CONVOLUTIONS OF EQUICONTRACTIVE SELF-SIMILAR MEASURES ON THE LINE

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Abstract. Let μ be a self-similar measure on $\mathbb{R}$ generated by an equicontractive iterated function system. We prove that the Hausdorff dimension of μ^{*n} tends to 1 as n tends to infinity, where μ^{*n} denotes the n-fold convolution of μ. Similar results hold for the L^q dimension and the entropy dimension of μ^{*n}.

1. Introduction

Let $\mu_1, ..., \mu_n$ $(n \geq 2)$ be a family of Borel probability measures on $\mathbb{R}$. Recall that the *convolution* $\mu_1 * \ldots * \mu_n$ of $\mu_1, ..., \mu_n$ is defined by

$$\mu_1 * \ldots * \mu_n(E) = \int_{\mathbb{R}^n} \chi_E(x_1 + \ldots + x_n) d\mu_1(x_1) \ldots d\mu_n(x_n)$$

for any Borel set $E \subset \mathbb{R}$, where χ_E denotes the characteristic function of E. In particular if $\mu_1 = ... = \mu_n = \mu$, then

$$\mu^{*n} := \underbrace{\mu * \ldots * \mu}_{n}$$

is called the *n-fold convolution* of μ.

It is well known that if μ is absolutely continuous with a density function f, then μ^{*n} is absolutely continuous with the density f^{*n} for each $n \geq 2$, where f^{*n} denotes the n-fold convolution of f. However if μ is a singular measure, μ^{*n} may be still singular for all n. In this case it is interesting to describe the asymptotic behavior of the "degree of singularity" of μ^{*n} as n tends to infinity. There are some widely used indices for describing the degree of singularity of measures, such as the Hausdorff dimension, the L^q dimension and the entropy dimension.

Recall that for a Borel probability measure η on $\mathbb{R}$, the *upper Hausdorff dimension* and the *lower Hausdorff dimension* of η are defined respectively by

$$\overline{\dim}_H \eta = \inf\{\dim_H E : \ E \text{ is a Borel set with } \eta(E) = 1\}$$

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and

$$\underline{\dim}_H \eta = \inf\{\dim_H E : \ E \text{ is a Borel set with } \eta(E) > 0\},$$

where $\dim_H E$ denotes the Hausdorff dimension of E (see [1, 2, 8] for the definition and properties of the Hausdorff dimension). For $q > 1$, the *upper* L^q*-dimension* of η is defined by

$$\overline{\dim}_q \eta = \limsup_{r \to 0} \frac{\log \int \eta([x-r, x+r])^q dx}{(q-1)\log r} - \frac{1}{q-1}.$$

The *lower* L^q*-dimension* $\underline{\dim}_q \eta$ can be defined similarly by taking the lower limit. The *upper entropy dimension* of η is defined by

$$\overline{\dim}_e \eta = \limsup_{n \to \infty} \frac{H_n(\eta)}{\log 2^n},$$

where

$$H_n(\eta) = -\sum_{k=-\infty}^{\infty} \eta\left([2^{-n}k, 2^{-n}(k+1))\right) \log \eta\left([2^{-n}k, 2^{-n}(k+1))\right).$$

The *lower entropy dimension* $\underline{\dim}_e \eta$ is defined similarly by taking the lower limit.

As we will show, the sequences $\underline{\dim}_H \mu^{*n}$, $\overline{\dim}_H \mu^{*n}$, $\underline{\dim}_q \mu^{*n}$, $\overline{\dim}_q \mu^{*n}$, $\underline{\dim}_e \mu^{*n}$ and $\overline{\dim}_e \mu^{*n}$ are increasing on n and bounded from above by 1 (see Corollary 2.4). However, it is a rather subtle question to determine the limits of these sequences in general. In this paper, we provide precise values of the above limits for the class of equicontractive self-similar measures on $\mathbb{R}$. Suppose

