VARIATIONAL PRINCIPLES FOR TOPOLOGICAL ENTROPIES OF SUBSETS

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ABSTRACT. Let (X, T) be a topological dynamical system. We define the measure-theoretical lower and upper entropies $\underline{h}_\mu(T)$, $\overline{h}_\mu(T)$ for any $\mu \in M(X)$, where $M(X)$ denotes the collection of all Borel probability measures on X. For any non-empty compact subset K of X, we show that

$$h_{\text{top}}^B(T, K) = \sup\{\underline{h}_\mu(T) : \mu \in M(X), \ \mu(K) = 1\},$$
$$h_{\text{top}}^P(T, K) = \sup\{\overline{h}_\mu(T) : \mu \in M(X), \ \mu(K) = 1\},$$

where $h_{\text{top}}^B(T, K)$ denotes Bowen's topological entropy of K, and $h_{\text{top}}^P(T, K)$ the packing topological entropy of K. Furthermore, when $h_{\text{top}}(T) < \infty$, the first equality remains valid when K is replaced by any analytic subset of X. The second equality always extends to any analytic subset of X.

1. INTRODUCTION

Throughout this paper, by a *topological dynamical system* (TDS) (X, T) we mean a compact metric space X together with a continuous self-map $T : X \to X$. Let $M(X)$, $M(X, T)$, and $E(X, T)$ denote respectively the sets of all Borel probability measures, T-invariant Borel probability measures, and T-invariant ergodic Borel probability measures on X. By a *measure theoretical dynamical system* (m.t.d.s.) we mean $(Y, \mathcal{C}, \nu, T)$, where Y is a set, $\mathcal{C}$ is a σ-algebra over Y, ν is a probability measure on $\mathcal{C}$ and T is a measure preserving transformation. A probability measure $\mu \in M(X, T)$ induces a m.t.d.s. $(X, \mathcal{B}_X, \mu, T)$ or just (X, μ, T), where $\mathcal{B}_X$ is the σ-algebra of Borel subsets of X.

In 1958 Kolmogorov [17] associated to any m.t.d.s. $(Y, \mathcal{C}, \nu, T)$ an isomorphic invariant, namely the measure-theoretical entropy $h_\nu(T)$. Later on in 1965, Adler, Konheim and McAndrew [1] introduced for any TDS (X, T) an analogous notion of topological entropy $h_{\text{top}}(T)$, as an invariant of topological conjugacy. There is a basic relation between topological entropy and measure-theoretic entropy: if (X, T) is a TDS, then $h_{\text{top}}(T) = \sup\{h_\mu(T) : \mu \in M(X, T)\}$. This variational principle

was proved by Goodman [12], and plays a fundamental role in ergodic theory and dynamical systems (cf. [24, 26]).

In 1973, Bowen [4] introduced the topological entropy $h_{\mathrm{top}}^B(T, Z)$ for any set Z in a TDS (X, T) in a way resembling Hausdorff dimension, which we call *Bowen's topological entropy* (see Sect. 2 for the definition). In particular, $h_{\mathrm{top}}^B(T, X) = h_{\mathrm{top}}(T)$. Bowen's topological entropy plays a key role in topological dynamics and dimension theory [24].

A question arises naturally whether there is certain variational relation between Bowen's topological entropy and measure-theoretic entropy for arbitrary non-invariant compact set, or Borel set in general. However, when $K \subseteq X$ is T-invariant but not compact, or K is compact but not T-invariant, it may happen that $h_{\mathrm{top}}^B(T, K) > 0$ but $\mu(K) = 0$ for any $\mu \in M(X, T)$ (see Example 1.5). Hence we don't expect to have such variational principle on the class $M(X, T)$. For our purpose, we need to define the measure-theoretic entropy for elements in $M(X)$.

Fix a compatible metric d on X. For any $n \in \mathbb{N}$, the *n-th Bowen metric d_n on X* is defined by

$$(1.1) \qquad d_n(x, y) = \max \left\{ d\left(T^k(x), T^k(y)\right) : \ k = 0, \ldots, n-1 \right\}.$$

For every $\epsilon > 0$ we denote by $B_n(x, \epsilon), \overline{B}_n(x, \epsilon)$ the open (resp. closed) ball of radius ϵ in the metric d_n around x, i.e.,

$$(1.2) \quad B_n(x, \epsilon) = \{y \in X : \ d_n(x, y) < \epsilon\}, \quad \overline{B}_n(x, \epsilon) = \{y \in X : \ d_n(x, y) \leq \epsilon\}.$$

Following the idea of Brin and Katok [6], we give the following.

Definition 1.1. Let $\mu \in M(X)$. The *measure-theoretical lower and upper entropies* of μ are defined respectively by

$$\underline{h}_\mu(T) = \int \underline{h}_\mu(T, x) \, d\mu(x), \quad \overline{h}_\mu(T) = \int \overline{h}_\mu(T, x) \, d\mu(x),$$

where

$$\underline{h}_\mu(T, x) = \lim_{\epsilon \to 0} \liminf_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)),$$

$$\overline{h}_\mu(T, x) = \lim_{\epsilon \to 0} \limsup_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon)).$$

Brin and Katok [6] proved that for any $\mu \in M(X, T)$, $\underline{h}_\mu(T, x) = \overline{h}_\mu(T, x)$ for μ-a.e $x \in X$, and $\int \underline{h}_\mu(T, x) \, d\mu(x) = h_\mu(T)$. Hence for $\mu \in M(X, T)$,

$$\underline{h}_\mu(T) = \overline{h}_\mu(T) = h_\mu(T).$$

To formulate our results, we need to introduce an additional notion. A set in a metric space is said to be *analytic* if it is a continuous image of the set $\mathcal{N}$ of

infinite sequences of natural numbers (with its product topology). It is known that in a Polish space, the analytic subsets are closed under countable unions and intersections, and any Borel set is analytic (cf. Federer [11, 2.2.10]).

The main results of this paper are the following two theorems.

Theorem 1.2. *Let (X,T) be a TDS.*

(i) *If $K \subseteq X$ is non-empty and compact, then*
$$h_{\text{top}}^B(T,K) = \sup\{\underline{h}_\mu(T) : \mu \in M(X),\ \mu(K) = 1\}.$$

(ii) *Assume that $h_{\text{top}}(T) < \infty$. If $Z \subseteq X$ is analytic, then*

(1.3) $$h_{\text{top}}^B(T,Z) = \sup\{h_{\text{top}}^B(T,K) :\ K \subseteq Z \text{ is compact }\}.$$

Theorem 1.3. *Let (X,T) be a TDS.*

(i) *If $K \subseteq X$ is non-empty and compact, then*
$$h_{\text{top}}^P(T,K) = \sup\{\overline{h}_\mu(T) : \mu \in M(X),\ \mu(K) = 1\},$$
where $h_{\text{top}}^P(T,K)$ denotes the packing topological entropy of K (see Sect. 2 for the definition).

(ii) *If $Z \subseteq X$ is analytic, then*

(1.4) $$h_{\text{top}}^P(T,Z) = \sup\{h_{\text{top}}^P(T,K) :\ K \subseteq Z \text{ is compact }\}.$$

The above two theorems establish the variational principles for Bowen and packing topological entropies of arbitrary Borel sets in a dual manner. They provide as a kind of extension of the classical variational principle for topological entropy of compact invariant sets. In the reminder of this section, we give two examples which motivated this paper.

