THE SMOOTHNESS OF L^q-SPECTRUM OF SELF-SIMILAR MEASURES WITH OVERLAPS

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Abstract. Let μ be the self-similar measure for a linear function system $S_jx = \rho x + b_j$ $(j = 1, 2, \dots, m)$ on the real line with the probability weight $\{p_j\}_{j=1}^m$. Under the condition that $\{S_j\}_{j=1}^m$ satisfies the finite type condition, the L^q-spectrum $\tau(q)$ of μ is shown to be differentiable on $(0, \infty)$; as an application, μ is exact dimensional and satisfies the multifractal formalism.

1. Introduction

Let ν be a finite Borel measure on $\mathbb{R}^n$ with compact support. For $q \in \mathbb{R}$, the L^q-spectrum of ν is defined by

$$\tau(q) = \tau(\nu, q) = \liminf_{\delta\downarrow 0} \frac{\log\left(\sup \sum_i \nu(B_\delta(x_i))^q\right)}{\log \delta},$$

where the supremum is taken over all the families of disjoint balls $B_\delta(x_i)$ of radius δ and center $x_i \in \mathrm{supp}(\nu)$.

The L^q-spectrum of a measure is one of the basic ingredients in the study of multifractal phenomena. It is well known that if μ is the self-similar measure defined by a family of contractive similitudes $\{S_j\}_{j=1}^m$ which satisfies the *open set condition* [8], $\tau(q)$ can be calculated by an explicit formula and it is analytic on $\mathbb{R}$ ([2, 18]). Moreover, the Legender transform of $\tau(q)$ (i.e., $\tau^*(\alpha) = \inf\{q\alpha - \tau(q) : \ q \in \mathbb{R}\}$) equals the Hausdorff dimension of the set

$$K(\alpha) = \Big\{x \in \mathrm{supp}(\mu) : \ \lim_{\delta\to 0} \frac{\log \mu(B_\delta(x))}{\log \delta} = \alpha\Big\}.$$

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The relationship between $\tau(q)$ and the dimension of $K(\alpha)$ as above is the well-known *multifractal formalism*. One may refer to [3, 18, 21, 22] for some further properties of L^q-spectrum and multifractal formalism.

Following the terminology of Barnsley [1], we call the above family of contractive similitudes $\{S_j\}_{j=1}^m$ an *iterated function system* (IFS). If the family does not satisfy the open set condition, it is much harder to obtain a formula for $\tau(q)$ and it is not known whether the multifractal formalism will hold in general. Nevertheless, Lau and Ngai proved in [13] that the multifractal formalism holds if the IFS $\{S_j\}_{j=1}^m$ satisfies the *weak separation condition* and in the mean time $\tau(q)$ is differentiable for $0 < q < \infty$. That is, $\dim_H K(\alpha) = \tau^*(\alpha)$ for any $\alpha = \tau'(q)$ with $q > 0$. The weak separation condition is strictly weaker than the open set condition and is satisfied by many interesting overlap cases. A question arises naturally whether or not $\tau(q)$ is differentiable on $(0, \infty)$ for every self-similar measure. To our best knowledge, except for a few examples (e.g. [12, 14, 23]), there is no general theorem to guarantee the differentiability of $\tau(q)$ for self-similar measures with overlaps.

In this paper, we provide a rigorous proof of the smoothness of $\tau(q)$ on $(0, \infty)$ for a class of self-similar measures with overlaps. We say that a family of similitudes

$$S_j(x) = \rho x + b_j, \quad 0 < \rho < 1, \quad b_j \in \mathbb{R}, \quad j = 1, \ldots, m$$

satisfies the *finite type condition* if there is a finite set Γ such that for each integer $n > 0$ and any two indices $J = j_1 \ldots j_n$ and $J' = j'_1 \ldots j'_n$,

$$\text{either} \qquad \rho^{-n}|S_J(0) - S_{J'}(0)| > c \qquad \text{or} \qquad \rho^{-n}|S_J(0) - S_{J'}(0)| \in \Gamma, \tag{1.1}$$

where S_J denotes the composition $S_{j_1} \circ \ldots \circ S_{j_n}$ and

$$c = (1 - \rho)^{-1}(\max_{1 \le j \le m} b_j - \min_{1 \le i \le m} b_i).$$

Denotes by K the self-similar set generated by $\{S_j\}_{j=1}^m$ (see [8]). It is not hard to see $c = \operatorname{diam}(K)$. The finite type condition defined under the present setting is equivalent to the more general definition introduced in [16] where the contraction ratios ρ_j can be different for different S_j and the domain of S_j is $\mathbb{R}^d$. It was proved by Nguyen [17] that an IFS of finite type always satisfies the weak separation condition.

Under the finite type condition, the Hausdorff dimension of K has been studied in [7, 9, 16, 24, 26, 28].

For a given probability weight $\{p_j\}_{j=1}^m$, it is well known (see [8]) that there is a unique one probability measure μ on $\mathbb{R}$ satisfying the relation

$$\mu = \sum_{j=1}^{m} p_j \mu \circ S_j^{-1}. \tag{1.2}$$

This measure is often called the *self-similar measure* generated by $\{S_i\}_{i=1}^m$.

Now we can formulate our main results as follows:

Theorem 1.1. *Let μ be the self-similar measure on $\mathbb{R}$ generated by an IFS $S_j x = \rho x + b_j$ $(j = 1, 2 \ldots, m)$ satisfying the finite type condition with the probability weight $\{p_j\}_{j=1}^m$. Then the L^q-spectrum $\tau(q)$ of μ is differentiable on $(0, \infty)$.*

This combining with the result of Lau and Ngai (Theorem B of [13]) or a recent result of Feng and Lau (Theorem 3.4 of [5]) yields immediately

Theorem 1.2. *Under the condition of Theorem 1.1, the multifractal formalism holds for μ. That is,*

$$\dim_H K(\alpha) = \inf\{\alpha t - \tau(t) : \ t \in \mathbb{R}\} = \alpha q - \tau(q), \qquad \forall \alpha = \tau'(q) \textit{ for } q > 0.$$

Recall a Borel measure μ on $\mathbb{R}^n$ is called *exact dimensional* (or more precisely d exact dimensional) if there exists a constant d such that

$$\lim_{\delta \to 0} \frac{\log \mu(B_\delta(x))}{\log \delta} = d$$

for μ almost all $x \in \mathbb{R}^n$. In [15] Ngai proved that if μ is a compactly supported probability Borel measure on $\mathbb{R}^n$ and the L^q spectrum of μ is differentiable at 1, then μ is exact dimensional and

$$\dim_H \mu = \dim_e \mu = \tau'(1),$$

where $\dim_H \mu$ denotes the Hausdorff dimension of μ and $\dim_e \mu$ denotes the entropy dimension of μ (see [22, 27] for the definitions of different dimensions of a measure). The above result has also been obtained (in generalized form) by Heurteaux [6] and Olsen [19]. This combining with Theorem 1.1 yields

Theorem 1.3. *Under the condition of Theorem 1.1, μ is exact dimensional with $\dim_H \mu = \dim_e \mu = \tau'(1)$.*

The above results can be applied directly to the *classical Bernoulli convolutions* associated with Pisot numbers. Let μ be the self-similar measure generated by

$$S_1(x) = \rho x, \qquad S_2(x) = \rho x + (1 - \rho)$$

with probability weight $\{1/2, 1/2\}$, where $1/2 < \rho < 1$. Such measures are known as the classical Bernoulli convolutions and have been studied for a long time (see [11, 20] and references therein). It is known that if ρ^{-1} is a Pisot number, the corresponding maps $\{S_1,\ S_2\}$ satisfies the finite type condition (see e.g. [7, 16]). Recall that $\beta > 1$ is called a *Pisot number* if β is an algebraic integer and all its conjugates have moduli less than 1.

Theorem 1.3 generalizes a result of Lalley. In [10] Lalley showed that the Bernoulli convolutions associated with Pisot numbers are exact dimensional, and $\dim_H \mu$ can be expressed as the top Lyapunov exponent of certain random matrix products.

We remark that under the condition of Theorem 1.1, the function $\tau(q)$ may be not differentiable for some $q < 0$. In [4], the author gave a complete explicit formula of $\tau(q)$ ($q \in \mathbb{R}$) for the Bernoulli convolution associated with $\rho = \frac{\sqrt{5}-1}{2}$ and showed that $\tau(q)$ is not differentiable at one point $q_0 < 0$.