$$\phi_i(x) = \rho x + d_i \qquad (i = 1, \ldots, m)$$

is a family of equicontractive similitudes on $\mathbb{R}$ with $0 < \rho < 1$, $m \geq 2$ and $d_1 < d_2 < \ldots < d_m$. Usually, $\{\phi_i\}_{i=1}^m$ is called an *equicontractive iterated function system*. For a given probability weight $\{p_i\}_{i=1}^m$ (i.e., $p_i > 0$ and $\sum_i p_i = 1$), it was proved by Hutchinson [5] that there is a unique one Borel probability measure ν on $\mathbb{R}$ such that

$$\nu = \sum_{i=1}^m p_i \nu \circ \phi_i^{-1}. \tag{1.1}$$

The measure ν is called an *equicontractive self-similar measure.*

We can formulate our result as follows

Theorem 1.1. *Let ν be an equicontractive self-similar measure on $\mathbb{R}$. Then*

$$\lim_{n\to\infty} \underline{\dim}_H \nu^{*n} = \lim_{n\to\infty} \overline{\dim}_H \nu^{*n} = \lim_{n\to\infty} \underline{\dim}_e \nu^{*n} = \lim_{n\to\infty} \overline{\dim}_e \nu^{*n} = 1 \tag{1.2}$$

and

$$\lim_{n\to\infty} \underline{\dim}_q \nu^{*n} = \lim_{n\to\infty} \overline{\dim}_q \nu^{*n} = 1 \qquad (1 < q \leq 2). \tag{1.3}$$

We remark that under the condition of Theorem 1.1, ν^{*n} is an equicontractive self-similar measure for each $n \geq 1$ (cf. [3, Proposition 3.1]). It follows from a result of Peres and Solomyak ([9, Theorem 1.1]) that

$$\underline{\dim}_e \nu^{*n} = \overline{\dim}_e \nu^{*n}, \qquad \underline{\dim}_q \nu^{*n} = \overline{\dim}_q \nu^{*n} \quad (q > 1).$$

However we do not know whether $\underline{\dim}_H \nu^{*n}$ and $\overline{\dim}_H \nu^{*n}$ coincide.

Lindenstrauss, Meiri and Peres have considered the measure-theoretic entropy of convolutions of ergodic measures on the circle $\mathbb{R}/\mathbb{Z}$ [7]. Let $\{\mu_i\}$ be a sequence of invariant and ergodic measures on $\mathbb{R}/\mathbb{Z}$ with respect to the transformation $\sigma_p : x \mapsto px \pmod 1$, where p is an integer greater than 1. They proved that the measure-theoretic entropy $h(\mu_1 * \cdots * \mu_n, \sigma_p)$ tends to $\log p$ as n tends to infinity, under a sharp condition

$$\sum_{i=1}^{\infty} \frac{h_i}{|\log h_i|} = \infty,$$

where $h_i = h(\mu_i, \sigma_p)/\log p$. We remark that one can use the above deep result to deduce (1.2), if ν is a self-similar measure for the special iterated function system

$$\phi_i(x) = \frac{1}{p}(x + i - 1), \qquad i = 1, \cdots, p.$$

We organize the paper as follows. In Section 2 we establish a sufficient condition for a probability measure on $\mathbb{R}$ to satisfy the result of Theorem 1.1. This condition will be verified for equicontractive self-similar measures in Section 3 to demonstrate Theorem 1.1. Our proof is based on some classical properties of Fourier transforms of Borel probability measures as well as some basic properties of energy functions. We have also used some properties of Fourier transforms of self-similar measures developed by Strichartz [10, 11, 12] and Lau and Wang [6].

2. Probability Measures satisfying (1.2) and (1.3)

For a Borel probability measure η, the Fourier transformation $\widehat{\eta}$ is a complex-valued function on $\mathbb{R}$ defined by

$$\widehat{\eta}(t) = \int e^{-itx} d\eta(x).$$

For any integer $n > 0$ denote

$$\alpha_n = \alpha_n(\eta) = \limsup_{T \to \infty} \frac{\log \int_{|t|<T} |\widehat{\eta}(t)|^n dt}{\log T}. \tag{2.1}$$

In this section we establish the following fact, which is the first step in our proof of Theorem 1.1.

