SOME RELATIONS BETWEEN PACKING PREMEASURE AND PACKING MEASURE

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Abstract

Let K be a compact subset of $\mathbb{R}^n$, $0 \le s \le n$. Let P_0^s, $\mathcal{P}^s$ denote s-dimensional packing premeasure and measures, respectively. We discuss in this paper the relation between P_0^s and $\mathcal{P}^s$. We prove: if $P_0^s(K) < \infty$, then $\mathcal{P}^s(K) = P_0^s(K)$; and if $P_0^s(K) = \infty$, then for any $\epsilon > 0$, there exists a compact subset F of K, such that $\mathcal{P}^s(F) = P_0^s(F)$ and $\mathcal{P}^s(F) \ge \mathcal{P}^s(K) - \epsilon$.

1 Introduction

Let $E \subset \mathbb{R}^n$. A δ-packing of the set E is a countable family of disjoint closed balls of radii at most δ and with centers in E. For $s \ge 0$, the s-dimensional packing premeasure is defined as

$$P_0^s(E) = \inf_{\delta>0}\{P_\delta^s(E)\}, \tag{1}$$

where $P_\delta^s(E) = \sup\{\sum_{B_i\in\mathcal{R}} |B_i|^s : \mathcal{R} \text{ a } \delta\text{-packing of } E\}$, $|B_i|$ denotes the diameter of B_i.

The s-dimensional packing measure is defined as

$$\mathcal{P}^s(E) = \inf\{\sum_{i=1}^{\infty} P_0^s(E_i) : \ E \subset \bigcup_{i=1}^{\infty} E_i\}. \tag{2}$$

It is known that $\mathcal{P}^s$ is countably subadditive, but P_0^s is only finitely subadditive. For any $E \subset \mathbb{R}^n$, $P_0^s(E) \ge \mathcal{P}^s(E)$ and $P_0^s(E) = P_0^s(\overline{E})$, where $\overline{E}$ is the closure of E.

The upper box-counting dimension $\overline{\dim}_B$ and the packing dimension $\dim_{\mathcal{P}}$ can be induced respectively by packing premeasure and packing measure by

$$\overline{\dim}_B(E) = \inf\{s \ge 0 : \ P_0^s(E) = 0\} = \sup\{s \ge 0 : \ P_0^s(E) = \infty\};$$

$$\dim_{\mathcal{P}}(E) = \inf\{s \ge 0 : \ \mathcal{P}^s(E) = 0\} = \sup\{s \ge 0 : \ \mathcal{P}^s(E) = \infty\}.$$

For further properties of above measures and dimensions, we refer to Tricot[6], Sullivan [5], Mattila [4] and Falconer [2].

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As parameters to describe non-smooth sets, the packing measure, packing premeasure, upper box-counting dimension and packing dimension play an important role in the study of fractal geometry in a manner dual to the Hausdorff measure and Hausdorff dimension.

From the above definitions, we see that estimating the packing premeasure (and thus the upper box-counting dimension) is much easier than estimating the packing measure (and thus the packing dimension). It is therefore natural to look for relationships between the packing premeasure and measure. This is the main aim of this paper and we prove the following main result:

Let K be a compact set of $\mathbb{R}^n$ and let $s \geq 0$.

(1) *If $P_0^s(K) < \infty$, then $\mathcal{P}^s(K) = P_0^s(K)$;*

(2) *If $P_0^s(K) = \infty$, then for any $\epsilon > 0$, there exists a compact subset F of K, such that $\mathcal{P}^s(F) = P_0^s(F)$, and $\mathcal{P}^s(F) \geq \mathcal{P}^s(K) - \epsilon$.*

As a corollary, We show that there are compact sets with infinite packing premeasure which contain no subsets of positive finite packing premeasure. This is in contrast to the existence results of Joyce and Preiss [3] for packing measure, and Bescovitch [1] for Hausdorff measure.

2 Main result and the proof

In this section, we will prove our main result: the packing measure and the packing premeasure of a compact set coincide if the packing premeasure of this set is finite.

Lemma 2.1 *Let $K \subset \mathbb{R}^n$ be a compact set, $s \geq 0$ and $P_0^s(K) < \infty$. Then for any subset F of K, and any $\epsilon > 0$, there exists an open set U such that $U \supset F$ and $P_0^s(U \cap K) < P_0^s(F) + \epsilon$.*

Proof. Since F has the same packing premeasure as its closure, we can assume F to be compact. The case $s = 0$ is trivial, so we assume $s > 0$.

For $\epsilon > 0$, define the ϵ-parallel body F_ϵ of F by

$$F_\epsilon = \{x \in \mathbb{R}^n : |x - y| < \epsilon \text{ for some } y \in F\}.$$

By the definition, F_ϵ is open and $F_\epsilon \supset F$.