Let us give a brief description of our idea in the proof of Theorem 1.1. First we define a family of so-called basic net intervals which has a net structure. Using the finite type condition, we construct a symbolic space with finite states and a family of transition matrices (maybe not squared) on these states, so that each basic net interval can be identified as an admissible string in the symbolic space, and the distribution of the measure μ (written in a vector form) on each basic net interval can be expressed as a product of these matrices. Using an additional technique, we construct a family of non-negative squared matrices so that their sum is irreducible, and the measure μ can be re-expressed as a product of these squared matrices on a subclass of basic net intervals. In this way we can show that $\tau(q) = \frac{P(q)}{-\log \rho}$ for $q > 0$, where $P(q)$ is the pressure function for these squared matrices (see Section 5 for the definition). A recent result of Feng and Lau [5] shows that $P(q)$ is always differentiable on $(0, \infty)$ under the irreducible condition. This leads to our differential result for $\tau(q)$.

The method used above for constructing the symbolic space and corresponding transition matrices extends an idea in [4], and it is different from that of Lalley in [10]. In fact, it seems hard to set up completely the relationship between $\tau(q)$ and the pressure function for the matrices derived from Lalley's method.

We organize the paper as follows. In Section 2, we study the structure of basic net intervals and give the symbolic expressions (i.e. Markov strings in a subshift space) for them. In section 3, we express the distribution of μ (written in a vector form) on each basic net interval as a product of some non-negative matrices (maybe not squared). In Section 4, we re-express it as a product of some squared matrices, and prove the irreducibility of the sum of these matrices. In Section 5, we set up the relationship between $\tau(q)$ and $P(q)$, which completes the proof of Theorem 1.1.

2. Basic net intervals and their symbolic expressions

Let $S_j x = \rho x + b_j$ $(j = 1, 2, \ldots, m)$ be an IFS satisfying the finite type condition and μ the self-similar measure generated by $\{S_j\}_{j=1}^m$ with the probability weight $\{p_j\}_{j=1}^m$. Without loss of generality, here and afterwards we always assume

$$0 = b_1 < b_2 < \ldots < b_m = 1 - \rho.$$

Under this assumption, the convex hull of K is just the interval $[0, 1]$, where K is the self-similar set generated by $\{S_j\}_{j=1}^m$. And also the constant c in (1.1) equals 1. In what follows we will define basic net intervals and their symbolic expressions.

Write $\mathcal{A} = \{1, \ldots, m\}$. For $n > 0$ denote by $\mathcal{A}_n$ the collection of all indices $j_1 \ldots j_n$ of length n over $\mathcal{A}$. We define two families of sets P_n^0, P_n^1 $(n \geq 0)$ in the following way: $P_0^0 = \{0\}$, $P_0^1 = \{1\}$, and $P_n^0 = \{S_\sigma(0) : \ \sigma \in \mathcal{A}_n\}$, $P_n^1 = \{S_\sigma(1) : \ \sigma \in \mathcal{A}_n\}$ for $n \geq 1$. Define $P_n = P_n^0 \bigcup P_n^1$ for $n \geq 0$. Let $h_1, \ldots, h_{s_n}$ be all the elements of P_n ranked in the increasing order. Define

$$\mathcal{F}_n = \{[h_j, h_{j+1}] : \ 1 \leq j < s_n, \ (h_j, h_{j+1}) \cap K \neq \emptyset\}.$$

Each element in $\mathcal{F}_n$ is called a *n-th basic net interval.*

The following facts about basic net intervals can be checked easily: (i) $\bigcup_{\Delta\in\mathcal{F}_n}\Delta\supset K$ for any $n\geq 0$; (ii) For any $\Delta_1,\Delta_2\in\mathcal{F}_n$ with $\Delta_1\neq\Delta_2$, $\mathrm{int}(\Delta_1)\cap\mathrm{int}(\Delta_2)=\emptyset$; (iii) For any $\Delta\in\mathcal{F}_n$ $(n\geq 1)$, there is a unique element $\widehat{\Delta}\in\mathcal{F}_{n-1}$ such that $\widehat{\Delta}\supset\Delta$.

For each $\Delta=[a,b]\in\mathcal{F}_n$ $(n\geq 0)$, we will define a positive number $\ell_n(\Delta)$, a vector $V_n(\Delta)$ and a positive integer $r_n(\Delta)$. If $\Delta=[0,1]\in\mathcal{F}_0$, we define $\ell_0(\Delta)=1$, $V_0(\Delta)=0$ and $r_0(\Delta)=1$. Otherwise for $n\geq 1$, we define $\ell_n(\Delta)$ and $V_n(\Delta)$ directly by

$$\ell_n(\Delta)=\rho^{-n}(b-a)$$

and

$$V_n(\Delta)=(a_1,\ldots,a_k).$$

where $a_1,\ldots,a_k$ (ranked in the increasing order) are all the element of the following set

$$\{\rho^{-n}(a-S_\sigma(0)):\ \sigma\in\mathcal{A}_n,\ S_\sigma(K)\cap(a,b)\neq\emptyset\}.$$

Denote by $v_n(\Delta)$ the dimension of $V_n(\Delta)$, that is, $v_n(\Delta)=k$. We define $r_n(\Delta)$ in the following way: let $\widehat{\Delta}$ be the unique one interval in $\mathcal{F}_{n-1}$ containing Δ, and $\Delta_1,\ldots,\Delta_k$ (ranked in the increasing order) be all the elements in $\mathcal{F}_n$ satisfying $\Delta_j\subset\widehat{\Delta}$, $\ell_n(\Delta_j)=\ell_n(\Delta)$, $V_n(\Delta_j)=V_n(\Delta)$ for $1\leq j\leq k$. Define $r_n(\Delta)$ to be the integer r so that $\Delta_r=\Delta$.

For convenience, we call the triple

$$\mathcal{C}_n(\Delta):=(\ell_n(\Delta),V_n(\Delta),r_n(\Delta))$$

the *n-th characteristic vector* of Δ, or simply *characteristic vector* of Δ. The vector $\mathcal{C}_n(\Delta)$ contains the information about the length and neighborhood relation of Δ. The following elementary but important fact is our start point.

Lemma 2.1. *For a given $\Delta\in\mathcal{F}_n(n\geq 0)$, let $\Delta_1,\ldots,\Delta_k$ (ranked in the increasing order) be all the elements in $\mathcal{F}_{n+1}$ which are subintervals of Δ. Then the number k, the vectors $\mathcal{C}_{n+1}(\Delta_i)$ $(1\leq i\leq k)$ are determined by $\ell_n(\Delta)$ and $V_n(\Delta)$ (thus they are determined by $\mathcal{C}_n(\Delta)$).*

Proof. Let $\Delta=[a,b]\in\mathcal{F}_n$. Write $V_n(\Delta)=(a_1,\ldots,a_{v_n(\Delta)})$.

To determine the subintervals of Δ which belong to $\mathcal{F}_{n+1}$, we first determine the points in $[a,b] \cap P_{n+1}$. Assume $\sigma = j_1 \dots j_{n+1} \in \mathcal{A}_{n+1}$ such that $S_\sigma(0)$ or $S_\sigma(1)$ belongs to the interval (a,b). Then $S_\sigma(K) \cap (a,b) \neq \emptyset$, and consequently $S_{\hat{\sigma}}(K) \cap (a,b) \neq \emptyset$, where $\hat{\sigma} = j_1 \dots j_n \in \mathcal{A}_n$. Hence $S_{\hat{\sigma}}(0) \in \{a - \rho^n a_i : \ 1 \leq i \leq v_n(\Delta)\}$ and therefore

$$S_\sigma(0) \in \{a - \rho^n a_i + \rho^n b_s : \ 1 \leq i \leq v_n(\Delta), \ 1 \leq s \leq m\}$$

and

$$S_\sigma(1) \in \left\{a - \rho^n a_i + \rho^n b_s + \rho^{n+1} : \ 1 \leq i \leq v_n(\Delta), \ 1 \leq s \leq m\right\}.$$

This implies that

$$\begin{aligned}(a,b) \cap P_{n+1} &= (a, a + \rho^n \ell_n(\Delta)) \\ &\quad \cap \left\{a - \rho^n a_i + \rho^n b_s + \epsilon \rho^{n+1} : \ 1 \leq i \leq v_n(\Delta), \ 1 \leq s \leq m, \ \epsilon = 0 \text{ or } 1\right\}.\end{aligned}$$

Denote by $a + \rho^n c_j$ $(1 \leq j \leq u)$ all the elements of $[a,b] \cap P_{n+1}$ ranked in the increasing order. The above equality shows that the points c_j $(1 \leq j \leq u)$ are determined completely by $\ell_n(\Delta)$ and $V_n(\Delta)$ (independent of a and n).