Example 1.4. Let (X,T) denote the one-sided full shift over a finite alphabet $\{1, 2, \ldots, \ell\}$, where ℓ is an integer ≥ 2. Endow X with the metric $d(x,y) = e^{-n}$ for $x = (x_j)_{j=1}^\infty$ and $y = (y_j)_{j=1}^\infty$, where n is the largest integer such that $x_j = y_j$ $(1 \leq j \leq n)$. It is easy to check by definition that for any $E \subseteq X$,
$$h_{\text{top}}^B(T,E) = \dim_H E, \quad h_{\text{top}}^P(T,E) = \dim_P E,$$
where $\dim_H E, \dim_P E$ denote respectively the Hausdorff dimension and the packing dimension of E in the ultra-metric space (X,d) (cf. [23]). It is a well known fact in geometric measure theory (cf. [23]) that, for any analytic set $Z \subseteq X$ with $\dim_H Z > 0$, and any $0 \leq s < \dim_H Z$, $0 \leq t < \dim_P Z$, there exist compact sets $K_1, K_2 \subset Z$ such that
$$0 < \mathcal{H}^s(K_1) < \infty, \quad 0 < \mathcal{P}^t(K_2) < \infty,$$

where $\mathcal{H}^s$, $\mathcal{P}^s$ denote respectively the s-dimensional Hausdorff measure and packing measure, and hence $\dim_H K_1 = s$, $\dim_P K_2 = t$. Furthermore, for $\mathcal{H}^s$-a.e $x \in K_1$, and $\mathcal{P}^t$-a.e $y \in K_2$,

$$\liminf_{r \to 0} \frac{\log \mathcal{H}^s(K_1 \cap B_r(x))}{\log r} = s, \quad \limsup_{r \to 0} \frac{\log \mathcal{P}^t(K_2 \cap B_r(x))}{\log r} = t,$$

where $B_r(x)$ denotes the open ball centered at x of radius r. This can derive Theorems 1.2-1.3 in the full shift case with some additional density arguments as in [23, p.99, Exercises 6-7].

Example 1.5. Again let (X, T) denote the one-sided full shift over a finite alphabet $\{1, 2, \ldots, \ell\}$. Define $\varphi : X \to \mathbb{R}$ as

$$\varphi(x) = \begin{cases} 1 & \text{if } x_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

for $x = (x_i)_{i=1}^\infty \in X$. Let E denote the set of "non-typical points" associated with the Birkhoff average of φ, i.e.,

$$E = \left\{ x \in X : \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) \neq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) \right\}.$$

It is easy to see that E is T-invariant and Borel. By the Birkhoff ergodic theorem, $\mu(E) = 0$ for any $\mu \in M(X, T)$. However $h_{\text{top}}^B(T, E) = h_{\text{top}}(T) = \log \ell$ (cf. [2]). Furthermore, as we mention in Example 1.4 that for any $0 \le s < \log \ell$, there exists a compact set $K \subset E$ such that $h_{\text{top}}^B(T, K) = \dim_H K = s$.

In our proofs of Theorems 1.2-1.3, we use and extend some ideas and techniques in geometric measure theory and topological dynamical systems. We remark that the assumption $h_{\text{top}}^B(T) < \infty$ in Theorem 1.2(ii) can be weaken somehow (see Remark 3.15). However it remains open whether this assumption can be removed.

The paper is organized as follows. In Sect. 2 we give the definitions and some basic properties of several topological entropies of subsets in a TDS: upper capacity topological entropy, Bowen's topological entropy, the packing topological entropy. In Sect. 3.3, we prove Theorem 1.2. In Sect. 4, we prove Theorem 1.3.

2. Topological entropies of subsets

In this section, we give the definitions and some basic properties of several topological entropies of subsets in a TDS: upper capacity topological entropy, Bowen's topological entropy and packing topological entropy.

Let (X, d) be a compact metric space and $T : X \to X$ a continuous transformation. Let d_n and $B_n(x, \epsilon)$ be defined as in (1.1)-(1.2).

2.1. **Upper capacity topological entropy.** Let $Z \subseteq X$ be a non-empty set. For $\epsilon > 0$, a set $E \subset Z$ is called a (n, ϵ)-*separated set* of Z if $x, y \in E, x \neq y$ implies $d_n(x, y) > \epsilon$; $E \subseteq X$ is called (n, ϵ)-*spanning set* of Z, if for any $x \in Z$, there exists $y \in E$ with $d_n(x, y) \leq \epsilon$. Let $r_n(Z, \epsilon)$ denote the largest cardinality of (n, ϵ)-separated sets for Z, and $\tilde{r}_n(Z, \epsilon)$ the smallest cardinality of (n, ϵ)-separated sets of Z. The *upper capacity topological entropy of T restricted on Z*, or simply, the *upper capacity topological entropy of Z* is defined as

$$h_{\mathrm{top}}^{UC}(T, Z) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(Z, \epsilon) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \tilde{r}_n(Z, \epsilon).$$

We remark that the second equality holds for each $Z \subseteq X$ (cf. [26, P. 169]). The quantity $h_{\mathrm{top}}^{UC}(T, Z)$ is the straightforward generalization of the Adler-Konheim-McAndrew definition [1] of the topological entropy to arbitrary subsets.

2.2. **Bowen's topological entropy.** Suppose that $\mathcal{U}$ is a finite open cover of X. Denote $\mathrm{diam}(\mathcal{U}) := \max\{\mathrm{diam}(U) : U \in \mathcal{U}\}$. For $n \geq 1$ we denote by $\mathcal{W}_n(\mathcal{U})$ the collection of strings $\mathbf{U} = U_1 \ldots U_n$ with $U_i \in \mathcal{U}$. For $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$ we call the integer $m(\mathbf{U}) = n$ the *length of* $\mathbf{U}$ and define

$$\begin{aligned} X(\mathbf{U}) &= U_1 \cap T^{-1}U_2 \cap \ldots \cap T^{-(n-1)}U_n \\ &= \left\{ x \in X : T^{j-1}x \in U_j \text{ for } j = 1, \ldots, n \right\}. \end{aligned}$$

Let $Z \subseteq X$. We say that $\Lambda \subset \bigcup_{n \geq 1} \mathcal{W}_n(\mathcal{U})$ *covers* Z if $\bigcup_{\mathbf{U} \in \Lambda} X(\mathbf{U}) \supset Z$. For $s \in \mathbb{R}$, define

$$\mathcal{M}_N^s(\mathcal{U}, Z) = \inf_{\Lambda} \sum_{\mathbf{U} \in \Lambda} e^{-sm(\mathbf{U})},$$

where the infimum is taken over all $\Lambda \subset \bigcup_{j \geq N} \mathcal{W}_j(\mathcal{U})$ that cover Z. Clearly $\mathcal{M}_N^s(\mathcal{U}, \cdot)$ is a finite outer measure on X, and

$$(2.1) \qquad \mathcal{M}_N^s(\mathcal{U}, Z) = \inf\{\mathcal{M}_N^s(\mathcal{U}, G) : G \supset Z, G \text{ is open}\}.$$

Note that $\mathcal{M}_N^s(\mathcal{U}, Z)$ increases as N increases. Define $\mathcal{M}^s(\mathcal{U}, Z) = \lim_{N \to \infty} \mathcal{M}_N^s(\mathcal{U}, Z)$ and

$$h_{\mathrm{top}}^B(T, \mathcal{U}, Z) = \inf\{s : \mathcal{M}^s(\mathcal{U}, Z) = 0\} = \sup\{s : \mathcal{M}^s(\mathcal{U}, Z) = +\infty\}.$$

Set

$$(2.2) \qquad h_{\mathrm{top}}^B(T, Z) = \sup_{\mathcal{U}} h_{\mathrm{top}}^B(T, \mathcal{U}, Z),$$

where $\mathcal{U}$ runs over finite open covers of Z. We call $h_{\mathrm{top}}^B(T, Z)$ the *Bowen's topological entropy of T restricted to Z* or, simply, the *topological entropy of Z*. This quantity was first introduced by Bowen in [4]. It is known (see, i.e. [24, Theorem 11.1]) that

$$(2.3) \qquad \sup_{\mathcal{U}} h_{\mathrm{top}}^B(T, \mathcal{U}, Z) = \lim_{\mathrm{diam}(\mathcal{U}) \to 0} h_{\mathrm{top}}^B(T, \mathcal{U}, Z).$$

Bowen's topological entropy of subsets can be defined in an alternative way. For $Z \subseteq X$, $s \geq 0$, $N \in \mathbb{N}$ and $\epsilon > 0$, define

$$\mathcal{M}^s_{N,\epsilon}(Z) = \inf \sum_i \exp(-sn_i),$$

where the infimum is taken over all finite or countable families $\{B_{n_i}(x_i, \epsilon)\}$ such that $x_i \in X$, $n_i \geq N$ and $\bigcup_i B_{n_i}(x_i, \epsilon) \supseteq Z$. The quantity $\mathcal{M}^s_{N,\epsilon}(Z)$ does not decrease as N increases and ϵ decreases, hence the following limits exist:

$$\mathcal{M}^s_\epsilon(Z) = \lim_{N \to \infty} \mathcal{M}^s_{N,\epsilon}(Z), \quad \mathcal{M}^s(Z) = \lim_{\epsilon \to 0} \mathcal{M}^s_\epsilon(Z).$$

Bowen's topological entropy $h^B_{\text{top}}(T, Z)$ can be equivalently defined as a critical value of the parameter s, where $\mathcal{M}^s(Z)$ jumps from ∞ to 0, i.e.

$$\mathcal{M}^s(Z) = \begin{cases} 0, & s > h^B_{\text{top}}(T, Z), \\[2mm] \infty, & s < h^B_{\text{top}}(T, Z). \end{cases}$$

For details, see [24, Page 74].