Proposition 2.1. *Suppose that η is a Borel probability measure on $\mathbb{R}$ with compact support. If $\lim_{n\to\infty}\alpha_n = 0$, then η staisfies (1.2) and (1.3), where ν is replaced by η.*

Although the condition in the above proposition looks rather technical and hard to check, we can verify it for the class of equicontractive self-similar measures which proves our Theorem 1.1.

We prove several lemmas before giving the proof of Proposition 2.1.

Lemma 2.2. *Let η_1 and η_2 be Borel probability measures on $\mathbb{R}$. Then*

(i) $\underline{\dim}_H \eta_1 * \eta_2 \geq \underline{\dim}_H \eta_1$, $\overline{\dim}_H \eta_1 * \eta_2 \geq \overline{\dim}_H \eta_1$.

(ii) *For any $q > 1$, $\overline{\dim}_q \eta_1 * \eta_2 \geq \overline{\dim}_q \eta_1$ and $\underline{\dim}_q \eta_1 * \eta_2 \geq \underline{\dim}_q \eta_1$.*

(iii) *If furthermore η_1 and η_2 are compactly supported, then $\overline{\dim}_e \eta_1 * \eta_2 \leq 1$, and $\overline{\dim}_e \eta_1 * \eta_2 \geq \overline{\dim}_e \eta_1$, $\underline{\dim}_e \eta_1 * \eta_2 \geq \underline{\dim}_e \eta_1$.*

Proof. Suppose $\eta_1 * \eta_2(E) > 0$ for some Borel set $E \subset \mathbb{R}$. Then

$$\int \eta_1(E - x) d\eta_2(x) = \eta_1 * \eta_2(E) > 0,$$

which implies that $\eta_1(E-x) > 0$ for a set of x with positive η_2 measure. Thus there is at least one point $x_0 \in \mathbb{R}$ such that $\eta_1(E - x_0) > 0$. Hence $\dim_H E = \dim_H(E - x_0) \geq \underline{\dim}_H \eta_1$, from which we obtain $\underline{\dim}_H \eta_1 * \eta_2 \geq \underline{\dim}_H \eta_1$.

Now suppose $\eta_1 * \eta_2(F) = 1$ for some Borel set $F \subset \mathbb{R}$. Then

$$\int \eta_1(F - y) d\eta_2(y) = \eta_1 * \eta_2(F) = 1,$$

which implies that $\eta_1(F-y) = 1$ for η_2 almost all $y \in \mathbb{R}$. Thus there is at least one point $y_0 \in \mathbb{R}$ such that $\eta_1(F - y_0) = 1$. Hence $\dim_H F = \dim_H(F - y_0) \geq \overline{\dim}_H \eta_1$, from which we obtain $\overline{\dim}_H \eta_1 * \eta_2 \geq \overline{\dim}_H \eta_1$.

To see (ii), by the Hölder inequality we have

$$\begin{aligned}
\int \eta_1 * \eta_2([x-r, x+r])^q dx &= \int \left(\int \eta_1([x-y-r, x-y+r]) d\eta_2(y) \right)^q dx \\
&\leq \int\int \eta_1([x-y-r, x-y+r])^q d\eta_2(y) dx \\
&= \int\int \eta_1([x-y-r, x-y+r])^q dx d\eta_2(y) \\
&= \int \eta_1([x-r, x+r])^q dx,
\end{aligned}$$

which implies (ii).

To prove (iii), define $f(x) = -x \log x$ for $x \in \mathbb{R}^+$. It is easy to see that

$$f(x+y) \leq f(x) + f(y) \leq 2f\left(\frac{x+y}{2}\right) = f(x+y) + (x+y)\log 2 \tag{2.2}$$

for all $x, y \in \mathbb{R}^+$. Since η_1 is compactly supported,

$$\sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k+z, 2^{-n}(k+1)+z])\right) < \infty, \qquad \forall n \in \mathbb{N},\ z \in \mathbb{R}.$$