Let $\Delta_1, \dots, \Delta_k$ (ranked in the increasing order) be all the elements in $\mathcal{F}_{n+1}$ which are subintervals of Δ. Then Δ_i $(1 \leq i \leq k)$ are exact the intervals in the following collection:

$$\{[a + \rho^n c_j, a + \rho^n c_{j+1}] : \ 1 \leq j \leq u-1, \ (a + \rho^n c_j, a + \rho^n c_{j+1}) \cap K \neq \emptyset\}.$$

Note that for a given j,

$$\begin{aligned} & (a + \rho^n c_j, a + \rho^n c_{j+1}) \cap K \neq \emptyset \\ \Longleftrightarrow \quad & (a + \rho^n c_j, a + \rho^n c_{j+1}) \cap \left(\bigcup_{\sigma \in \mathcal{A}_n : \ S_\sigma(K) \cap (a,b) \neq \emptyset} S_\sigma(K) \right) \neq \emptyset \\ \Longleftrightarrow \quad & (a + \rho^n c_j, a + \rho^n c_{j+1}) \cap \left(\bigcup_{i=1}^{v_n(\Delta)} (\rho^n K + a - \rho^n a_i) \right) \neq \emptyset \\ \Longleftrightarrow \quad & (c_j, c_{j+1}) \cap \left(\bigcup_{i=1}^{v_n(\Delta)} (K - a_i) \right) \neq \emptyset \end{aligned}$$

It implies that whether or not $[a+\rho^n c_j, a+\rho^n c_{j+1}]$ is a $(n+1)$-th basic net interval is determined by $\ell_n(\Delta)$ and $V_n(\Delta)$. Therefore if we write $\Delta_i = [a+\rho^n d_i, a+\rho^n d_{i+1}]$ $(i = 1, \ldots, k)$, then d_i $(1 \le i \le k+1)$ are determined by $\ell_n(\Delta)$ and $V_n(\Delta)$.

Recall that

$$\begin{aligned} & \{S_\sigma(0) : \sigma \in \mathcal{A}_{n+1},\ S_\sigma(K) \cap (a,b) \ne \emptyset\} \\ \subset \ & \{a - \rho^n a_i + \rho^n b_s :\ 1 \le i \le v_n(\Delta),\ 1 \le s \le m\}. \end{aligned}$$

By the definition of characteristic vector and the analysis in the preceding paragraph, we know that the vectors $\mathcal{C}_{n+1}(\Delta_i)$ $(1 \le i \le k)$ are determined by $\ell_n(\Delta)$ and $V_n(\Delta)$. □

In the following we would like to use a finite sequence of characteristic vectors to identify a basic net interval. For each $\Delta \in \mathcal{F}_n$ $(n \ge 0)$, we list the intervals

$$\Delta^0, \Delta^1, \ldots, \Delta^n$$

such that $\Delta^n = \Delta$, and Δ^j $(j = 0, \ldots, n-1)$ is the unique element in $\mathcal{F}_j$ such that $\Delta^j \supset \Delta^{j+1}$. The sequence

$$\mathcal{C}_0(\Delta^0), \mathcal{C}_1(\Delta^1), \ldots, \mathcal{C}_n(\Delta^n)$$

is called the *symbolic expression* for Δ.

For a given $\Delta \in \mathcal{F}_n (n \ge 0)$, let $\Delta_1, \ldots, \Delta_k$ (ranked in the increasing order) be all the elements in $\mathcal{F}_{n+1}$ which are subintervals of Δ. The introduction of the third term in a characteristic vector guarantees that $C_{n+1}(\Delta_j)$ $(1 \le j \le k)$ are distinct with each other. By induction, we have

Lemma 2.2. *For any $\Delta_1, \Delta_2 \in \mathcal{F}_n (n \ge 1)$ with $\Delta_1 \ne \Delta_2$, the symbolic expression of Δ_1 is different from that of Δ_2.* □

Define

$$\Omega = \{\mathcal{C}_n(\Delta) :\ n \ge 0,\ \Delta \in \mathcal{F}_n\}. \tag{2.1}$$

For any $\alpha \in \Omega$, we write for simplicity

$$\ell(\alpha) = \ell_n(\Delta),\ V(\alpha) = V_n(\Delta),\ v(\alpha) = v_n(\Delta),\ r(\alpha) = r_n(\Delta), \tag{2.2}$$

if $\Delta \in \mathcal{F}_n$ and $\mathcal{C}_n(\Delta) = \alpha$.

The finite type condition of $\{S_i\}_{i=1}^m$ implies

Lemma 2.3. *The set Ω is finite.*

Proof. It suffices to prove the finiteness of $\{\ell_n(\Delta) :\ n \geq 0,\ \Delta \in \mathcal{F}_n\}$, $\{V_n(\Delta) :\ n \geq 0,\ \Delta \in \mathcal{F}_n\}$ and $\{r_n(\Delta) :\ n \geq 0,\ \Delta \in \mathcal{F}_n\}$ respectively. For simplicity, we only prove that of $\{V_n(\Delta) :\ n \geq 0,\ \Delta \in \mathcal{F}_n\}$. To prove this, take any $\Delta = [a, b] \in \mathcal{F}_n$ and $e \in V_n(\Delta)$. There exists $\sigma \in \mathcal{A}_n$ such that $S_\sigma(K) \cap (a, b) \neq \emptyset$ and $e = \rho^{-n}(a - S_\sigma(0))$. By the definition of basic net intervals, $S_\sigma(0) \notin (a, b)$. Therefore $a - \rho^n \leq S_\sigma(0) \leq a$. It follows that $e \in \Gamma$ whenever $a \in P_n^0$, and $1 - e \in \Gamma$ whenever $a \in P_n(1)$, where Γ is defined as in (1.1). By the finiteness of Γ, the set $\{V_n(\Delta) :\ n \geq 0,\ \Delta \in \mathcal{F}_n\}$ is finite. $\square$

Now we are going to define a natural map ζ from Ω to Ω^*, where Ω^* denotes the collection of all finite words over Ω. For any $\alpha \in \Omega$, pick n and $\Delta \in \mathcal{F}_n$ such that $\alpha = \mathcal{C}_n(\Delta)$. Let $\Delta_1, \ldots, \Delta_k$ (ranked in the increasing order) be all the elements in $\mathcal{F}_{n+1}$ which are subintervals of Δ. Write $\alpha_j = \mathcal{C}_{n+1}(\Delta_j)$ for $1 \leq j \leq k$. By Lemma 2.1, the word $\alpha_1 \ldots \alpha_k$ depend only on α (independent of the choice of n and Δ). We define ζ by

$$\zeta(\alpha) = \alpha_1 \ldots \alpha_k.$$

Define a 0-1 matrix A on $\Omega \times \Omega$ in the following way:

$$A_{\alpha,\beta} = \begin{cases} 1 & \text{if } \beta \text{ is a letter of } \zeta(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

A word $\beta_1 \ldots \beta_n \in \Omega^*$ is called a admissible word if $A_{\beta_j,\beta_{j+1}} = 1$ for $1 \leq j < n$.

For our convenience, denote by $\gamma_0 = \mathcal{C}_0([0, 1])$. Combining Lemma 2.2 and the above definitions, we have

Lemma 2.4. *Any $\Delta \in \mathcal{F}_n (n \geq 0)$ can be identified (via its symbolic expression) as an admissible word in Ω^* of length $n + 1$ starting from the letter γ_0.* $\square$

3. The distribution of μ on basic net intervals

In this section, we will analyze the distribution of μ on basic net intervals. We construct a family of non-negative matrices (maybe not squared), such that the distribution of μ (written in a vector form) on any basic net interval can be expressed as a product of these matrices.