2.3. **Packing topological entropy.** Let $Z \subseteq X$. For $s \geq 0$, $N \in \mathbb{N}$ and $\epsilon > 0$, define

$$P^s_{N,\epsilon}(Z) = \sup \sum_i \exp(-sn_i),$$

where the supremum is taken over all finite or countable pairwise disjoint families $\{\overline{B}_{n_i}(x_i, \epsilon)\}$ such that $x_i \in Z$, $n_i \geq N$ for all i, where

$$\overline{B}_n(x, \epsilon) := \{y \in X : d_n(x, y) \leq \epsilon\}.$$

The quantity $P^s_{N,\epsilon}(Z)$ does not decrease as N, ϵ decrease, hence the following limits exist:

$$P^s_\epsilon(Z) = \lim_{N \to \infty} P^s_{N,\epsilon}(Z).$$

Define

$$\mathcal{P}^s_\epsilon(Z) = \inf \left\{ \sum_{i=1}^\infty P^s_\epsilon(Z_i) : \bigcup_{i=1}^\infty Z_i \supseteq Z \right\}.$$

Clearly, $\mathcal{P}^s_\epsilon$ satisfies the following property: if $Z \subseteq \bigcup_{i=1}^\infty Z_i$, then $\mathcal{P}^s_\epsilon(Z) \leq \sum_{i=1}^\infty \mathcal{P}^s_\epsilon(Z_i)$. There exists a critical value of the parameter s, which we will denote by $h^P_{\text{top}}(T, Z, \epsilon)$, where $\mathcal{P}^s_\epsilon(Z)$ jumps from ∞ to 0, i.e.

$$\mathcal{P}^s_\epsilon(Z) = \begin{cases} 0, & s > h^P_{\text{top}}(T, Z, \epsilon), \\[2mm] \infty, & s < h^P_{\text{top}}(T, Z, \epsilon). \end{cases}$$

Note that $h^P_{\text{top}}(T, Z, \epsilon)$ increases when ϵ decreases. We call

$$h^P_{\text{top}}(T, Z) := \lim_{\epsilon \to 0} h^P_{\text{top}}(T, Z, \epsilon)$$

the *packing topological entropy of T restricted to Z* or, simply, the *packing topological entropy of Z*, when there is no confusion about T. This quantity is defined in way which resembles the packing dimension. We remark that an equivalent definition of packing topological entropy was given earlier in [15].

2.4. Some basic properties.

Proposition 2.1.   (i) *For $Z \subseteq Z'$,*

$$h_{\text{top}}^{UC}(T, Z) \le h_{\text{top}}^{UC}(T, Z'),\ h_{\text{top}}^{B}(T, Z) \le h_{\text{top}}^{B}(T, Z'),\ h_{\text{top}}^{P}(T, Z) \le h_{\text{top}}^{P}(T, Z').$$

(ii) *For $Z \subseteq \bigcup_{i=1}^{\infty} Z_i$, $s \ge 0$ and $\epsilon > 0$, we have*

$$\mathcal{M}_{\epsilon}^{s}(Z) \le \sum_{i=1}^{\infty} \mathcal{M}_{\epsilon}^{s}(Z_i),\ h_{\text{top}}^{B}(T, Z) \le \sup_{i \ge 1} h_{\text{top}}^{B}(T, Z_i),\ h_{\text{top}}^{P}(T, Z) \le \sup_{i \ge 1} h_{\text{top}}^{P}(T, Z_i).$$

(iii) *For any $Z \subseteq X$, $h_{\text{top}}^{B}(T, Z) \le h_{\text{top}}^{P}(T, Z) \le h_{\text{top}}^{UC}(T, Z)$.*

(iv) *Furthermore, if Z is T-invariant and compact, then*

$$h_{\text{top}}^{B}(T, Z) = h_{\text{top}}^{P}(T, Z) = h_{\text{top}}^{UC}(T, Z).$$

Proof. (i) and (ii) follow directly from the definitions of topological entropies. To see (iii), let $Z \subseteq X$ and assume $0 < s < h_{\text{top}}^{B}(T, Z)$. For any $n \in \mathbb{N}$ and $\epsilon > 0$, let $R = R_n(Z, \epsilon)$ be the largest number so that there is a disjoint family $\{\overline{B}_n(x_i, \epsilon)\}_{i=1}^{R}$ with $x_i \in Z$. Then it is easy to see that for any $\delta > 0$,

$$\bigcup_{i=1}^{R} \overline{B}_n(x_i, 2\epsilon + \delta) \supseteq Z,$$

which implies that $\mathcal{M}_{n, 2\epsilon+\delta}^{s}(Z) \le R e^{-ns} \le P_{n,\epsilon}^{s}(Z)$ for any $s \ge 0$, and hence $\mathcal{M}_{2\epsilon+\delta}^{s}(Z) \le P_{\epsilon}^{s}(Z)$. By (ii), $\mathcal{M}_{2\epsilon+\delta}^{s}(Z) \le \mathcal{P}_{\epsilon}^{s}(Z)$. Since $0 < s < h_{\text{top}}^{B}(T, Z)$, we have $\mathcal{M}^{s}(Z) = \infty$ and thus $\mathcal{M}_{2\epsilon+\delta}^{s}(Z) \ge 1$ when ϵ and δ are small enough. Hence $\mathcal{P}_{\epsilon}^{s}(Z) \ge 1$ and $h_{\text{top}}^{P}(T, Z, \epsilon) \ge s$ when ϵ is small. Therefore $h_{\text{top}}^{P}(T, Z) = \lim_{\epsilon \to 0} h_{\text{top}}^{P}(T, Z, \epsilon) \ge s$. This implies that $h_{\text{top}}^{B}(T, Z) \le h_{\text{top}}^{P}(T, Z)$.

Next we show that $h_{\text{top}}^{P}(T, Z) \le h_{\text{top}}^{UC}(T, Z)$. Our argument is modified slightly from the proof of [10, Lemma 3.7]. Assume that $h_{\text{top}}^{P}(T, Z) > 0$; otherwise there is nothing left to prove. Choose $0 < t < s < h_{\text{top}}^{P}(T, Z)$. Then there exists $\delta > 0$ such that for $0 < \epsilon < \delta$, $h_{\text{top}}^{P}(T, Z, \epsilon) > s$ and thus $P_{\epsilon}^{s}(Z) \ge \mathcal{P}_{\epsilon}^{s}(Z) = \infty$. Thus for any N, there exists a countable pairwise disjoint families $\{\overline{B}_{n_i}(x_i, \epsilon)\}$ such that $x_i \in Z$, $n_i \ge N$ for all i, and $1 < \sum_i e^{-n_i s}$. For each k, let m_k be the number of i so that $n_i = k$. Then we have

$$1 < \sum_{k=N}^{\infty} m_k e^{-ks}.$$

There must be some $k \geq N$ with $m_k > e^{kt}(1 - e^{t-s})$, otherwise the above sum is at most $\sum_{k=1}^{\infty} e^{kt-ks}(1 - e^{t-s}) < 1$. Let $r_k(Z, \epsilon)$ denote the largest cardinality of (k, ϵ)-separated sets for Z. Then $r_k(Z, \epsilon) \geq m_k > e^{kt}(1 - e^{t-s})$. Hence $\limsup_{n\to\infty} \frac{1}{n} \log r_n(Z, \epsilon) \geq t$. Letting $\epsilon \to 0$, we obtain $h_{\text{top}}^{UC}(T, Z) \geq t$. This is true for any $0 < t < h_{\text{top}}^{P}(T, Z)$ so $h_{\text{top}}^{UC}(T, Z) \geq h_{\text{top}}^{P}(T, Z)$.