Now fix n and z. Denote by z_0 the unique real number satisfying $0 \leq z_0 < 2^{-n}$ and $2^n(z_0 - z) \in \mathbb{Z}$. Using (2.2), we have

$$\begin{aligned}
& \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k+z, 2^{-n}(k+1)+z))\right) \\
= & \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k+z_0, 2^{-n}(k+1)+z_0))\right) \\
\geq & \sum_{k=-\infty}^{\infty} \Big[f\left(\eta_1([2^{-n}k+z_0, 2^{-n}(k+1)))\right) + f\left(\eta_1([2^{-n}(k+1), 2^{-n}(k+1)+z_0))\right) \\
& \qquad -\eta_1([2^{-n}k+z_0, 2^{-n}(k+1)+z_0)\log 2\Big] \\
= & \sum_{k=-\infty}^{\infty} \Big[f\left(\eta_1([2^{-n}k+z_0, 2^{-n}(k+1)))\right) + f\left(\eta_1([2^{-n}k, 2^{-n}k+z_0))\right)\Big] - \log 2 \\
\geq & \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k, 2^{-n}(k+1)))\right) - \log 2 = H_n(\eta_1) - \log 2.
\end{aligned}$$

A similar argument yields

$$H_n(\eta_1) \geq \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k+z, 2^{-n}(k+1)+z))\right) - \log 2.$$

Therefore we have

$$\left|H_n(\eta_1) - \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k+z, 2^{-n}(k+1)+z))\right)\right| \leq \log 2. \tag{2.3}$$

Similarly using (2.2) again, we can deduce that

$$H_n(\eta_1) \leq H_{n+1}(\eta_1) \leq H_n(\eta_1) + \log 2.$$

By the above inequality and the definition of entropy dimension, we have $\overline{\dim}_e \eta_1 \leq 1$. Note that $\eta_1 * \eta_2$ is also compactly supported, therefore

$$\overline{\dim}_e \eta_1 * \eta_2 \leq 1.$$

By the convexity of f, we have

$$\begin{aligned}
H_n(\eta_1 * \eta_2) &= \sum_{k=-\infty}^{\infty} f\left(\eta_1 * \eta_2([2^{-n}k, 2^{-n}(k+1)))\right) \\
&= \sum_{k=-\infty}^{\infty} f\left(\int \eta_1([2^{-n}k - z, 2^{-n}(k+1) - z))d\eta_2(z)\right) \\
&\geq \sum_{k=-\infty}^{\infty} \int f\left(\eta_1([2^{-n}k - z, 2^{-n}(k+1) - z))\right) d\eta_2(z) \\
&= \int \sum_{k=-\infty}^{\infty} f\left(\eta_1([2^{-n}k - z, 2^{-n}(k+1) - z))\right) d\eta_2(z) \\
&\geq \int (H_n(\eta_1) - \log 2)\, d\eta_2(z) = H_n(\eta_1) - \log 2,
\end{aligned}$$

from which the last two inequalities in (iii) follow. $\square$

In the following we cite some known facts about the relationship between various dimensions of a measure.

Lemma 2.3. *Suppose η is a Borel probability measure on $\mathbb{R}$ with compact support. Then*

(i) $\underline{\dim}_q \eta \leq \underline{\dim}_H \eta \leq \underline{\dim}_e \eta \leq \overline{\dim}_e \eta \leq 1$ *for any* $q > 1$.

(ii) $\overline{\dim}_q \eta \leq 1$ *for any* $q > 1$. *Furthermore* $\underline{\dim}_q \eta$ *and* $\overline{\dim}_q \eta$ *are monotone decreasing on* $q > 1$.

We remark that part (i) of the above lemma was proved by Fan, Lau and Rao [4, Theorem 1.4], while part (ii) was proved by Strichartz [12, Theorem 2.8 and Lemma 2.9].

As a corollary of Lemma 2.2 and Lemma 2.3, we have

Corollary 2.4. *Suppose η is a Borel probability measure on $\mathbb{R}$ with compact support. Then the sequences* $\overline{\dim}_H \eta^{*n}$, $\underline{\dim}_H \eta^{*n}$, $\overline{\dim}_q \eta^{*n}$, $\underline{\dim}_q \eta^{*n}$, $\overline{\dim}_e \eta^{*n}$ *and* $\underline{\dim}_e \eta^{*n}$ *are increasing on n. Each of them are bounded from above by* 1.