Let $\Delta = [a, b]$ be a n-th basic net interval. Iterating (1.2) n times we obtain

$$\mu(\Delta) = \sum_{\sigma \in \mathcal{A}_n} p_\sigma \mu(S_\sigma^{-1}(\Delta)),$$

where p_σ denotes the product $p_{j_1} \dots p_{j_n}$ for $\sigma = j_1 \dots j_n$. Since μ is a non-atomic measure supported on K, we have

$$\mu(\Delta) \;=\; \sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(K) \cap (a,b) \neq \emptyset} p_\sigma \mu(S_\sigma^{-1}(\Delta)). \tag{3.1}$$

Write $V_n(\Delta) = (a_1, \dots, a_{v_n(\Delta)})$. By the definition of $V_n(\Delta)$, we can rewrite (3.1) as

$$\begin{aligned} \mu(\Delta) &= \sum_{i=1}^{v_n(\Delta)} \sum_{\sigma \in \mathcal{A}_n:\ \rho^{-n}(a - S_\sigma(0)) = a_i} p_\sigma \mu(S_\sigma^{-1}(\Delta)) \\ &= \sum_{i=1}^{v_n(\Delta)} \mu([a_i, a_i + \ell_n(\Delta)]) \sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_i} p_\sigma. \end{aligned} \tag{3.2}$$

Now we define a $v_n(\Delta)$-dimensional row vector $Q_n(\Delta) = (q_1, \dots, q_{v_n(\Delta)})$ by

$$q_i = \mu([a_i, a_i + \ell_n(\Delta)]) \sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_i} p_\sigma, \qquad i = 1, \dots, v_n(\Delta). \tag{3.3}$$

By (3.2), $\mu(\Delta) = \|Q_n(\Delta)\| := \sum_{i=1}^{v_n(\Delta)} q_i$. We call $Q_n(\Delta)$ the *vector form* of μ on Δ.

Lemma 3.1. *$Q_n(\Delta)$ is a positive $v_n(\Delta)$-dimensional vector for any $n \geq 0$ and $\Delta \in \mathcal{F}_n$.*

Proof. Let q_i be defined as in (3.3). It suffices to prove $q_i > 0$ for any $1 \leq i \leq v_n(\Delta)$. For any given i, since there exists a $\delta \in \mathcal{A}_n$ so that $S_\delta(0) = a - \rho^n a_i$ and $S_\delta^{-1}(a, b) \cap K \neq \emptyset$, it follows that $\sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_i} p_\sigma \geq p_\delta > 0$, and $\mu([a_i, a_i + \ell_n(\Delta)]) = \mu(S_\delta^{-1}(a, b)) > 0$. Thus $q_i > 0$. $\square$

The following result is essential in our analysis.

Lemma 3.2. *For any $\Delta \in \mathcal{F}_n$ $(n \geq 1)$, denote by $\widehat{\Delta}$ the unique element in $\mathcal{F}_{n-1}$ so that $\widehat{\Delta} \supset \Delta$. There is a $v_{n-1}(\widehat{\Delta}) \times v_n(\Delta)$ matrix $T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta))$ which depends only on $\mathcal{C}_{n-1}(\widehat{\Delta})$ and $\mathcal{C}_n(\Delta)$ such that*

$$Q_n(\Delta) = Q_{n-1}(\widehat{\Delta}) T(\mathcal{C}_{n-1}(\widehat{\Delta}), \mathcal{C}_n(\Delta)).$$

Proof. Assume $\Delta = [a, b]$ and $\widehat{\Delta} = [c, d]$. Write $V_n(\Delta) = (a_1, \ldots, a_{v_n(\Delta)})$ and $V_{n-1}(\widehat{\Delta}) = (c_1, \ldots, c_{v_{n-1}(\widehat{\Delta})})$. Also Write $Q_n(\Delta) = (q_1, \ldots, q_{v_n(\Delta)})$ and $Q_{n-1}(\widehat{\Delta}) = (u_1, \ldots, u_{v_{n-1}(\widehat{\Delta})})$. By the definition of $Q_n(\Delta)$ and $Q_{n-1}(\widehat{\Delta})$,

$$q_i = \mu([a_i, a_i + \ell_n(\Delta)]) \sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_i} p_\sigma, \qquad i = 1, \ldots, v_n(\Delta),$$

and

$$u_j = \mu([c_j, c_j + \ell_{n-1}(\widehat{\Delta})]) \sum_{\sigma' \in \mathcal{A}_{n-1}:\ S_{\sigma'}(0) = c - \rho^{n-1} c_j} p_{\sigma'}, \qquad j = 1, \ldots, v_{n-1}(\widehat{\Delta}).$$

For $\sigma = i_1 \ldots i_n \in \mathcal{A}_n$, write $\hat{\sigma} = i_1 \ldots i_{n-1}$. By the definition of basic net intervals, we see that if $S_\sigma(0) = a - \rho^n a_i$ for some i, then

$$S_{\hat{\sigma}}(0) \in \left\{ c - \rho^{n-1} c_j :\ 1 \leq j \leq v_{n-1}(\widehat{\Delta}) \right\}.$$

Now define for any $i \in \{1, \ldots, v_n(\Delta)\}$ and $j \in \{1, \ldots, v_{n-1}(\widehat{\Delta})\}$,

$$w_{j,i} = \begin{cases} p_s & \exists s \in \mathcal{A} \text{ so that } c - \rho^{n-1} c_j + \rho^{n-1} b_s = a - \rho^n a_i, \\ 0 & \text{otherwise.} \end{cases}$$

That is $w_{j,i} = p_s$ if and only if there is $\sigma = i_1 \ldots i_n \in \mathcal{A}_n$ with $i_n = s$ such that $S_\sigma(0) = a - \rho^n a_i$ and $S_{i_1 \ldots i_{n-1}}(0) = c - \rho^{n-1} c_j$. Therefore

$$\sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_i} p_\sigma = \sum_{j=1}^{v_{n-1}(\widehat{\Delta})} w_{j,i} \sum_{\sigma' \in \mathcal{A}_{n-1}:\ S_{\sigma'}(0) = c - \rho^{n-1} c_j} p_{\sigma'}, \qquad i = 1, \ldots, v_n(\Delta).$$

Define a $v_{n-1}(\widehat{\Delta}) \times v_n(\Delta)$ matrix $T = (t_{j,i})$ by

$$t_{j,i} = \frac{w_{j,i} \mu\left([a_i, a_i + \ell_n(\Delta)]\right)}{\mu\left([c_j, c_j + \ell_{n-1}(\widehat{\Delta})]\right)}, \qquad 1 \leq j \leq v_{n-1}(\widehat{\Delta}),\ 1 \leq i \leq v_n(\Delta).$$

We have

$$Q_n(\Delta) = Q_{n-1}(\widehat{\Delta}) T.$$

Since $\rho^{-n}(c-a)$ depends only on $\mathcal{C}_{n-1}(\widehat{\Delta})$ and $\mathcal{C}_n(\Delta)$, so does $(w_{j,i})$. Thus T depends only on $\mathcal{C}_{n-1}(\widehat{\Delta})$ and $\mathcal{C}_n(\Delta)$. This completes the proof. □

The above result, together with the fact $Q_0([0,1]) = 1$, yields immediately

Theorem 3.3. *There exists a family of non-negative matrices* $\{T(\alpha,\beta):\ \alpha,\beta \in \Omega,\ A_{\alpha,\beta} = 1\}$, *such that for any* $\Delta \in \mathcal{F}_n$,

$$Q_n(\Delta) = T(\gamma_0,\gamma_1)\dots T(\gamma_{n-1},\gamma_n),$$

where $\gamma_0\dots\gamma_n$ *is the symbolic expression of* Δ. □

As a corollary of Theorem 3.3 and Lemma 3.1, we have

Corollary 3.4. *Suppose* $\alpha_1\alpha_2\dots\alpha_n$ *is an admissible word in* Ω^*. *Denote by* $\mathbf{e}(\alpha_1)$ *the* $v(\alpha_1)$*-dimensional row vector of which each coordinate equals* 1. *Then*

$$\mathbf{e}(\alpha_1)T(\alpha_1,\alpha_2)T(\alpha_2,\alpha_3)\dots T(\alpha_{n-1},\alpha_n) \tag{3.4}$$

is a positive $v(\alpha_n)$*-dimensional row vector.*

Proof. Since $\alpha_1\alpha_2\dots\alpha_n$ is an admissible word in Ω^*, there exists $\gamma_1,\dots,\gamma_l$ such that

$$\gamma_0\gamma_1\dots\gamma_l\alpha_1\alpha_2\dots\alpha_n$$

is an admissible word in Ω^* starting from γ_0. Therefore by Lemma 2.4 there is $\Delta \in \mathcal{F}_{n+l}$ such that the symbolic expression of Δ is $\gamma_0\gamma_1\dots\gamma_l\alpha_1\alpha_2\dots\alpha_n$. By Theorem 3.3 and Lemma 3.1,

$$T(\gamma_0,\gamma_1)\dots T(\gamma_{l-1},\gamma_l)T(\gamma_l,\alpha_1)T(\alpha_1,\alpha_2)\dots T(\alpha_{n-1},\alpha_n)$$

is a positive $v(\alpha_n)$-dimensional row vector, which implies that (3.4) is positive. □