Set

$$A = \inf_{\epsilon > 0} P_0^s(F_\epsilon \cap K); \tag{3}$$

then $0 \leq A < \infty$.

Let $\mathcal{B}$ be the collection of all closed balls in R^n. Define a mapping $\phi: \mathcal{B} \to [0, 1] \subset \mathbb{R}$ as follows:

$$\phi(B(x, r)) = \begin{cases} 0, & \text{if } x \in F, \\ \sup\{r'/r : 0 < r' \leq r, \ B(x, r') \cap F = \emptyset\}, & \text{otherwise,} \end{cases} \tag{4}$$

where $B(x,r)$ denotes the closed ball with center x and radius r.

From the above definition, if $\phi(B(x,r)) \neq 0$, then for any $0<t<1$, $B(x, t\phi(B(x,r))r) \cap F = \emptyset$; if $\phi(B(x,r)) \neq 1$, then by the compactness of F, there exists $y \in F$, such that $B(y, (1-\phi(B(x,r)))r) \subset B(x,r)$.

Given any $\omega > 0$, there exists, by (3), $\epsilon_1 > 0$ such that

$$A \leq P_0^s(F_{\epsilon_1} \cap K) \leq A + \omega. \tag{5}$$

By the definition of the packing premeasure, we can choose $\delta_1 > 0$ such that

$$P_{\delta_1}^s(F_{\epsilon_1} \cap K) < P_0^s(F_{\epsilon_1} \cap K) + \omega, \tag{6}$$

then select a δ_1-packing $\{B_i\}_{i=1}^m$ of $F_{\epsilon_1} \cap K$ such that

$$\sum_{i=1}^m |B_i|^s > P_{\delta_1}^s(F_{\epsilon_1} \cap K) - \omega. \tag{7}$$

Since $\sum\limits_{i=1}^m |B_i|^s \leq P_{\delta_1}^s(F_{\epsilon_1} \cap K)$ and $P_{\delta_1}^s(F_{\epsilon_1} \cap K) \geq P_0^s(F_{\epsilon_1} \cap K)$, we obtain from (5)-(7) that

$$A - \omega \leq \sum_{i=1}^m |B_i|^s \leq A + 2\omega. \tag{8}$$

Set $I = \{i: \ 1 \leq i \leq m, \ \phi(B_i) < 1\}$. Then, as mentioned, for any $i \in I$ there exists $y \in F$, such that

$$B(y, \frac{1}{2}(1-\phi(B_i))|B_i|) \subset B_i. \tag{9}$$

Let $\widehat{B}_i := B(y, \frac{1}{2}(1-\phi(B_i))|B_i|)$, $i \in I$. Since $\{B_i\}$ are disjoint, the finite family of the balls $\{\widehat{B}_i: \ i \in I\}$ is a δ_1-packing of F. Hence

$$P_{\delta_1}^s(F) \geq \sum_{i\in I} |\widehat{B}_i|^s = \sum_{i \in I}(1-\phi(B_i))^s |B_i|^s = \sum_{i=1}^m (1-\phi(B_i))^s|B_i|^s. \tag{10}$$

Set $J = \{j: \ 1 \leq j \leq m, \ \phi(B_j) > 0\}$ and let x_j be the center of B_j. Then, as we have already seen, for any $0<t<1$ and $j \in J$,

$$B(x_j, \frac{1}{2}t\phi(B_j)|B_j|) \cap F = \emptyset. \tag{11}$$

For $0<t<1$, $j \in J$, define $B_{t,j}^\star = B(x_j, \frac{1}{2}t\phi(B_j)|B_j|)$. Since F and $B_{t,j}^\star$ $(j \in J)$ are compact, by (11) we can choose $0 < \epsilon_2(t) < \min\{\epsilon_1, \delta_1\}$, such that

$$B_{t,j}^\star \cap F_{\epsilon_2(t)} = \emptyset, \ \ j \in J. \tag{12}$$

Now, let $\epsilon_3(t) = \epsilon_2(t)/2$, and suppose $\{C_i\}_{i=1}^\infty$ is an arbitrary $\epsilon_3(t)$-packing of $F_{\epsilon_3(t)} \cap K$, then $C_i \subset F_{\epsilon_2(t)}$, $1 \leq i < \infty$. Thus, from (12), $\{B_{t,j}^\star: \ j \in J\} \cup \{C_i: \ 1 \leq i < \infty\}$ is a δ_1-packing of $F_{\epsilon_1} \cap K$, and hence

$$\begin{aligned} P_{\delta_1}^s(F_{\epsilon_1} \cap K) &\geq \sum_{j\in J} |B_{t,j}^\star|^s + \sum_{i=1}^\infty |C_i|^s \\ &= t^s \sum_{j \in J} (\phi(B_j))^s |B_i|^s + \sum_{i=1}^\infty |C_i|^s \\ &= t^s \sum_{i=1}^m (\phi(B_i))^s|B_i|^s + \sum_{i=1}^\infty |C_i|^s. \end{aligned} \tag{13}$$