The following lemma is used to prove Proposition 2.1.

Lemma 2.5. *For a Borel probability measure η on $\mathbb{R}$ with compact support, we have*

$$\underline{\dim}_H \eta \geq 1 - \alpha, \ \text{ and } \ \underline{\dim}_2 \eta = 1 - \alpha,$$

where $\alpha = \alpha_2 = \limsup_{T \to \infty} \dfrac{\log \int_{|t|<T} |\widehat{\eta}(t)|^2 dt}{\log T}$.

Although the statement $\underline{\dim}_H \eta \geq 1 - \alpha$ can be obtained by $\underline{\dim}_2 \eta = 1 - \alpha$ and Lemma 2.3 (i), we will prove the both statements directly for the self-containedness. We divide the proof into three parts, i.e., Claims 2.6-2.8 given below. In the proof of Claim 2.7 and 2.8, we adopt some ideas due to Lau and Wang [6].

Claim 2.6. $\underline{\dim}_H \eta \geq 1 - \alpha$.

Proof. Recall that for $t \geq 0$, the t-energy $I_t(\eta)$ of η is defined by

$$I_t(\eta) = \iint |x-y|^{-t} d\eta(x) d\eta(y).$$

It is well known (cf. [8, Theorem 8.7]) that if E is a Borel set with $\eta(E) > 0$, then $I_s(\eta) = \infty$ for any $s > \dim_H E$. It implies that

$$\underline{\dim}_H \eta \geq \sup\{s \geq 0 : \ I_s(\eta) < \infty\}.$$

Recall that (cf. [8, Lemma 12.12]) for each $0 < t < 1$, there is a positive constant $c(t)$ (independent of η) such that

$$I_t(\eta) = c(t) \int |x|^{t-1} |\widehat{\eta}(x)|^2 dx.$$

Therefore

$$\underline{\dim}_H \eta \geq \sup\{s \in (0,1) : \int |x|^{s-1} |\widehat{\eta}(x)|^2 dx < \infty\}.$$

Consequently, to prove $\underline{\dim}_H \eta \geq 1 - \alpha$ it suffices to establish the following inequality

$$\int |x|^{\beta-1} |\widehat{\eta}(x)|^2 dx < \infty \text{ for any } \beta \in (0, 1-\alpha). \tag{2.4}$$

To see (2.4) take $\epsilon > 0$ so that $\beta < 1 - \alpha - 2\epsilon$. By the definition of α, there exists an integer $N > 0$ such that

$$\int_{|x|<T} |\widehat{\eta}|^2 dx \leq T^{\alpha+\epsilon} \text{ for any } T > N.$$

It follows that

$$\begin{aligned}
\int_{|x|\geq N} |x|^{\beta-1} |\widehat{\eta}(x)|^2 dx &\leq \sum_{i=1}^{\infty} \int_{N+i-1\leq|x|\leq N+i} |x|^{-\alpha-2\epsilon} |\widehat{\eta}(x)|^2 dx \\
&\leq \sum_{i=1}^{\infty} (N+i-1)^{-\alpha-2\epsilon} \int_{N+i-1\leq|x|\leq N+i} |\widehat{\eta}(x)|^2 dx \\
&\leq \sum_{i=1}^{\infty} (N+i-1)^{-\alpha-2\epsilon} (N+i)^{\alpha+\epsilon} < \infty.
\end{aligned}$$

Since $\beta > 0$, we have

$$\int_{|x|<N} |x|^{\beta-1} |\widehat{\eta}|^2 dx \leq \int_{|x|<N} |x|^{\beta-1} dx < \infty.$$

The above two inequalities prove (2.4). $\square$

Claim 2.7. $\underline{\dim}_2\eta \geq 1-\alpha$.