By the construction of the matrices $T(\alpha,\beta)$, we can express precisely the entries of the product $T(\alpha_1,\alpha_2)\dots T(\alpha_{n-1},\alpha_n)$ for a given admissible word $\alpha_1\dots\alpha_n$. To see this, choose $t \in \mathbb{N}$ and $\Delta = [a,b] \in \mathcal{F}_t$ so that $\mathcal{C}_t(\Delta) = \alpha_1$. Assume that the symbolic expression of Δ is $\gamma_0\dots\gamma_{t-1}\alpha_1$. By Lemma 2.4, there is a unique one $\Delta' = [e,f] \in \mathcal{F}_{t+n-1}$ whose symbolic expression is $\gamma_0\dots\gamma_{t-1}\alpha_1\dots\alpha_n$. Write $V_t(\Delta) = (a_1,\dots,a_{v_t(\Delta)})$ and $V_{t+n-1}(\Delta') = (e_1,\dots,e_{v_{t+n-1}(\Delta')})$. Denote for simplicity

$X = T(\alpha_1, \alpha_2) \ldots T(\alpha_{n-1}, \alpha_n)$. Then from the construction of $T(\alpha, \beta)$, we have by induction that

Proposition 3.5. *For any* $1 \le j \le v_t(\Delta)$*, and* $1 \le i \le v_{t+n-1}(\Delta')$*,*

$$X_{j,i} = \frac{\mu([e_i, e_i + \ell_{t+n-1}(\Delta')]}{\mu([a_j, a_j + \ell_t(\Delta)])} \cdot \sum_{\xi \in \mathcal{A}_{n-1}:\ a - \rho^t a_j + \rho^t S_\xi(0) = e - \rho^{t+n-1} e_i} p_\xi.$$

□

4. Products of squared matrices

Let Ω be the set defined as in (2.1). A non-empty subset $\widehat{\Omega}$ of Ω is said to be an *essential class* of Ω if it satisfies: (i) $\{\beta \in \Omega :\ A_{\alpha,\beta} = 1\} \subset \widehat{\Omega}$ for any $\alpha \in \widehat{\Omega}$; (ii) for any $\alpha, \beta \in \widehat{\Omega}$, there exist $\gamma_1, \ldots, \gamma_n \in \widehat{\Omega}$ such that $\gamma_1 = \alpha$, $\gamma_n = \beta$ and $A_{\gamma_i,\gamma_{i+1}} = 1$ for $1 \le i \le n-1$. The existence of at least one essential class is well known (see, e.g. Lemma 1.1 of [25]).

Now fix an essential class $\widehat{\Omega}$ of Ω. Let $\eta_1, \ldots, \eta_s$ be all the elements in $\widehat{\Omega}$. Set

$$d = \sum_{i=1}^{s} v(\eta_i),$$

where $v(\cdot)$ is defined as in (2.2). In the following we construct a family of $d \times d$ matrices $\{M_i\}_{i=1}^s$. For any $1 \le i \le s$, define M_i to be the partitioned matrix

$$\begin{bmatrix} U^i_{1,1} & U^i_{1,2} & \cdots & U^i_{1,s} \\ U^i_{2,1} & U^i_{2,2} & \cdots & U^i_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ U^i_{s,1} & U^i_{s,2} & \cdots & U^i_{s,s} \end{bmatrix},$$

where for each $1 \le j, k \le s$, $U^i_{j,k}$ is a $v(\eta_j) \times v(\eta_k)$ matrix defined by

$$U^i_{j,k} = \begin{cases} T(\eta_j, \eta_i) & \text{if } k = i \text{ and } A_{\eta_j,\eta_i} = 1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Choose an integer n_0 and $I_0 \in \mathcal{F}_{n_0}$ so that $\mathcal{C}_{n_0}(I_0) = \eta_1$. Denote by $\Theta = \gamma_0 \ldots \gamma_{n_0-1}\eta_1$ the symbolic expression of I_0. In the following we consider the distribution of μ on basic net intervals which are contained in I_0.

Given $\Delta \in \mathcal{F}_n$ $(n \geq n_0)$ with $\Delta \subset I_0$, define $\widehat{Q}_n(\Delta)$ to be the partitioned vector $(W_1, \dots, W_s)$, where W_i is a $v(\eta_i)$-dimensional row vector defined by

$$W_i = \begin{cases} Q_n(\Delta) & \text{if } \eta_i = \mathcal{C}_n(\Delta), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

It is clear that $\widehat{Q}_n(\Delta)$ is a d-dimensional row vector, which we call the *uniform vector form* of μ on Δ. By Lemma 3.2, Theorem 3.3 and the product formula of partitioned matrices, we have

Lemma 4.1. *(1) Given $\Delta \in \mathcal{F}_{n_0+k}$ $(k \geq 1)$ with $\Delta \subset I_0$, we have*

$$\widehat{Q}_{n_0+k}(\Delta) = \widehat{Q}_{n_0}(I_0) M_{i_1} \dots M_{i_k},$$

where $\Theta\eta_{i_1} \dots \eta_{i_k}$ is the symbolic expression of Δ.

(2) $M_{i_1} \dots M_{i_k} \neq \mathbf{0}$ if and only if $\eta_{i_1} \dots \eta_{i_k}$ is an admissible sequence. $\square$

In the remain part of this section, we will prove the following proposition, which is needed in our proof of Theorem 1.1.

Proposition 4.2. *The matrix $H := \sum_{i=1}^{s} M_i$ is irreducible. That is, there exists an integer $r > 0$ such that $H^r > \mathbf{0}$.*

The proof of the above result is based on several lemmas.

Let $\{T(\alpha, \beta) : \alpha, \beta \in \Omega,\ A_{\alpha,\beta} = 1\}$ be the family of matrices we constructed in Section 3. By the definition of the matrices M_i $(1 \leq i \leq s)$ and the product formula of partitioned matrices, we have immediately

Lemma 4.3. *Given an admissible word $\eta_{i_1} \dots \eta_{i_n}$ with $n \geq 2$, write the matrix $M_{i_2} \dots M_{i_n}$ in the form of the partioned matrix $\left(U_{i,j}\right)_{1 \leq i,j \leq s}$, where $U_{i,j}$ is a $v(\eta_i) \times v(\eta_j)$ matrix. Then $U_{i_1,i_n} = T(i_1, i_2) \dots T(i_{n-1}, i_n)$.* $\square$

The following lemma is a key part for the proof of Proposition 4.2.

Lemma 4.4. *Given $i \in \{1, \dots, s\}$ and $k \in \{1, \dots, v(\eta_i)\}$, for each $j \in \{1, \dots, s\}$ there exists an admissible sequence $\eta_{i_1}\eta_{i_2} \dots \eta_{i_n}$ such that $\eta_{i_1} = \eta_i$, $\eta_{i_n} = \eta_j$ and all the entries of the k-th row of the matrix $T(\eta_{i_1}, \eta_{i_2}) \dots T(\eta_{i_{n-1}}, \eta_{i_n})$ are positive.*

Proof. Suppose i, j, k are given. Choose $n > n_0$ and $\Delta = [a, b] \in \mathcal{F}_n$ so that $\Delta \subset I_0$ and $\mathcal{C}_n(\Delta) = \eta_i$.

Write $V_n(\Delta) = (a_1, \ldots, a_{v_n(\Delta)})$. By the definition of $V_n(\Delta)$, there exists $\sigma \in \mathcal{A}_n$ with $S_\sigma(0) = a - \rho^n a_k$ and $S_\sigma(K) \cap (a, b) \neq \emptyset$. Find a large integer l and $\phi \in \mathcal{A}_l$ so that $S_{\sigma\phi}(K) \subset (a, b)$ and thus $S_{\sigma\phi}([0, 1]) \subset (a, b)$, where $\sigma\phi$ denotes the concatenation of σ and ϕ.