When $Z \subseteq X$ is T-invariant and compact, Bowen [4] proved that $h_{\text{top}}^{B}(T, Z) = h_{\text{top}}^{UC}(T, Z)$; this together with (iii) yields (iv). $\qquad \square$

3. Variational principle for Bowen's topological entropy of subsets

3.1. **Weighted topological entropy.** For any function $f : X \to [0, \infty)$, $N \in \mathbb{N}$ and $\epsilon > 0$, define

$$(3.1) \qquad \mathcal{W}_{N,\epsilon}^{s}(f) = \inf \sum_i c_i \exp(-sn_i),$$

where the infimum is taken over all finite or countable families $\{(B_{n_i}(x_i, \epsilon), c_i)\}$ such that $0 < c_i < \infty$, $x_i \in X$, $n_i \geq N$ and

$$\sum_i c_i \chi_{B_i} \geq f,$$

where $B_i := B_{n_i}(x_i, \epsilon)$, and χ_A denotes the characteristic function of A, i.e, $\chi_A(x) = 1$ if $x \in A$ and 0 if $x \in X \backslash A$.

For $Z \subseteq X$ and $f = \chi_Z$ we set $\mathcal{W}_{N,\epsilon}^{s}(Z) = \mathcal{W}_{N,\epsilon}^{s}(\chi_Z)$. The quantity $\mathcal{W}_{N,\epsilon}^{s}(Z)$ does not decrease as N increases and ϵ decreases, hence the following limits exist:

$$\mathcal{W}_{\epsilon}^{s}(Z) = \lim_{N\to\infty} \mathcal{W}_{N,\epsilon}^{s}(Z), \quad \mathcal{W}^{s}(Z) = \lim_{\epsilon\to 0} \mathcal{W}_{\epsilon}^{s}(Z).$$

We remark that $\mathcal{W}^s$ is defined in a way which resembles the weighted Hausdorff measure in geometric measure theory (cf. [11, 23]). Clearly, there exists a critical value of the parameter s, which we will denote by $h_{\text{top}}^{WB}(T, Z)$, where $\mathcal{W}^s(Z)$ jumps from ∞ to 0, i.e.

$$\mathcal{W}^{s}(Z) = \begin{cases} 0, & s > h_{\text{top}}^{WB}(T, Z), \\ \infty, & s < h_{\text{top}}^{WB}(T, Z). \end{cases}$$

We call $h_{\text{top}}^{WB}(T, Z)$ the *weighted Bowen's topological entropy of T restricted to Z* or, simply, the *weighted Bowen's topological entropy of Z*.

3.2. **Equivalence of h_{top}^{B} and h_{top}^{WB}.** The following properties about $\mathcal{M}^s$ (cf. Sect. 2.2) and $\mathcal{W}^s$ can be verifies directly from the definitions.

Proposition 3.1.   (i) *For any $s \geq 0$, $N \in \mathbb{N}$ and $\epsilon > 0$, both $\mathcal{M}_{N,\epsilon}^{s}$ and $\mathcal{W}_{N,\epsilon}^{s}$ are outer measures on X.*

   (ii) *For any $s \geq 0$, both $\mathcal{M}^s$ and $\mathcal{W}^s$ are metric outer measures on X.*

We remark that $\mathcal{M}^s$ and $\mathcal{W}^s$ depend not only s but also the TDS (X, T). However, $\mathcal{M}^s$ and $\mathcal{W}^s$ are purely topological and independent of the special choice of the metric d.

The main result of this subsection is the following.

Proposition 3.2. *Let $Z \subseteq X$. Then for any $s \geq 0$ and $\epsilon, \delta > 0$, we have*
$$\mathcal{M}_{N,6\epsilon}^{s+\delta}(Z) \leq \mathcal{W}_{N,\epsilon}^s(Z) \leq \mathcal{M}_{N,\epsilon}^s(Z),$$
when N is large enough. As a result, $\mathcal{M}^{s+\delta}(Z) \leq \mathcal{W}^s(Z) \leq \mathcal{M}^s(Z)$ and $h_{\text{top}}^B(T, Z) = h_{\text{top}}^{WB}(T, Z)$.

To prove Proposition 3.2, we need the following lemma.

Lemma 3.3 ([23], Theorem 2.1). *Let (X, d) be a compact metric space and $\mathcal{B} = \{B(x_i, r_i)\}_{i \in \mathcal{I}}$ be a family of closed (or open) balls in X. Then there exists a finite or countable subfamily $\mathcal{B}' = \{B(x_i, r_i)\}_{i \in \mathcal{I}'}$ of pairwise disjoint balls in $\mathcal{B}$ such that*
$$\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{i \in \mathcal{I}'} B(x_i, 5r_i).$$

Proof of Proposition 3.2. Let $Z \subseteq X$, $s \geq 0$, $\epsilon, \delta > 0$. Taking $f = \chi_Z$ and $c_i \equiv 1$ in the definition (3.1), we see that $\mathcal{W}_{N,\epsilon}^s(Z) \leq \mathcal{M}_{N,\epsilon}^s(Z)$ for each $N \in \mathbb{N}$. In the following, we prove that $\mathcal{M}_{N,6\epsilon}^{s+\delta}(Z) \leq \mathcal{W}_{N,\epsilon}^s(Z)$ when N is large enough.

Assume that $N \geq 2$ such that $n^2 e^{-n\delta} \leq 1$ for $n \geq N$. Let $\{(B_{n_i}(x_i, \epsilon), c_i)\}_{i \in \mathcal{I}}$ be a family so that $\mathcal{I} \subseteq \mathbb{N}$, $x_i \in X$, $0 < c_i < \infty$, $n_i \geq N$ and
$$(3.2) \qquad\qquad \sum_i c_i \chi_{B_i} \geq \chi_Z,$$
where $B_i := B_{n_i}(x_i, \epsilon)$. We show below that
$$(3.3) \qquad\qquad \mathcal{M}_{N,6\epsilon}^{s+\delta}(Z) \leq \sum_{i \in \mathcal{I}} c_i e^{-n_i s},$$
which implies $\mathcal{M}_{N,6\epsilon}^{s+\delta}(Z) \leq \mathcal{W}_{N,\epsilon}^s(Z)$.

Denote $\mathcal{I}_n := \{i \in \mathcal{I} : n_i = n\}$ and $\mathcal{I}_{n,k} = \{i \in \mathcal{I}_n : i \leq k\}$ for $n \geq N$ and $k \in \mathbb{N}$. Write for brevity $B_i := B_{n_i}(x_i, \epsilon)$ and $5B_i := B_{n_i}(x_i, 5\epsilon)$ for $i \in \mathcal{I}$. Obviously we may assume $B_i \neq B_j$ for $i \neq j$. For $t > 0$, set
$$Z_{n,t} = \left\{ x \in Z : \sum_{i \in \mathcal{I}_n} c_i \chi_{B_i}(x) > t \right\} \quad \text{and}$$
$$Z_{n,k,t} = \left\{ x \in Z : \sum_{i \in \mathcal{I}_{n,k}} c_i \chi_{B_i}(x) > t \right\}.$$

We divide the proof of (3.3) into the following three steps.

Step 1. For each $n \geq N$, $k \in \mathbb{N}$ and $t > 0$, there exists a finite set $\mathcal{J}_{n,k,t} \subseteq \mathcal{I}_{n,k}$ such that the balls B_i ($i \in \mathcal{J}_{n,k,t}$) are pairwise disjoint, $Z_{n,k,t} \subseteq \bigcup_{i \in \mathcal{J}_{n,k,t}} 5B_i$ and

$$\#(\mathcal{J}_{n,k,t})e^{-ns} \leq \frac{1}{t} \sum_{i \in \mathcal{I}_{n,k}} c_i e^{-ns}.$$