Proof. Let

$$V_\gamma(r;\eta) = \frac{1}{r^{1+\gamma}}\int \eta([x-r,x+r])^2dx \qquad \text{for any } \gamma, r \geq 0.$$

The claim is a simple consequence of the following fact, proved by Lau and Wang (see the proof of Proposition 3.2 in [6]):

$$V_\gamma(r;\eta) \leq C(\gamma)I_\gamma(\eta) \qquad \text{for every } r>0, \tag{2.5}$$

where $C(\gamma)$ is a positive constant depending on γ only.

For the reader's convenience, we include a brief proof of (2.5):

$$\begin{aligned}
V_\gamma(r;\eta) &= \frac{1}{r^{1+\gamma}}\int \eta([x-r,x+r])^2dx \\
&= \frac{1}{r^{1+\gamma}}\int\int\int \chi_{[x-r,x+r]}(y)\chi_{[x-r,x+r]}(z)d\eta(y)d\eta(z)dx \\
&= \frac{1}{r^{1+\gamma}}\int\int \mathcal{L}^1([y-r,y+r]\cap[z-r,z+r])d\eta(y)d\eta(z) \\
&\leq \frac{1}{r^{1+\gamma}}\int\int_{|y-z|\leq 2r} 2rd\eta(y)d\eta(z) \\
&\leq 2^{1+\gamma}\int\int \frac{1}{|y-z|^\gamma}d\eta(y)d\eta(z) = 2^{1+\gamma}I_\gamma(\eta),
\end{aligned}$$

which proves (2.5).

Now take $\beta < 1-\alpha$. Since $I_\beta(\eta)<\infty$, $V_\beta(r;\eta)$ has a uniform upper bound, by the definition of $\underline{\dim}_2\eta$ we have $\underline{\dim}_2\eta \geq \beta$. Since $\beta < 1-\alpha$ is arbitrary, $\underline{\dim}_2\eta \geq 1-\alpha$. □

Claim 2.8. $\underline{\dim}_2\eta \leq 1-\alpha$.

Proof. First we prove

$$\int \eta([x-r,x+r])^2dx = \frac{2}{\pi}\int |\widehat{\eta}(t)|^2\frac{\sin^2(rt)}{t^2}dt \qquad \forall r>0. \tag{2.6}$$

To see (2.6), fix $r>0$ and define $f(x)=\eta([x-r,x+r])$. Then $f(x)$ is a Borel measurable function with compact support. By Fubini Theorem,

$$\begin{aligned}
\widehat{f}(t) &= \int e^{-itx}f(x)dx = \int e^{-itx}\int_{|x-y|\leq r} d\eta(y)dx \\
&= \int\int_{|x-y|\leq r} e^{-itx}dxd\eta(y) \\
&= \int \frac{2e^{-ity}\sin(tr)}{t}d\eta(y) = \frac{2\sin(tr)}{t}\widehat{\eta}(t).
\end{aligned}$$

Therefore (2.6) follows from the following equality, known as the Plancherel formula (cf. [8])

$$\int |\widehat{f}(t)|^2 dx = 2\pi \int |f(x)|^2 dx.$$

Now since $\sin^2(tr) \geq \frac{4}{\pi^2}(tr)^2$ for $|tr| \leq 1$, by (2.6) we have

$$\frac{8\pi^3}{r^2} \int \eta([x-r, x+r])^2 dx \geq \int_{|t|\leq 1/r} |\widehat{\eta}(t)|^2 dt.$$

Therefore by the definition of $\underline{\dim}_2 \eta$, we have $\underline{\dim}_2 \eta \leq 1 - \alpha$. $\square$

Proof of Proposition 2.1. Since $|\widehat{\eta^{*n}}(x)| = |\widehat{\eta}(x)|^n$, by Lemma 2.5 we have

$$\underline{\dim}_H \eta^{*n} \geq 1 - \alpha_{2n} \quad \text{and} \quad \underline{\dim}_2 \eta^{*n} = 1 - \alpha_{2n}.$$

Since $\lim_{n\to\infty} \alpha_n = 0$,

$$\lim_{n\to\infty} \underline{\dim}_H \eta^{*n} = 1 \text{ and } \underline{\dim}_2 \eta^{*n} = 1.$$

Combining it with Lemma 2.3 yields the desired result.