Pick $i_0 \in \{1, \ldots, s\}$ such that (i) $\ell(\eta_{i_0}) = \min\{\ell(\eta_u) : \ 1 \leq u \leq s\}$; (ii) $v(\eta_{i_0}) = \max\{v(\eta_u) : \ 1 \leq u \leq s, \ell(u) = \ell(\eta_{i_0})\}$, where $v(\cdot)$ and $\ell(\cdot)$ are defined as in (2.2). Choose $n_1 \in \mathbb{N}$ and $\Delta_1 = [c, d] \in \mathcal{F}_{n_1}$ so that $\Delta_1 \subset I_0$ and $\mathcal{C}_{n_1}(\Delta_1) = \eta_{i_0}$. Write $V_{n_1}(\Delta_1) = (c_1, \ldots, c_{v_{n_1}(\Delta_1)})$.

Denote $\Delta_2 = S_{\sigma\phi}(\Delta_1)$. It is clear $\Delta_2 \subset (a, b)$ since $S_{\sigma\phi}([0, 1]) \subset (a, b)$. We claim that $\Delta_2 \in \mathcal{F}_{n+l+n_1}$ with $V_{n+l+n_1}(\Delta_2) = V_{n_1}(\Delta_1)$ and $\ell_{n+l+n_1}(\Delta_2) = \ell_{n_1}(\Delta_1)$. First we show $\Delta_2 \in \mathcal{F}_{n+l+n_1}$ and $\ell_{n+l+n_1}(\Delta_2) = \ell_{n_1}(\Delta_1)$. To see this, we observe that the two endpoints of Δ_2 belong to the set P_{n+l+n_1} since those of Δ_1 belong to P_{n_1}; and $\Delta_2 \cap K \neq \emptyset$ by $\Delta_1 \cap K \neq \emptyset$. Therefore, Δ_2 contains at least an elements in $\mathcal{F}_{n+l+n_1}$. On the other hand the minimality of $\ell(\eta_{i_0})$ shows that each $(n+l+n_1)$-th basic net interval contained in I_0 has length at least $\rho^{n+l+n_1}\ell(\eta_{i_0})$, i.e., the length of Δ_2. Combining these two facts we have $\Delta_2 \in \mathcal{F}_{n+l+n_1}$ and $\ell_{n+l+n_1}(\Delta_2) = \ell(\eta_{i_0}) = \ell_{n_1}(\Delta_1)$. To show $V_{n+l+n_1}(\Delta_2) = V_{n_1}(\Delta_1)$, by the maximum of $v(\eta_{i_0})$ it suffices to show each coordinate of the vector $V_{n_1}(\Delta_1)$ is a coordinate of $V_{n+l+n_1}(\Delta_2)$. To prove this, note that for any $1 \leq u \leq v_{n_1}(\Delta_1)$, there exists $\psi \in \mathcal{A}_{n_1}$ such that $S_\psi(0) = c - \rho^{n_1} c_u$ and $S_\psi(K) \cap (c, d) \neq \emptyset$. Therefore, $S_{\sigma\phi\psi}(0) = S_{\sigma\phi}(c) - \rho^{n+l+n_1} c_u$ and $S_{\sigma\phi\psi}(K) \cap \mathrm{int}(S_{\sigma\phi}(\Delta_1)) \neq \emptyset$. Note that $\Delta_2 = S_{\sigma\phi}(\Delta_1)$ and $S_{\sigma\phi}(c)$ is the left endpoint of Δ_2. By the definition of $V_{n+l+n_1}(\Delta_2)$, c_u is a coordinate of $V_{n+l+n_1}(\Delta_2)$. This finishes the proof of the above claim.

Let e be the unique integer in $\{1, \ldots, s\}$ so that $\mathcal{C}_{n+l+n_1}(\Delta_2) = \eta_e$. By the above claim, $V(\eta_e) = V(\eta_{i_0})$ and $\ell(\eta_e) = \ell(\eta_{i_0})$. Denote by $\gamma_0 \ldots \gamma_{n-1}\eta_i$ the symbolic expression of Δ. Since $\Delta_2 \subset \Delta$, we can denote by $\gamma_0 \ldots \gamma_{n-1}\eta_i\eta_{i_2} \ldots \eta_{i_{\ell+n_1}}\eta_e$ the symbolic expression of Δ_2. Denote

$$X = T(\eta_i, \eta_{i_2}) \ldots T(\eta_{i_{l+n_1}}, \eta_e).$$

By Proposition 3.5, for any $1 \le u \le v(\eta_e)$,

$$X_{k,u} = \frac{\mu([c_u, c_u + \ell(\eta_e)])}{\mu([a_k, a_k + \ell(\eta_i)])} \cdot \sum_{\xi \in \mathcal{A}_{l+n_1}:\ a - \rho^n a_k + \rho^n S_\xi(0) = S_{\sigma\phi}(c) - \rho^{n+l+n_1} c_u} p_\xi. \tag{4.1}$$

Recall we have proved in last paragraph that for each $1 \le u \le v(\eta_e)$, there exists $\psi \in \mathcal{A}_{n_1}$ such that $S_{\sigma\phi\psi}(0) = S_{\sigma\phi}(c) - \rho^{n+l+n_1} c_u$. Note that

$$S_{\sigma\phi\psi}(0) = S_\sigma(0) + \rho^n S_{\phi\psi}(0) = a - \rho^n a_k + \rho^n S_{\phi\psi}(0).$$

By (4.1), $X_{k,u} > 0$. Therefore

$$\mathbf{e}_{i,k} T(\eta_i, \eta_{i_2}) \dots T(\eta_{i_{l+n_1}}, \eta_e) > \mathbf{0}, \tag{4.2}$$

where $\mathbf{e}_{i,k}$ denotes the $v(\eta_i)$-dimensional row vector whose k-th coordinate is 1 and all other coordinates are 0.

Choose an admissible sequence $\eta_{j_1} \dots \eta_{j_t}$ such that $\eta_{j_1} = \eta_e$ and $\eta_{j_t} = \eta_j$. By (4.2) and (3.4),

$$\mathbf{e}_{i,k} T(\eta_i, \eta_{i_2}) \dots T(\eta_{i_{l+n_1}}, \eta_e) T(\eta_e, \eta_{j_2}) \dots T(\eta_{j_{t-1}}, \eta_j) > \mathbf{0}, \tag{4.3}$$

That is, all the entries of the k-th row of the matrix

$$T(\eta_i, \eta_{i_2}) \dots T(\eta_{i_{l+n_1}}, \eta_e) T(\eta_e, \eta_{j_2}) \dots T(\eta_{j_{t-1}}, \eta_j)$$

are positive, which completes the proof of the lemma. $\square$

Proof of Proposition 4.2: To show that $H = \sum_{i=1}^s M_i$ is irreducible, it is equivalent to show that for any $1 \le u, l \le d$, there exists $i_1, i_2, \dots, i_n$ such that the (u, l)-entry of the matrix $M_{i_1} M_{i_2} \dots M_{i_n}$ is positive.

Now fix u, l. Let $i, j \in \{1, \dots, s\}$ and $k, k_1 \in \{1, \dots, v(\eta_i)\}$ be the integers such that

$$u = \sum_{t \le i-1} v(\eta_t) + k, \quad \text{and} \quad l = \sum_{t \le j-1} v(\eta_t) + k_1.$$

By Lemma 4.4, there exists an admissible sequence $\eta_i \eta_{i_1} \dots \eta_{i_n}$ with $i_n = j$ so that the (k, k_1)-entry of the matrix $T(\eta_i, \eta_{i_1}) \dots T(\eta_{i_{n-1}}, \eta_{i_n})$ is positive. By Lemma 4.3, the (u, l)-entry of the matrix $M_{i_1} \dots M_{i_n}$ is positive, which finishes the proof. $\square$

5. Proof of Theorem 1.1

Let $M_1, \dots, M_s$ be the $d \times d$ non-negative matrices we constructed in Section 4. For $q \in \mathbb{R}$, define

$$P(q) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum \|M_{i_1} \dots M_{i_n}\|^q \right),$$

where the summation is taken over all indices $i_1 \dots i_n$ over $\{1, \dots, s\}$ such that $M_{i_1} \dots M_{i_n} \neq \mathbf{0}$. We remark that the limit in the above definition exists under the condition that $\sum_{i=1}^s M_i$ is irreducible (cf. [5]). The function $P(q)$ is called the *pressure function* of $M_1, \dots, M_s$. The following result (for general non-negative matrices) was proved by Feng and Lau [5]:

Proposition 5.1. *(Theorem 3.3 of* [5]*) The pressure function $P(q)$ is differentiable for $q > 0$ under the condition that $\sum_{i=1}^s M_i$ is irreducible.*

By Proposition 4.2 and Proposition 5.1, $P(q)$ is differentiable. This combining the following theorem yields Theorem 1.1:

Theorem 5.2. *Under the condition of Theorem 1.1, $\tau(q) = P(q)/\log\rho$ for any $q > 0$.* $\square$

In the remaining part of this section we will prove the above theorem.