To prove the above result, we adopt the method of Federer [11, 2.10.24] used in the study of weighted Hausdorff measures (see also Mattila [23, Lemma 8.16]). Since $\mathcal{I}_{n,k}$ is finite, by approximating the c_i's from above, we may assume that each c_i is a positive rational, and then multiplying with a common denominator we may assume that each c_i is a positive integer. Let m be the least integer with $m \geq t$. Denote $\mathcal{B} = \{B_i, \ i \in \mathcal{I}_{n,k}\}$ and define $u : \mathcal{B} \to \mathbb{Z}$ by $u(B_i) = c_i$. We define by induction integer-valued functions $v_0, v_1, \ldots, v_m$ on $\mathcal{B}$ and sub-families $\mathcal{B}_1, \ldots, \mathcal{B}_m$ of $\mathcal{B}$ starting with $v_0 = u$. Using Lemma 3.3 (in which we take the metric d_n instead of d) we find a pairwise disjoint subfamily $\mathcal{B}_1$ of $\mathcal{B}$ such that $\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{B \in \mathcal{B}_1} 5B$, and hence $Z_{n,k,t} \subseteq \bigcup_{B \in \mathcal{B}_1} 5B$. Then by repeatedly using Lemma 3.3, we can define inductively for $j = 1, \ldots, m$, disjoint subfamilies $\mathcal{B}_j$ of $\mathcal{B}$ such that

$$\mathcal{B}_j \subseteq \{B \in \mathcal{B} : \ v_{j-1}(B) \geq 1\}, \quad Z_{n,k,t} \subseteq \bigcup_{B \in \mathcal{B}_j} 5B$$

and the functions v_j such that

$$v_j(B) = \begin{cases} v_{j-1}(B) - 1 & \text{for } B \in \mathcal{B}_j, \\ v_{j-1}(B) & \text{for } B \in \mathcal{B}\backslash\mathcal{B}_j. \end{cases}$$

This is possible since for $j < m$, $Z_{n,k,t} \subseteq \left\{x : \sum_{B \in \mathcal{B}: \ B \ni x} v_j(B) \geq m - j\right\}$, whence every $x \in Z_{n,k,t}$ belongs to some ball $B \in \mathcal{B}$ with $v_j(B) \geq 1$. Thus

$$\sum_{j=1}^{m} \#(\mathcal{B}_j)e^{-ns} = \sum_{j=1}^{m} \sum_{B \in \mathcal{B}_j} (v_{j-1}(B) - v_j(B))e^{-ns}$$

$$\leq \sum_{B \in \mathcal{B}} \sum_{j=1}^{m} (v_{j-1}(B) - v_j(B))e^{-ns} \leq \sum_{B \in \mathcal{B}} u(B)e^{-ns} = \sum_{i \in \mathcal{I}_{n,k}} c_i e^{-ns}.$$

Choose $j_0 \in \{1, \ldots, m\}$ so that $\#(\mathcal{B}_{j_0})$ is the smallest. Then

$$\#(\mathcal{B}_{j_0})e^{-ns} \leq \frac{1}{m} \sum_{i \in \mathcal{I}_{n,k}} c_i e^{-ns} \leq \frac{1}{t} \sum_{i \in \mathcal{I}_{n,k}} c_i e^{-ns}.$$

Hence $\mathcal{J}_{n,k,t} = \{i \in \mathcal{I} : \ B_i \in \mathcal{B}_{j_0}\}$ is desired.

Step 2. For each $n \geq N$ and $t > 0$, we have

(3.4) $$\mathcal{M}_{N,6\epsilon}^{s+\delta}(Z_{n,t}) \leq \frac{1}{n^2 t} \sum_{i \in \mathcal{I}_n} c_i e^{-ns}.$$

To see this, assume $Z_{n,t} \neq \emptyset$; otherwise this is nothing to prove. Since $Z_{n,k,t} \uparrow Z_{n,t}$, $Z_{n,k,t} \neq \emptyset$ when k is large enough. Let $\mathcal{J}_{n,k,t}$ be the sets constructed in Step 1. Then

$\mathcal{J}_{n,k,t} \neq \emptyset$ when k is large enough. Define $E_{n,k,t} = \{x_i : i \in \mathcal{J}_{n,k,t}\}$. Note that the family of all non-empty compact subsets of X is compact with respect to the Hausdorff distance (cf. Federer [11, 2.10.21]). It follows that there is a subsequence (k_j) of natural numbers and a non-empty compact set $E_{n,t} \subset X$ such that $E_{n,k_j,t}$ converges to $E_{n,t}$ in the Hausdorff distance as $j \to \infty$. Since any two points in $E_{n,k,t}$ have a distance (with respect to d_n) not less than ϵ, so do the points in $E_{n,t}$. Thus $E_{n,t}$ is a finite set, moreover, $\#(E_{n,k_j,t}) = \#(E_{n,t})$ when j is large enough. Hence

$$\bigcup_{x \in E_{n,t}} B_n(x, 5.5\epsilon) \supseteq \bigcup_{x \in E_{n,k_j,t}} B_n(x, 5\epsilon) = \bigcup_{i \in \mathcal{J}_{n,k_j,t}} 5B_i \supseteq Z_{n,k_j,t}$$

when j is large enough, and thus $\bigcup_{x \in E_{n,t}} B_n(x, 6\epsilon) \supseteq Z_{n,t}$. By the way, since $\#(E_{n,k_j,t}) = \#(E_{n,t})$ when j is large enough, we have $\#(E_{n,t})e^{-ns} \leq \frac{1}{t} \sum_{i \in \mathcal{I}_n} c_i e^{-ns}$. This forces

$$\mathcal{M}_{N,6\epsilon}^{s+\delta}(Z_{n,t}) \leq \#(E_{n,t})e^{-n(s+\delta)} \leq \frac{1}{e^{n\delta}t} \sum_{i \in \mathcal{I}_n} c_i e^{-ns} \leq \frac{1}{n^2 t} \sum_{i \in \mathcal{I}_n} c_i e^{-ns}.$$

Step 3. For any $t \in (0,1)$, we have $\mathcal{M}_{N,6\epsilon}^{s+\delta}(Z) \leq \frac{1}{t} \sum_{i \in \mathcal{I}} c_i e^{-n_i s}$. As a result, (3.3) holds.

To see this, fix $t \in (0,1)$. Note that $\sum_{n=N}^{\infty} n^{-2} < 1$. It follows that $Z \subseteq \bigcup_{n=N}^{\infty} Z_{n,n^{-2}t}$ from (3.2). Hence by Proposition 3.1(i) and (3.4), we have

$$\mathcal{M}_{N,6\epsilon}^{s+\delta}(Z) \leq \sum_{n=N}^{\infty} \mathcal{M}_{N,6\epsilon}^{s+\delta}(Z_{n,n^{-2}t}) \leq \sum_{n=N}^{\infty} \frac{1}{t} \sum_{i \in \mathcal{I}_n} c_i e^{-ns} = \frac{1}{t} \sum_{i \in \mathcal{I}} c_i e^{-n_i s},$$

which finishes the proof of the proposition. $\qquad\qquad\square$

3.3. A dynamical Frostman's lemma and the proof of Theorem 1.2 (i).

To prove Theorem 1.2(i), we need the following dynamical Frostman's lemma, which is an analogue of the classical Frostman's lemma in in compact metric space. Our proof is adapt from Howroyd's elegant argument (cf. [13, Theorem 2], [23, Theorem 8.17]).

Lemma 3.4. *Let K be a non-empty compact subset of X. Let $s \geq 0$, $N \in \mathbb{N}$ and $\epsilon > 0$. Suppose that $c := \mathcal{W}_{N,\epsilon}^s(K) > 0$. Then there is a Borel probability measure μ on X such that $\mu(K) = 1$ and*

$$\mu(B_n(x, \epsilon)) \leq \frac{1}{c} e^{-ns}, \quad \forall\, x \in X,\, n \geq N.$$

Proof. Clearly $c < \infty$. We define a function p on the space $C(X)$ of continuous real-valued functions on X by

$$p(f) = (1/c)\mathcal{W}_{N,\epsilon}^s(\chi_K \cdot f),$$

where $\mathcal{W}_{N,\epsilon}^s$ is defined as in (3.1).

Let $\mathbf{1} \in C(X)$ denote the constant function $\mathbf{1}(x) \equiv 1$. It is easy to verify that

(1) $p(f + g) \leq p(f) + p(g)$ for any $f, g \in C(X)$.
(2) $p(tf) = tp(f)$ for any $t \geq 0$ and $f \in C(X)$.
(3) $p(\mathbf{1}) = 1$, $0 \leq p(f) \leq \|f\|_\infty$ for any $f \in C(X)$, and $p(g) = 0$ for $g \in C(X)$ with $g \leq 0$.