Since $\{C_i : 1 \le i < \infty\}$ is an arbitrary $\epsilon_3(t)$-packing of $F_{\epsilon_3(t)} \cap K$, we obtain from (3) and (13) that

$$\begin{aligned} P^s_{\delta_1}(F_{\epsilon_1} \cap K) &\ge t^s \sum_{i=1}^m (\phi(B_i))^s |B_i|^s + P^s_{\epsilon_3(t)}(F_{\epsilon_3(t)} \cap K) \\ &\ge t^s \sum_{i=1}^m (\phi(B_i))^s |B_i|^s + P^s_0(F_{\epsilon_3(t)} \cap K) \\ &\ge t^s \sum_{i=1}^m (\phi(B_i))^s |B_i|^s + A. \end{aligned}$$

Since $0 < t < 1$ is arbitrary, we have

$$P^s_{\delta_1}(F_{\epsilon_1} \cap K) \ge \sum_{i=1}^m (\phi(B_i))^s |B_i|^s + A. \tag{14}$$

Hence, from (5), (6) and (14), we get

$$\sum_{i=1}^m (\phi(B_i))^s |B_i|^s \le 2\omega. \tag{15}$$

Let $l = \omega^{1/2s}$, $M = \{i : \ 1 \le i \le m, \ \phi(B_i) \ge l\}$, and $M^c = \{i : \ 1 \le i \le m, \ \phi(B_i) < l\}$. Then, from (15),

$$\sum_{i \in M} |B_i|^s \le l^{-s} \sum_{i \in M} (\phi(B_i))^s |B_i|^s \le l^{-s} \sum_{i=1}^m (\phi(B_i))^s |B_i|^s \le l^{-s}(2\omega) = 2\sqrt{\omega}. \tag{16}$$

From (8) and (16) we see that

$$\sum_{i \in M^c} |B_i|^s = \sum_{i=1}^m |B_i|^s - \sum_{i \in M} |B_i|^s \ge \sum_{i=1}^m |B_i|^s - 2\sqrt{\omega} \ge A - \omega - 2\sqrt{\omega}. \tag{17}$$

Therefore, by (10) and (17),

$$\begin{aligned} P^s_{\delta_1}(F) &\ge \sum_{i=1}^m (1 - \phi(B_i))^s |B_i|^s \\ &\ge \sum_{i \in M^c} (1 - \phi(B_i))^s |B_i|^s \\ &\ge (1 - l)^s (A - \omega - 2\sqrt{\omega}) \\ &= (1 - \omega^{1/2s})^s (A - \omega - 2\sqrt{\omega}). \end{aligned} \tag{18}$$

Let $\delta_1 \downarrow 0$, we get by (18)

$$P^s_0(F) \ge (1 - \omega^{\frac{1}{2s}})^s (A - \omega - 2\sqrt{\omega}),$$

and since ω can be picked arbitrary small, we get finally

$$P^s_0(F) \ge A,$$

which yields the conclusion of the lemma. □

Lemma 2.2 *(Proposition 2 of [3]). If $M \subset \mathbb{R}^n$ is a compact set, $s \geq 0$, and if for every $\epsilon > 0$, every $\delta > 0$ and every subset S of M one can find an open set $G \supset S$ such that $P_0^s(G \cap M) \leq P_\delta^s(S) + \epsilon$, then $\mathcal{P}^s(M) = P_0^s(M)$.*

Theorem 2.3 *Let K be a compact subset of $\mathbb{R}^n$ and let $s \geq 0$, $P_0^s(K) < \infty$. Then $\mathcal{P}^s(K) = P_0^s(K)$.*

Proof. This follows immediately from Lemma 2.1 and Lemma 2.2. □

From Theorem 2.3, we get immediately

Corollary 2.4 *Let $E \in \mathbb{R}^n$ and $s \geq 0$.*

1). Assume that $0 < P_0^s(E) < \infty$. Then $0 < \mathcal{P}^s \overline{E} < \infty$. In particular, $\overline{\dim}_B E = \dim_{\mathcal{P}} \overline{E} = s$.

2). Assume that E is compact and $s > \dim_{\mathcal{P}} E$, then either $P_0^s(E) = 0$, or $P_0^s(E) = \infty$.

The following corollary shows that the Theorem of Joyce and Preiss [3] does not hold for the packing premeasure.