Let I_0 be given as in the last section. Denote $\mu_0 = \mu|_{I_0}$, i.e., $\mu_0(A) = \mu(I_0 \cap A)$ for all Borel set $A \subset \mathbb{R}$. Let $\tau(\mu_0, q)$ be the L^q-spectrum of μ_0.

Lemma 5.3. *$\tau(q) = \tau(\mu_0, q)$ for any $q \geq 0$.*

Proof. Fix $q \geq 0$. Since $\mu(A)^q \geq \mu_0(A)^q$ for each Borel set $A \subset \mathbb{R}$, it follows from the definition of the L^q-spectrum that

$$\tau(q) \leq \tau(\mu_0, q).$$

To show the reverse inequality, write $I_0 = [a_0, b_0]$. Find $\delta_0 > 0$, $n \in \mathbb{N}$ and $\phi \in \mathcal{A}_n$ such that $S_\phi([0, 1]) \subset [a_0 - \delta_0, b_0 + \delta_0]$. For each $0 < \delta < \delta_0$ and a family of disjoint intervals $[x_i - \delta, x_i + \delta]$ with $x_i \in K$, observe that $\{S_\phi([x_i - \delta, x_i + \delta])\}$ is a family of disjoint intervals of radius $\rho^n\delta$ and with centers in $\mathrm{supp}(\mu_0)$. It follows that

$$\sum_i \mu_0(S_\phi([x_i-\delta, x_i+\delta]))^q = \sum_i \mu(S_\phi([x_i-\delta, x_i+\delta]))^q \geq p_\phi^q \sum_i \mu([x_i-\delta, x_i+\delta])^q,$$

which combining the definition of the L^q spectrum yields

$$\tau(\mu_0, q) \leq \liminf_{\delta\to 0} \frac{\log p_\phi^q}{\log \delta} + \tau(q) = \tau(q).$$

This completes the proof. $\square$

Lemma 5.4. *For any $n \in \mathbb{N}$, $\Delta = [a,b] \in \mathcal{F}_n$, let $V_n(\Delta) = (a_1, \dots, a_{v_n(\Delta)})$. For each $j \in \{1, \dots, v_n(\Delta)\}$, pick $\sigma \in \mathcal{A}_n$ with $S_\sigma(0) = a - \rho^n a_j$. There is an integer k_0 (independent of n, Δ, j and σ) such that there is $\omega \in \mathcal{A}_{k_0}$ satisfying*

$$S_{\sigma\omega}([0,1]) \subset (a,b) \quad \textit{and} \quad S_{\sigma\omega}(0) - a \geq |S_{\sigma\omega}([0,1])|.$$

Proof. For any $\alpha \in \Omega$, pick $n \in \mathbb{N}$ and $\Delta = [a,b] \in \mathcal{F}_n$ with $\mathcal{C}_n(\Delta) = \alpha$. Write $V_n(\Delta) = (a_1, \dots, a_{v_n(\Delta)})$. For each $j \in \{1, \dots, v_n(\Delta)\}$, pick $\sigma \in \mathcal{A}_n$ with $S_\sigma(0) = a - \rho^n a_j$. Since $S_\sigma(K) \cap (a,b) \neq \emptyset$, there is $k = k(\alpha) \in \mathbb{N}$ and $\phi = \phi(\alpha) \in \mathcal{A}_k$ such that

$$S_{\sigma\phi}([0,1]) \subset (a,b) \quad \text{and} \quad S_{\sigma\phi}(0) - a \geq |S_{\sigma\phi}([0,1])|. \tag{5.1}$$

Observe that for any other $\Delta_1 = [c,d] \in \mathcal{F}_{n_1}$ with $\mathcal{C}_{n_1}(\Delta_1) = \alpha$, if pick $\sigma_1 \in \mathcal{A}_{n_1}$ with $S_\sigma(0) = c - \rho^{n_1} a_j$, we still have

$$S_{\sigma_1\phi}([0,1]) \subset (c,d) \quad \text{and} \quad S_{\sigma_1\phi}(0) - c \geq |S_{\sigma_1\phi}([0,1])|.$$

Let $k_0 = \max_{\alpha\in\Omega} k(\alpha)$. And choose $\hat{\phi}(\alpha) \in \mathcal{A}_{k_0}$ so that $\phi(\alpha)$ is the prefix of $\hat{\phi}(\alpha)$. It is clear that (5.1) still holds if in which ϕ is replaced by $\hat{\phi}(\alpha)$. This completes the proof. $\square$

Lemma 5.5. *There exist two constants $C_1, C_2 > 0$ such that for each n and $\Delta \in \mathcal{F}_n$ there is a subinterval $[x - C_1\rho^n, x + C_1\rho^n]$ of Δ with $x \in K$ and $\mu([x - C_1\rho^n, x + C_1\rho^n]) \geq C_2\mu(\Delta)$.*

Proof. Suppose $\Delta = [a,b] \in \mathcal{F}_n$. Write $V_n(\Delta) = (a_1, \ldots, a_{v_n(\Delta)})$. Recall that

$$\mu(\Delta) = \sum_{i=1}^{v_n(\Delta)} \mu([a_i, a_i + \ell_n(\Delta)]) \sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_i} \rho_\sigma.$$

Choose $j \in \{1, \ldots, v_n(\Delta)\}$ such that

$$\sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_j} \rho_\sigma = \max_{1 \le i \le v_n(\Delta)} \sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_i} \rho_\sigma.$$

We have

$$\mu(\Delta) \le v_n(\Delta) \sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_j} \rho_\sigma. \tag{5.2}$$

Now pick $\sigma_0 \in \mathcal{A}_n$ so that $S_{\sigma_0}(0) = a - \rho^n a_j$. By Lemma 5.4 we can find $\omega \in \mathcal{A}_{k_0}$ such that

$$S_{\sigma_0\omega}([0,1]) \subset (a,b), \quad S_{\sigma_0\omega}(0) - a \ge |S_{\sigma_0\omega}([0,1])|. \tag{5.3}$$

Set $x = S_{\sigma_0\omega}(0)$. Then $x \in K$ since $0 \in K$. By (5.3), we have

$$[x - \rho^{n+k_0}, x + \rho^{n+k_0}] \subset (a,b).$$

Note that

$$\begin{aligned}
\mu([x - \rho^{n+k_0}, x + \rho^{n+k_0}]) &\ge \mu(S_{\sigma_0\omega}([0,1])) \\
&= \sum_{\gamma \in \mathcal{A}_{n+k_0}} p_\gamma \mu(S_\gamma^{-1}(S_{\sigma\omega}([0,1]))) \\
&\ge \sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_j} p_{\sigma\omega} \mu(S_{\sigma\omega}^{-1}(S_{\sigma_0\omega}([0,1]))) \\
&= \sum_{\sigma \in \mathcal{A}_n:\ S_\sigma(0) = a - \rho^n a_j} p_{\sigma\omega} \\
&\ge p_\omega \frac{\mu(\Delta)}{v_n(\Delta)} \qquad (\text{ by } (5.2)\ ) \\
&\ge \frac{\min_{\omega' \in \mathcal{A}_{k_0}} p_{\omega'}}{\max_{\alpha \in \Omega} v(\alpha)} \mu(\Delta).
\end{aligned}$$

Letting $C = \dfrac{\min_{\omega' \in \mathcal{A}_{k_0}} p_{\omega'}}{\max_{\alpha \in \Omega} v(\alpha)}$, we complete the proof. $\square$

Proposition 5.6. *For each* $q \in \mathbb{R}$*,*

$$\tau(q) = \liminf_{n\to\infty} \frac{1}{n\log\rho} \log \sum_{\Delta\in\mathcal{F}_n} \mu(\Delta)^q, \quad \tau(\mu_0, q) = \liminf_{n\to\infty} \frac{1}{n\log\rho} \log \sum_{\Delta\in\mathcal{F}_n,\ \Delta\subset I_0} \mu(\Delta)^q.$$

Proof. For simplicity we only prove the first equality. The second one follows by a similar argument.