By the Hahn-Banach theorem, we can extend the linear functional $t \mapsto tp(\mathbf{1})$, $t \in \mathbb{R}$, from the subspace of the constant functions to a linear functional $L : C(X) \to \mathbb{R}$ satisfying

$$L(\mathbf{1}) = p(\mathbf{1}) = 1 \text{ and } - p(-f) \leq L(f) \leq p(f) \text{ for any } f \in C(X).$$

If $f \in C(X)$ with $f \geq 0$, then $p(-f) = 0$ and so $L(f) \geq 0$. Hence combining the fact $L(\mathbf{1}) = 1$, we can use the Riesz representation theorem to find a Borel probability measure μ on X such that $L(f) = \int f d\mu$ for $f \in C(X)$.

Now we show that $\mu(K) = 1$. To see this, for any compact set $E \subseteq X \backslash K$, by the Uryson lemma there is $f \in C(X)$ such that $0 \leq f \leq 1$, $f(x) = 1$ for $x \in E$ and $f(x) = 0$ for $x \in K$. Then $f \cdot \chi_K \equiv 0$ and thus $p(f) = 0$. Hence $\mu(E) \leq L(f) \leq p(f) = 0$. This shows $\mu(X \backslash K) = 0$, i.e. $\mu(K) = 1$.

In the end, we show that $\mu(B_n(x, \epsilon)) \leq (1/c)e^{-ns}$ for any $x \in X$ and $n \geq N$. To see this, for any compact set $E \subset B_n(x, \epsilon)$, by the Uryson lemma, there exists $f \in C(X)$ such that $0 \leq f \leq 1$, $f(y) = 1$ for $y \in E$ and $f(y) = 0$ for $y \in X \backslash B_n(x, \epsilon)$. Then $\mu(E) \leq L(f) \leq p(f)$. Since $f \cdot \chi_K \leq \chi_{B_n(x,\epsilon)}$ and $n \geq N$, we have $\mathcal{W}^s_{N,\epsilon}(\chi_K \cdot f) \leq e^{-ns}$ and thus $p(f) \leq \frac{1}{c}e^{-sn}$. Therefore $\mu(E) \leq \frac{1}{c}e^{-ns}$. It follows that

$$\mu(B_n(x, \epsilon)) = \sup\{\mu(E) : E \text{ is a compact subset of } B_n(x, \epsilon)\} \leq \frac{1}{c}e^{-sn}.$$

$\square$

Remark 3.5. There is a related known result (see, e.g. [22, 25]) that, for any Borel set $E \subset X$ and any Borel probability measure μ on E, if $\underline{h}_\mu(T, x) \leq s$ for all $x \in E$, then $h^B_{\text{top}}(T, E) \leq s$; conversely if $\underline{h}_\mu(T, x) \geq s$ for all $x \in E$, then $h^B_{\text{top}}(T, E) \geq s$, where $\underline{h}_\mu(T, x)$ is defined as in Sect. 1.

Now we are ready to prove Theorem 1.2(i).

Proof of Theorem 1.2(i). We first show that $h^B_{\text{top}}(T, K) \geq \underline{h}_\mu(T)$ for any $\mu \in M(X)$ with $\mu(K) = 1$. Let μ be a given such measure. Write

$$\underline{h}_\mu(T, x, \epsilon) = \liminf_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon))$$

for $x \in X, n \in \mathbb{N}$ and $\epsilon > 0$. Clearly $\underline{h}_\mu(T, x, \epsilon)$ is nonnegative and increases as ϵ decreases. Hence by the monotone convergence theorem,

$$\lim_{\epsilon \to 0} \int \underline{h}_\mu(T, x, \epsilon) d\mu = \int \underline{h}_\mu(T, x) d\mu = h_\mu(T).$$

Thus to show $h_{\text{top}}^B(T, K) \geq \underline{h}_\mu(T)$, it is sufficient to show $h_{\text{top}}^B(T, K) \geq \int \underline{h}_\mu(T, x, \epsilon) d\mu$ for each $\epsilon > 0$.

Fix $\epsilon > 0$ and $\ell \in \mathbb{N}$. Denote $u_\ell = \min\{\ell, \int \underline{h}_\mu(T, x, \epsilon) d\mu(x) - \frac{1}{\ell}\}$. Then there exist a Borel set $A_\ell \subset X$ with $\mu(A_\ell) > 0$ and $N \in \mathbb{N}$ such that

(3.5) $$\mu(B_n(x, \epsilon)) \leq e^{-nu_\ell}, \quad \forall \, x \in A_\ell, \, n \geq N.$$

Now let $\{B_{n_i}(x_i, \epsilon/2)\}$ be a countable or finite family so that $x_i \in X$, $n_i \geq N$ and $\bigcup_i B_{n_i}(x_i, \epsilon/2) \supset K \cap A_\ell$. We may assume that for each i, $B_{n_i}(x_i, \epsilon/2) \cap (K \cap A_\ell) \neq \emptyset$, and choose $y_i \in B_{n_i}(x_i, \epsilon/2) \cap (K \cap A_\ell)$. Then by (3.5),

$$\sum_i e^{-n_i u_\ell} \geq \sum_i \mu(B_{n_i}(y_i, \epsilon)) \geq \sum_i \mu(B_{n_i}(x_i, \epsilon/2))$$

$$\geq \mu(K \cap A_\ell) = \mu(A_\ell) > 0.$$

It follows that $\mathcal{M}^{u_\ell}(K) \geq \mathcal{M}_{N, \epsilon/2}^{u_\ell}(K) \geq \mathcal{M}_{N, \epsilon/2}^{u_\ell}(K \cap A_\ell) \geq \mu(A_\ell)$. Therefore $h_{\text{top}}^B(T, K) \geq u_\ell$. Letting $\ell \to \infty$, we have the desired inequality $h_{\text{top}}^B(T, K) \geq \int \underline{h}_\mu(T, x, \epsilon) d\mu$. Hence $h_{\text{top}}^B(T, K) \geq \underline{h}_\mu(T)$.

We next show that $h_{\text{top}}^B(T, K) \leq \sup\{\underline{h}_\mu(T) : \mu \in M(X), \mu(K) = 1\}$. We can assume that $h_{\text{top}}^B(T, K) > 0$, otherwise we have nothing to prove. By Proposition 3.2, $h_{\text{top}}^{BW}(T, K) = h_{\text{top}}^B(T, K)$. Let $0 < s < h_{\text{top}}^B(T, K)$. Then there exist $\epsilon > 0$ and $N \in \mathbb{N}$ such that $c := \mathcal{W}_{N, \epsilon}^s(K) > 0$. By Proposition 3.4, there exists $\mu \in M(X)$ with $\mu(K) = 1$ such that $\mu(B_n(x, \epsilon)) \leq \frac{1}{c} e^{-sn}$ for any $x \in X$ and $n \geq N$. Clearly $\underline{h}_\mu(T, x) \geq \underline{h}_\mu(T, x, \epsilon) \geq s$ for each $x \in X$ and hence $\underline{h}_\mu(T) \geq \int \underline{h}_\mu(T, x) d\mu(x) \geq s$. This finishes the proof of Theorem 1.2(i). $\qquad\square$

3.4. **The proof of Theorem 1.2(ii).** To prove Theorem 1.2(ii), we first prove the following.

Theorem 3.6. *Let (X, T) be a TDS. Assume that X is zero-dimensional, i.e., for any $\delta > 0$, X has a closed-open partition with diameter less than δ. Then for any analytic set $Z \subset X$,*

$$h_{\text{top}}(T, Z) = \sup\{h_{\text{top}}(T, K) : \ K \subset Z, \ K \text{ is compact}\}.$$

The following proposition is needed for the proof of Theorem 3.6.