Corollary 2.5 *There exists a compact set K and $s > 0$ with $P_0^s(K) = \infty$ such that K contains no subset with positive finite packing premeasure.*

Proof. Let $K = \{n^{-1} : n \in \mathbb{N}\} \cup \{0\}$ and $s = \frac{1}{4}$, then $\mathcal{P}^s(K) = 0$. Moreover, by a direct calculation, we obtain that $\overline{\dim}_B K = \frac{1}{2}$, and thus $P_0^s(K) = \infty$.

We conclude that for any $F \subset K$, $P_0^s(F) = 0$ or ∞. Otherwise, assume that $F \subset K$ with $0 < P_0^s(F) < \infty$. Then $0 < P_0^s(\overline{F}) < \infty$, thus by Theorem 2.3, $0 < \mathcal{P}^s(\overline{F}) < \infty$, which is impossible since $\overline{F}$ is a subset of K. □

3 Compact sets of infinite packing premeasure

In this section, we discuss the compact sets of infinite packing premeasure.

Theorem 3.1 *Let K be a compact subset of $\mathbb{R}^n$, $s \geq 0$ and $P_0^s(K) = \infty$. Then for any $\epsilon > 0$, there exists a compact subset F of K such that $P_0^s(F) = \mathcal{P}^s(F)$ and $\mathcal{P}^s(F) \geq \mathcal{P}^s(K) - \epsilon$.*

Proof. The case $\mathcal{P}^s(K) = \infty$ is trivial, so we assume in the following that $\mathcal{P}^s(K) < \infty$.

By the definition of $\mathcal{P}^s(K)$, there exist compact sets $\{K_i\}_{i=1}^\infty$, such that $\bigcup_{i=1}^\infty K_i = K$, and

$$\sum_{i=1}^\infty P_0^s(K_i) \leq \mathcal{P}^s(K) + \frac{\epsilon}{2}. \tag{19}$$

Since $\sum_{i=1}^\infty P_0^s(K_i) \geq \mathcal{P}^s(K)$, there exists $m \in \mathbb{N}$ such that

$$\sum_{i=1}^m P_0^s(K_i) \geq \mathcal{P}^s(K) - \frac{\epsilon}{2}. \tag{20}$$

From (19) and (20), we obtain

$$\sum_{i=m+1}^{\infty} P_0^s(K_i) \le \epsilon. \tag{21}$$

Let $F = \bigcup_{i=1}^{m} K_i$, then by the finite subadditivity of P_0^s and (19) we have

$$P_0^s(F) \le \sum_{i=1}^{m} P_0^s(K_i) < \infty,$$

thus from Theorem 2.3, we have $P_0^s(F) = \mathcal{P}^s(F)$.

Finally, by (21) we obtain

$$\mathcal{P}^s(K) - \mathcal{P}^s(F) \le \mathcal{P}^s(\bigcup_{i=m+1}^{\infty} K_i) \le \sum_{i=m+1}^{\infty} P_0^s(K_i) \le \epsilon. \qquad \square$$

Remark 3.2 The following example shows that the conclusion of Theorem 3.1 cannot be strengthened to: *under the conditions of Theorem 3.1, there exists a compact set $F \subset K$ such that $P_0^s(F) = \mathcal{P}^s(F) = \mathcal{P}^s(K)$.*

Example 3.3 Let $n \in \mathbb{N}$ and consider the intervals $I_n = [\frac{1}{n}, \frac{1}{n} + \frac{1}{n^3}]$. Since $\mathcal{P}^{1/4}(I_n) = \infty$ for each $n \in \mathbb{N}$, by the theorem of Joyce and Preiss [3], there exists a compact set $E_n \subset I_n$ such that $\mathcal{P}^{1/4}(E_n) = \frac{1}{n(n+1)}$. Set $K = (\bigcup_{n=1}^{\infty} E_n) \bigcup \{0\}$. Then K is compact and $\mathcal{P}^{1/4}(K) = 1$.

For any $n \in \mathbb{N}$, pick $x_n \in E_n$ arbitrarily, then by a simple calculation, we get $\overline{\dim}_B(\{x_n : n \in \mathbb{N}\}) = 1/2$, which yields that $P^{1/4}(K) \ge P^{1/4}(\{x_n : n \in \mathbb{N}\}) = \infty$.

Assume that $F \subset K$ satisfying $P^{1/4}(F) < \infty$, then by the analysis above, there must exist $m \in \mathbb{N}$, such that $F \cap E_m = \emptyset$, and we get therefore $\mathcal{P}^{1/4}(F) \le \mathcal{P}^{1/4}(K) - \mathcal{P}^{1/4}(E_m) = 1 - \frac{1}{m(m+1)} < 1$. That is, for any $F \subset K$ with $P^{1/4}(F) < \infty$, $\mathcal{P}^{1/4}(F) \ne \mathcal{P}^{1/4}(K)$.

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