First we show $\tau(q) \geq \liminf_{n\to\infty} \frac{1}{n\log\rho} \log \sum_{\Delta\in\mathcal{F}_n} \mu(\Delta)^q$. To see this, by Lemma 5.5, for each $\Delta \in \mathcal{F}_n$ pick $[x_\Delta - \rho^{n+k_0}, x_\Delta + \rho^{n+k_0}] \subset \mathrm{int}(\Delta)$ such that $x_\Delta \in K$ and

$$C\mu(\Delta) \leq \mu([x_\Delta - \rho^{n+k_0}, x_\Delta + \rho^{n+k_0}]) \leq \mu(\Delta).$$

Note that $\{[x_\Delta - \rho^{n+k_0}, x_\Delta + \rho^{n+k_0}] :\ \Delta \in \mathcal{F}_n\}$ is a family of disjoint intervals with $x_\Delta \in K$, we have for $q \in \mathbb{R}$,

$$\begin{aligned} \tau(q) &\leq \liminf_{n\to\infty} \frac{1}{(n+k_0)\log\rho} \log \sum_{\Delta\in\mathcal{F}_n} \big(\mu([x_\Delta - \rho^{n+k_0}, x_\Delta + \rho^{n+k_0}])\big)^q \\ &= \liminf_{n\to\infty} \frac{1}{n\log\rho} \log \sum_{\Delta\in\mathcal{F}_n} \mu(\Delta)^q. \end{aligned}$$

To see the reverse inequality, for any $0 < \delta < \rho$, let k be the integer so that $\rho^k < \delta \leq \rho^{k-1}$. Suppose that $\{[x_i - \delta, x_i + \delta]\}$ is a family of disjoint intervals with $x_i \in K$. Observe there is a constant D such that each $[x_i - \delta, x_i + \delta]$ intersects at most D many different $\Delta \in \mathcal{F}_k$, it follows that for $q \geq 0$,

$$\mu([x_i - \delta, x_i + \delta])^q \leq \Big(\sum_{\Delta\in\mathcal{F}_k,\ \Delta\cap[x_i-\delta,x_i+\delta]\neq\emptyset} \mu(\Delta)\Big)^q \leq D^q \sum_{\Delta\in\mathcal{F}_k,\ \Delta\cap[x_i-\delta,x_i+\delta]\neq\emptyset} \mu(\Delta)^q.$$

Taking the summation over i and observing that each $\Delta \in \mathcal{F}_k$ intersects at most two different intervals $[x_i - \delta, x_i + \delta]$, we have

$$\sum_i \mu([x_i - \delta, x_i + \delta])^q \leq 2D^q \sum_{\Delta\in\mathcal{F}_k} \mu(\Delta)^q, \quad \forall q \geq 0. \tag{5.4}$$

By the way each $[x_i - \delta, x_i + \delta]$ contains at least one $\Delta \in \mathcal{F}_k$, it follows that

$$\sum_i \mu([x_i - \delta, x_i + \delta])^q \leq \sum_{\Delta\in\mathcal{F}_k} \mu(\Delta)^q, \quad \forall q < 0. \tag{5.5}$$

Combining (5.4) and (5.5) we have

$$\tau(q) \geq \liminf_{n\to\infty} \frac{1}{n\log\rho} \log \sum_{\Delta\in\mathcal{F}_k} \mu(\Delta)^q,$$

which completes the proof. $\square$

Proposition 5.7. $\tau(\mu_0, q) = P(q)/\log\rho$, $\forall q \in \mathbb{R}$.

Proof. By Proposition 5.6 and Lemma 4.1,

$$\tau(\mu_0, q) = \liminf_{n\to\infty} \frac{1}{n\log\rho} \log \sum \|\widehat{Q}_{n_0}(I_0) M_{i_1} \dots M_{i_n}\|^q, \tag{5.6}$$

where the summation is taken over all indices $i_1 \dots i_n$ so that $\eta_1\eta_{i_1}\dots\eta_{i_n}$ is an admissible sequence. In the following we write for simplicity $M_I = M_{i_1}\dots M_{i_n}$ for $I = i_1 \dots i_n$.

For $i = 1, \dots, s$, write $\mathbf{e}_i$ to be the partitioned vector $(\mathbf{f}_{1,i}, \mathbf{f}_{2,i}, \dots, \mathbf{f}_{s,i})$, where $\mathbf{f}_{j,i}$ is a $v(\eta_j)$-dimensional row vector defined by

$$\mathbf{f}_{j,i} = \begin{cases} (\underbrace{1, \dots, 1}_{v(\eta_i)\text{'s } 1}) & \text{if } j = i \\ \mathbf{0} & \text{otherwise} \end{cases}$$

and write $\mathbf{e} = (\underbrace{1, \dots, 1}_{d\text{'s } 1})$. Since $\widehat{Q}_{n_0}(I_0) \approx \mathbf{e}_1 \approx \mathbf{e}M_1$ (here and afterwards we write $(a_1, \dots, a_d) \approx (b_1, \dots, b_d)$ if $Cb_i^{-1} \le a_i \le Cb_i$ for some $C > 0$), it follows from (5.6) and Lemma 4.1 that

$$\tau(\mu_0, q) = \liminf_{n\to\infty} \frac{1}{n\log\rho} \log \sum_{I\in\{1,\dots,s\}^n:\ M_{1I}\neq\mathbf{0}} \|M_{1I}\|^q.$$

To show $\tau(\mu_0, q) = \frac{P(q)}{\log\rho}$, it suffices to show

$$\liminf_{n\to\infty} \frac{1}{n} \log \sum_{I\in\{1,\dots,s\}^n:\ M_{1I}\neq\mathbf{0}} \|M_{1I}\|^q = \lim_{n\to\infty} \frac{1}{n} \log \sum_{I\in\{1,\dots,s\}^n:\ M_I\neq\mathbf{0}} \|M_I\|^q.$$

The part "$\le$" is clear. To prove the reverse part, denote by

$$R_n(q) = \sum_{I\in\{1,\dots,s\}^n:\ M_{1I}\neq\mathbf{0}} \|M_{1I}\|^q.$$

Using the fact $\|M_{iIj}\| \le \|M_{iI}\|\|M_j\|$, we have

$$R_n(q) \ge BR_{n+1}(q) \quad \text{for } q \ge 0 \qquad \text{and} \qquad R_n(q) \le BR_{n+1}(q) \quad \text{for } q < 0, \tag{5.7}$$

where $B > 0$ depends only on q. For each $j \in \{1, \ldots, s\}$, there is an admissible word $\eta_1\eta_{i_1}\ldots\eta_{i_{l_j}}\eta_j$ with length $l_j + 2$. Note that $\mathbf{e}M_1M_{i_1}\ldots M_{i_{l_j}}M_j \approx \mathbf{e}_j \approx \mathbf{e}M_j$, for any $I \in \{1, \ldots, s\}^n$ with $M_{jI} \neq 0$, we have

$$\frac{1}{C_j}\|M_{jI}\| \leq \|M_{1i_1\ldots i_{l_j}jI}\| \leq C_j\|M_{jI}\|,$$

where $C_j > 0$ is a constant independent of n and I. Therefore

$$\sum_{I\in\{1,\ldots,s\}^n:\ M_{jI}\neq\mathbf{0}} \|M_{jI}\|^q \leq (C_j)^{|q|}R_{n+l_j+1}(q).$$

Taking the summation over j and letting $C = \max_{1\leq j\leq s} C_j$, $l = \max_{1\leq j\leq s} l_j$, we have

$$\sum_{I\in\{1,\ldots,s\}^{n+1}:\ M_I\neq\mathbf{0}} \|M_I\|^q \leq sC^{|q|}\sum_{k=0}^{l} R_{n+k+1}(q).$$

Combining it with (5.7) yields

$$\sum_{I\in\{1,\ldots,s\}^{n+1}:\ M_I\neq\mathbf{0}} \|M_I\|^q \leq DR_n(q) \quad \text{for } q \geq 0$$

and

$$\sum_{I\in\{1,\ldots,s\}^{n+1}:\ M_I\neq\mathbf{0}} \|M_I\|^q \leq DR_{n+l+1}(q) \quad \text{for } q < 0,$$

where D is a positive constant depending on q. This implies the "$\geq$" part and the proof is completed. □

Proof of Theorem 5.2: It follows directly from Lemma 5.3 and Proposition 5.7. □

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