Proposition 3.7. *Assume $\mathcal{U}$ is a closed-open partition of X. Let $N \in \mathbb{N}$. Then*

(i) *If $E_i \uparrow E$, i.e., $E_{i+1} \supseteq E_i$ and $\bigcup_i E_i = E$, then*

$$\mathcal{M}_N^s(\mathcal{U}, E) = \lim_{i \to \infty} \mathcal{M}_N^s(\mathcal{U}, E_i).$$

(ii) *Assume $Z \subset X$ is analytic. Then*

$$\mathcal{M}_N^s(\mathcal{U}, Z) = \sup\{\mathcal{M}_N^s(\mathcal{U}, K) : K \subset Z, K \text{ is compact}\}.$$

Proof. We first show that (i) implies (ii). Assume that (i) holds. Let Z be analytic, i.e., there exists a continuous surjective map $\phi : \mathcal{N} \to Z$. Let $\Gamma_{n_1, n_2, \ldots, n_p}$ be the set of $(m_1, m_2, \ldots) \in \mathcal{N}$ such that $m_1 \leq n_1$, $m_2 \leq n_2$, ..., $m_p \leq n_p$ and let $Z_{n_1, \ldots, n_p}$ be the image of $\Gamma_{n_1, \ldots, n_p}$ under ϕ. Let (ϵ_p) be a sequence of positive numbers. Due to (i), we can pick a sequence (n_p) of positive integers recursively so that $\mathcal{M}_N^s(\mathcal{U}, Z_{n_1}) \geq \mathcal{M}_N^s(\mathcal{U}, Z) - \epsilon_1$ and

$$\mathcal{M}_N^s(\mathcal{U}, Z_{n_1, \ldots, n_p}) \geq \mathcal{M}_N^s(\mathcal{U}, Z_{n_1, \ldots, n_{p-1}}) - \epsilon_p, \quad p = 2, 3, \ldots$$

Hence $\mathcal{M}_N^s(\mathcal{U}, Z_{n_1, \ldots, n_p}) \geq \mathcal{M}_N^s(\mathcal{U}, Z) - \sum_{i=1}^\infty \epsilon_i$ for any $p \in \mathbb{N}$. Let

$$K = \bigcap_{p=1}^\infty \overline{Z_{n_1, \ldots, n_p}}.$$

Since ϕ is continuous, we can show that $\bigcap_{p=1}^\infty \overline{Z_{n_1, \ldots, n_p}} = \bigcap_{p=1}^\infty Z_{n_1, \ldots, n_p}$ by applying Cantor's diagonal argument. Hence K is a compact subset of Z. If $\Lambda \subset \bigcup_{j \geq N} \mathcal{W}_j(\mathcal{U})$ is a cover of K (of course it is an open cover), then it is a cover of $\overline{Z_{n_1, \ldots, n_p}}$ when p is large enough, which implies

$$\sum_{\mathbf{U} \in \Lambda} e^{-sm(\mathbf{U})} \geq \lim_{p \to \infty} \mathcal{M}_N^s(\mathcal{U}, Z_{n_1, \ldots, n_p}) \geq \mathcal{M}_N^s(\mathcal{U}, Z) - \sum_{i=1}^\infty \epsilon_i.$$

Hence $\mathcal{M}_N^s(\mathcal{U}, K) \geq \mathcal{M}_N^s(\mathcal{U}, Z) - \sum_{i=1}^\infty \epsilon_i$. Since $\sum_{i=1}^\infty \epsilon_i$ can be chosen arbitrarily small, we prove (ii).

Now we turn to prove (i). Our argument is modified from the classical proof of the "increasing sets lemma" for Hausdorff outer measures (cf. [7, Sect. II] and [9, Lemma 5.3]). Note that any two non-empty elements in $\mathcal{W}_n(\mathcal{U})$ are disjoint, and each element in $\mathcal{W}_{n+1}(\mathcal{U})$ is a subset of some element in $\mathcal{W}_n(\mathcal{U})$. We call this the *net property* of $(\mathcal{W}_n(\mathcal{U}))$.

Let $E_i \uparrow E$ be given. Let (δ_i) be a sequence of positive numbers to be specified later and for each i, choose a covering $\Lambda_i \subset \bigcup_{j \geq N} \mathcal{W}_j(\mathcal{U})$ of E_i such that

(3.6) $$\sum_{\mathbf{U} \in \Lambda_i} e^{-sm(\mathbf{U})} \leq M_N^s(\mathcal{U}, E_i) + \delta_i.$$

By the net property of $(\mathcal{W}_n(\mathcal{U}))$, we may assume that for each i, the elements in Λ_i are disjoint.

For any $x \in E$, choose $\mathbf{U}_x \in \bigcup_{i=1}^{\infty} \Lambda_i$ containing x such that $m(\mathbf{U}_x)$ is the smallest. By the net property of $(\mathcal{W}_n(\mathcal{U}))$, the collection $\{\mathbf{U}_x : x \in E\}$ consists of countable many disjoint elements. Relabel these elements by $\mathbf{U}_i$'s. Clearly $E \subset \bigcup_i \mathbf{U}_i$.

We now choose an integer k. Use $\mathcal{A}_1$ to denote the collection of those $\mathbf{U}_i$'s that are taken from Λ_1. They cover a certain subset Q_1 of E_k. The same subset is covered by a certain sub-collection of Λ_k, denoted as $\Lambda_{k,1}$. Since $\Lambda_{k,1}$ also covers the smaller set $Q_1 \cap E_1$, by (3.6),

$$(3.7) \qquad \sum_{\mathbf{U} \in \mathcal{A}_1} e^{-sm(\mathbf{U})} \leq \sum_{\mathbf{U} \in \Lambda_{k,1}} e^{-sm(\mathbf{U})} + \delta_1.$$

To see this, assume that (3.7) is false. Then by (3.6),

$$\sum_{\mathbf{U} \in (\Lambda_1 \backslash \mathcal{A}_1) \cup \Lambda_{k,1}} e^{-sm(\mathbf{U})} < M_N^s(\mathcal{U}, E_1),$$

which contradicts the fact that $(\Lambda_1 \backslash \mathcal{A}_1) \cup \Lambda_{k,1} \subset \bigcup_{j \geq N} \mathcal{W}_j(\mathbf{U})$ is an open cover of E_1. Next we use $\mathcal{A}_2$ to denote the collection of those $\mathbf{U}_i$'s that are taken from Λ_2 but not from Λ_1. Define $\Lambda_{k,2}$ similarly. As above, we find

$$(3.8) \qquad \sum_{\mathbf{U} \in \mathcal{A}_2} e^{-sm(\mathbf{U})} \leq \sum_{\mathbf{U} \in \Lambda_{k,2}} e^{-sm(\mathbf{U})} + \delta_2.$$

We repeat the argument until all coverings Λ_n, $n \leq k$, have been considered. Note that $\bigcup_{\mathbf{U} \in \Lambda_{k,i}} \mathbf{U} \subseteq \bigcup_{\mathbf{U} \in \mathcal{A}_i} \mathbf{U}$ for $i \leq k$. For different $i, i' \leq k$, the elements in $\Lambda_{k,i}$ are disjoint from those in $\Lambda_{k,i'}$. The k inequalities (3.7), (3.8), $\dots$, are added which yields

$$\sum_{\mathbf{U} \in \bigcup_{n=1}^{k} \mathcal{A}_n} e^{-sm(\mathbf{U})} \leq \sum_{\mathbf{U} \in \bigcup_{n=1}^{k} \Lambda_{k,n}} e^{-sm(\mathbf{U})} + \sum_{n=1}^{k} \delta_n \leq \mathcal{M}_N^s(\mathcal{U}, E_k) + \sum_{n=1}^{k} \delta_n + \delta_k.$$

Letting $k \to \infty$, we have

$$\sum_i e^{-sm(\mathbf{U}_i)} \leq \lim_{k \to \infty} \mathcal{M}_N^s(\mathcal{U}, E_k) + \sum_{n=1}^{\infty} \delta_n.$$

Since $\sum_{n=1}^{\infty} \delta_n$ can be chosen arbitrarily small we have

$$\mathcal{M}_N^s(\mathcal{U}, E) \leq \lim_{k \to \infty} \mathcal{M}_N^s(\mathcal{U}, E_k).$$

Since the opposite inequality is trivial we have proved (i). $\qquad \square$

Proof of Theorem 3.6. Let Z be an analytic subset of X with $h_{\text{top}}^B(T, Z) > 0$. Let $0 < s < h_{\text{top}}^B(T, Z)$. By (2.2), there exists a closed-open partition $\mathcal{U}$ so that $h_{\text{top}}^B(T, \mathcal{U}, Z) > s$ and thus $\mathcal{M}^s(\mathcal{U}, Z) = \infty$. Hence $\mathcal{M}_N^s(\mathcal{U}, Z) > 0$ for some $N \in \mathbb{N}$. By Proposition 3.7, we can find a compact set $K \subset Z$ such that $\mathcal{M}_N^s(\mathcal{U}, K) > 0$. It implies $h_{\text{top}}^B(T, K) \geq h_{\text{top}}^B(T, \mathcal{U}, K) \geq s$. $\qquad \square$

Before we prove Theorem 1.2(ii), we still need some notation and additional results in topological dynamical systems.