3. The proof of Theorem 1.1

Let ν be an equicontractive self-similar measure defined as in (1.1), and let $\alpha_n = \alpha_n(\nu)$ be defined as in (2.1). By Proposition 2.1, it suffices to prove $\lim_{n\to\infty} \alpha_n = 0$.

It is well known that the Fourier transform of ν is given by

$$\widehat{\nu}(x) = \prod_{n=0}^{\infty} P(\rho^n x),$$

where ρ is the common contractive ratio of ϕ_i and $P(x) = \sum_{j=1}^m p_j e^{-id_j x}$ (see [11, p. 342]). Note that $d_j \neq d_k$ for $j \neq k$ and

$$\begin{aligned}
|P(x)|^2 &= \sum_{j=1}^m p_j^2 + \sum_{1\leq k<j\leq m} 2p_k p_j \cos((d_j - d_k)x) \\
&= 1 - \sum_{1\leq k<j\leq m} 2p_k p_j \Big(1 - \cos((d_j - d_k)x)\Big).
\end{aligned}$$

We define $\Phi(x) = 1 - 2p_1p_2\Big(1 - \cos(2\pi x)\Big)$. Then Φ is a periodic function with period 1. By the above equality,

$$|P(x)|^2 \leq \Phi\left(\frac{d_2 - d_1}{2\pi} x\right).$$

Hence

$$|\widehat{\nu}(x)|^2 \leq \prod_{n=0}^{\infty} \Phi\left(\frac{d_2 - d_1}{2\pi} \rho^n x\right). \tag{3.1}$$

For a given positive integer ℓ and $0 < \delta < 1$, let $r = r(\ell, \delta)$ be a positive integer such that

$$\Phi_r(x) < \delta \qquad \text{for any } x \in \left[k + \frac{1}{3}\rho^\ell,\ k + 1 - \frac{1}{3}\rho^\ell\right] \text{ and } k \in \mathbb{Z}, \tag{3.2}$$

where $\Phi_r(x) := (\Phi(x))^r$. Let $q(\ell)$ be the smallest integer s $\geq \rho^{-\ell}$, and write $\Lambda = \{0, 1, \ldots, q(\ell) - 1\}$. For $j \in \Lambda$, define

$$I_j := \left[\frac{1-\rho^\ell}{q(\ell)-1} j, \quad \frac{1-\rho^\ell}{q(\ell)-1} j + \rho^\ell\right].$$

It is clear that $\bigcup_{j \in \Lambda} I_j = [0, 1]$ and for any $k \in \mathbb{Z}$, $y \in \mathbb{R}$ we have

$$\#\left\{j \in \Lambda : \ \left[k - \frac{1}{3}\rho^\ell, k + \frac{1}{3}\rho^\ell\right] \bigcap (I_j + y) \neq \emptyset\right\} \leq 2,$$

where $\#A$ denotes the cardinality of A. This combined with (3.2) yields

$$\#\left\{j \in \Lambda : \ \max_{x \in I_j + y} \Phi_r(x) \geq \delta\right\} \leq 2, \tag{3.3}$$

for any $y \in \mathbb{R}$.

Now define a family of maps $\{\psi_j\}_{j \in \Lambda}$ on $\mathbb{R}$ by

$$\psi_j(x) = \rho^\ell x + \frac{1-\rho^\ell}{q(\ell)-1} j, \qquad j \in \Lambda.$$

Then $\psi_j([0, 1]) = I_j$ and $[0, 1] = \bigcup_{j \in \Lambda} \psi_j([0, 1])$. Iterating the last equality n times we get

$$[0, 1] = \bigcup_{j_1, \ldots, j_n \in \Lambda} \psi_{j_1} \circ \ldots \circ \psi_{j_n}([0, 1]).$$

