{2^{-\alpha}D_0,\, 2^{-\alpha}D_1\}$. It follows that

$$\begin{aligned} d([a_0,b_0]) &= \frac{\mu([a_0,S_j(1)]) + \mu([S_j(1), 2c_0 - S_j(1)]) + \mu([2c_0 - S_j(1), b_0])}{\big(2\,|S_j(1)-a_0| + |2c_0 - 2S_j(1)|\big)^\alpha} \\ &> \frac{\mu([a_0,S_j(1)]) + \mu([S_j(1), 2c_0 - S_j(1)]) + \mu([2c_0 - S_j(1), b_0])}{\big(2\,|S_j(1)-a_0|\big)^\alpha + \big(|2c_0 - 2S_j(1)|\big)^\alpha} \\ &\ge \frac{\mu([a_0,S_j(1)]) + \mu([S_j(1), 2c_0 - S_j(1)])}{\big(2\,|S_j(1)-a_0|\big)^\alpha + \big(|2c_0 - 2S_j(1)|\big)^\alpha} \\ &\ge \min\left\{ \frac{\mu([a_0,S_j(1)])}{\big(2\,|S_j(1)-a_0|\big)^\alpha}, \frac{\mu\big([S_j(1), 2c_0 - S_j(1)]\big)}{\big(|2c_0 - 2S_j(1)|\big)^\alpha} \right\} \\ &= \min\big\{2^{-\alpha} d\big(S_j^{-1}\big([a_0, S_j(1)]\big),\ d\big([S_j(1), 2c_0 - S_j(1)]\big)\big\} \\ &\ge \min\big\{2^{-\alpha}D_1,\ d\big([S_j(1), 2c_0 - S_j(1)]\big)\big\}\,. \end{aligned}$$

Since $[S_j(1), 2c_0 - S_j(1)]$ is a subinterval of $[0,1]$ centered in K, the above inequality contradicts the facts that $d([a_0,b_0])$ attains the minimum value $d_{\min}$ $(< 2^{-\alpha}D_1)$. Thus the statement (i) is true.

By the statements (i) and (ii), we have

$$a_0 \in \big[S_{l_1}(1), S_{l_1+1}(0)\big)\,, \quad b_0 \in (S_{l_2}(1), S_{l_2+1}(0)] \tag{3.3}$$

for some $1 \le l_1 < l_2 < m$, which implies $m \ge 3$ immediately. Since $\mu([a_0,b_0]) = \sum_{k=l_1+1}^{l_2} \rho_k^\alpha$ and $d([a_0,b_0])$ attains to the minimum, it follows that a_0, b_0 are taken such that $b_0 - a_0$ is the largest under the conditions (3.3) and $(a_0+b_0)/2 \in K$. Thus we have

$$b_0 - a_0 = S_{l_2+1}(0) - S_{l_1}(1) - 2\,\mathrm{dist}\left(\frac{S_{l_2+1}(0) + S_{l_1}(1)}{2}, K\right)$$

and

$$d([a_0,b_0]) = \frac{\sum_{k=l_1+1}^{l_2} \rho_k^\alpha}{\Big(S_{l_2+1}(0) - S_{l_1}(1) - 2\,\mathrm{dist}\Big(\frac{S_{l_2+1}(0)+S_{l_1}(1)}{2}, K\Big)\Big)^\alpha}\,. \tag{3.4}$$

Therefore we complete the proof of Proposition 3.6. □

Proof of Theorem 1.1. It follows immediately from Proposition 2.3, Lemma 3.4, and Proposition 3.6. □

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