For simplicity we write $I_{j_1 \cdots j_n} = \psi_{j_1} \circ \ldots \circ \psi_{j_n}([0, 1])$. By (3.3) for any $k \in \mathbb{N}$ and $j_1, \ldots, j_k \in \Lambda$ we have

$$\#\left\{j_{k+1} \in \Lambda : \max_{x \in I_{j_1 \cdots j_k j_{k+1}}} \Phi_r(\rho^{-k\ell} x) \geq \delta\right\} \leq 2. \tag{3.4}$$

By (3.1), we have for any integer $n \in \mathbb{N}$,

$$\begin{aligned}
\int_0^{\frac{2\pi}{d_2-d_1}\rho^{-n\ell}} |\widehat{\nu}(x)|^{2r}dx &\leq \int_0^{\frac{2\pi}{d_2-d_1}\rho^{-n\ell}} \prod_{j=0}^{\infty} \Phi_r\left(\frac{(d_2-d_1)\rho^j x}{2\pi}\right)dx \\
&= \frac{2\pi}{d_2-d_1}\int_0^{\rho^{-n\ell}} \prod_{j=0}^{\infty} \Phi_r(\rho^j x)dx \\
&\leq \frac{2\pi}{d_2-d_1}\int_0^{\rho^{-n\ell}} \prod_{j=1}^{n} \Phi_r(\rho^{j\ell} x)dx \\
&= \frac{2\pi}{d_2-d_1}\rho^{-n\ell}\int_0^{1} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x)dx \\
&\leq \frac{2\pi}{d_2-d_1}\rho^{-n\ell}\sum_{j_1,\ldots,j_n\in\Lambda}\int_{I_{j_1\cdots j_n}} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x)dx \\
&\leq \frac{2\pi}{d_2-d_1}\sum_{j_1,\ldots,j_n\in\Lambda}\max_{x\in I_{j_1\cdots j_n}} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x).
\end{aligned}$$

Note that for any fixed indices $j_1, \ldots, j_{n-1}$ we have

$$\max_{x\in I_{j_1\cdots j_n}} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x) \leq \max_{x\in I_{j_1\cdots j_{n-1}}} \prod_{j=0}^{n-2} \Phi_r(\rho^{-j\ell} x) \max_{y\in I_{j_1\cdots j_n}} \Phi_r(\rho^{-(n-1)\ell} y).$$

Hence by (3.4),

$$\sum_{j_n\in\Lambda}\max_{x\in I_{j_1\cdots j_n}} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x) \leq \max_{x\in I_{j_1\cdots j_{n-1}}} \prod_{j=0}^{n-2} \Phi_r(\rho^{-j\ell} x)(2+\delta q(\ell)).$$

Thus by induction

$$\sum_{j_1,\ldots,j_n\in\Lambda}\max_{x\in I_{j_1\cdots j_n}} \prod_{j=0}^{n-1} \Phi_r(\rho^{-j\ell} x) \leq (2+\delta q(\ell))^n.$$

Therefore

$$\int_0^{\frac{2\pi}{d_2-d_1}\rho^{-n\ell}} |\widehat{\nu}(x)|^{2r}dx \leq \frac{2\pi}{d_2-d_1}(2+\delta q(\ell))^n.$$

Similarly

$$\int_{-\frac{2\pi}{d_2-d_1}\rho^{-n\ell}}^0 |\widehat{\nu}(x)|^{2r}dx \leq \frac{2\pi}{d_2-d_1}(2+\delta q(\ell))^n.$$

Thus

$$\int_{|x|<\frac{2\pi}{d_2-d_1}\rho^{-n\ell}} |\widehat{\nu}(x)|^{2r}dx \leq \frac{4\pi}{d_2-d_1}(2+\delta q(\ell))^n,$$

which implies (see (2.1))

$$\alpha_{2r} = \limsup_{T\to\infty} \frac{\log \int_{|x|<T} |\widehat{\nu}(x)|^{2r}dx}{\log T} \le \frac{\log(2+\delta q(\ell))}{\log \rho^{-\ell}}.$$

Now first let $\delta \to 0$ and then let $\ell \to \infty$ we finally obtain $\lim_{r\to\infty} \alpha_{2r} = 0$ and so $\lim_{r\to\infty} \alpha_r = 0$. Therefore by Proposition 2.1 we get the desired results.

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