Let us define the *natural extension* $(\widetilde{X}, \widetilde{T})$ of a TDS (X, T) with a metric d and a surjective map T where $\widetilde{X} = \{(x_1, x_2, \cdots) : T(x_{i+1}) = x_i, x_i \in X, i \in \mathbb{N}\}$ is a subspace of the product space $X^{\mathbb{N}} = \Pi_{i=1}^{\infty} X$ endowed with the compatible metric d_T as

$$d_T((x_1, x_2, \cdots), (y_1, y_2, \cdots)) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i},$$

$\widetilde{T} : \widetilde{X} \to \widetilde{X}$ is the shift homeomorphism with $\widetilde{T}(x_1, x_2, \cdots) = (T(x_1), x_1, x_2, \cdots)$, and $\pi_i : \widetilde{X} \to X$ is the projection to the i-th coordinate. Clearly, $\pi_i : (\widetilde{X}, \widetilde{T}) \to (X, T)$ is a factor map.

Lemma 3.8. *Let (X, T) be a TDS with a metric d and a surjective map T, $(\widetilde{X}, \widetilde{T})$ be the natural extension of (X, T) and $\pi_1 : \widetilde{X} \to X$ be the projection to the first coordinate. Then $\sup_{x \in X} h_{\mathrm{top}}^{UC}(\widetilde{T}, \pi_1^{-1}(x)) = 0$.*

Proof. Fix $x \in X$. For any $\epsilon > 0$, take $N \in \mathbb{N}$ large enough such that $\sum_{i=N}^{\infty} \frac{\mathrm{diam}(X)}{2^i} < \epsilon$.

Let $E_N \subseteq \pi_1^{-1}(x)$ be a finite (N, ϵ)-spanning set of $\pi_1^{-1}(x)$. Next we are to show that E_N is also a (n, ϵ)-spanning set of $\pi_1^{-1}(x)$ for $n > N$.

Fix $n \in \mathbb{N}$ with $n > N$. For any $\widetilde{y} \in \pi_1^{-1}(x)$, since E_N is a (N, ϵ)-spanning set of $\pi_1^{-1}(x)$ there exist $\widetilde{x} \in E_N$ such that $d_T(\widetilde{T}^i \widetilde{x}, \widetilde{T}^i \widetilde{y}) < \epsilon$ for $i = 0, 1, \cdots, N-1$. Now for $k \in \{N, N+1, \cdots, n-1\}$, we have $\pi_j(\widetilde{T}^k \widetilde{x}) = \pi_j(\widetilde{T}^k \widetilde{y})) = T^{k-j+1}(x)$ for $j = 1, \cdots, k, k+1$. Thus

$$d_T(\widetilde{T}^k \widetilde{x}, \widetilde{T}^k \widetilde{y}) = \sum_{j=1}^{\infty} \frac{d(\pi_j(\widetilde{T}^k \widetilde{x}), \pi_j(\widetilde{T}^k \widetilde{x}))}{2^j} = \sum_{j=k+2}^{\infty} \frac{d(\pi_j(\widetilde{T}^k \widetilde{x}), \pi_j(\widetilde{T}^k \widetilde{x}))}{2^j}$$

$$\leq \sum_{j=k+2}^{\infty} \frac{\mathrm{diam}(X)}{2^j} \leq \sum_{j=N}^{\infty} \frac{\mathrm{diam}(X)}{2^j} < \epsilon.$$

This implies $(d_T)_n(\widetilde{x}, \widetilde{y}) < \epsilon$. Hence E_N is also a (n, ϵ)-spanning set of $\pi_1^{-1}(x)$ for $n > N$. Let $\tilde{r}_n(\pi^{-1}(x), \epsilon)$ denote the smallest cardinality of (n, ϵ)-spanning sets of $\pi^{-1}(x)$. Then $\tilde{r}_n(\pi^{-1}(x), \epsilon) \leq \#(E_N)$. Hence

$$h_{\mathrm{top}}^{UC}(\widetilde{T}, \pi_1^{-1}(x)) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \tilde{r}_n(\pi^{-1}(x), \epsilon) \leq \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \#(E_N) = 0.$$

This ends the proof of the lemma. $\qquad \square$

In the following part we will lift general TDSs having finite topological entropy to zero dimensional TDSs by the so called *principal extensions*.

Definition 3.9. [18] *An extension $\pi : (Z, R) \to (X, T)$ between two TDSs is a principal extension if $h_\nu(R) = h_{\nu \circ \pi^{-1}}(T)$ for every $\nu \in M(Z, R)$.*

The following general result is needed in our proof of Theorem 1.2(ii).

Proposition 3.10 (Proposition 7.8 in [5])**.** *Every invertible TDS (X, T) with $h_{top}(T) < \infty$ has a zero dimensional principal extension (Z, R) with R being invertible.*

Let $\pi : (Y, S) \to (X, T)$ be a factor map between two TDSs. Bowen proved that $h_{\text{top}}(S) \le h_{\text{top}}(T) + \sup_{x \in X} h_{\text{top}}^{UC}(S, \pi^{-1}(x))$ (cf. [3, Theorem 17]). In fact, Bowen's proof is also valid for the following result (see, i.e. Theorem 7.3 in [14] for a detailed proof).

Theorem 3.11. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between two TDSs. Then for any $E \subseteq X$ one has*

$$(3.9) \qquad h_{\text{top}}^B(S, \pi(E)) \le h_{\text{top}}^B(T, E) \le h_{\text{top}}^B(S, \pi(E)) + \sup_{y \in Y} h_{\text{top}}^{UC}(T, \pi^{-1}(y)).$$

We also need the following variational principle of conditional entropies.

Proposition 3.12. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between two TDSs. Then we have*

$$(3.10) \qquad \sup_{y \in Y} h_{\text{top}}^{UC}(T, \pi^{-1}(y)) = \sup_{\mu \in M(X,T)} (h_\mu(T) - h_{\mu \circ \pi^{-1}}(S)).$$

Proof. It is the direct combination of [8, Theorem 3] and [19, Theorem 2.1]. $\square$

Lemma 3.13. *Let (X, T) be a TDS with $h_{\text{top}}(T) < \infty$. Then there exists a factor map $\pi : (H, \Gamma) \to (X, T)$ such that (H, Γ) is zero dimensional and*

$$\sup_{x \in X} h_{\text{top}}^{UC}(\Gamma, \pi^{-1}(x)) = 0.$$

Proof. First, we take $D = \{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\}$ and let $Z = X \times D$. Define $R : Z \to Z$ satisfying $R(x, \frac{1}{n+1}) = (x, \frac{1}{n})$, $n \in \mathbb{N}$; $R(x, 1) = (Tx, 1)$ and $R(x, 0) = (x, 0)$ for $x \in X$. Then (Z, R) is a TDS and R is surjective. If we identity $(x, 1)$ with x for each $x \in X$, then X can be viewed as a closed subset of Z and $R|_X = T$. It is also clear that $h_{\text{top}}(R) = h_{\text{top}}(T) < \infty$.

Let $(\widetilde{Z}, \widetilde{R})$ be the natural extension of (Z, R) and $\pi_1 : \widetilde{Z} \to Z$ be the projection to the first coordinate. Then

$$(3.11) \qquad \qquad \sup_{z \in Z} h_{\text{top}}^{UC}(\widetilde{R}, \pi_1^{-1}(z)) = 0$$

by Lemma 3.8, and, so $h_{\mathrm{top}}(\widetilde{R}) = h_{\mathrm{top}}(R) < \infty$. Since $\widetilde{R}$ is homeomorphism on $\widetilde{Z}$, by Lemma 3.10, there exists a factor map $\psi : (W, G) \to (\widetilde{Z}, \widetilde{R})$ such that (W, G) is a zero-dimensional TDS and ψ is principal extension.

Since $h_{\mathrm{top}}(\widetilde{R}) < \infty$ and ψ is principal extension, we have the following variational principle of condition entropy

$$(3.12) \qquad \sup_{\widetilde{z} \in \widetilde{Z}} h_{\mathrm{top}}^{UC}(G, \psi^{-1}(\widetilde{z})) = \sup_{\theta \in M(W,G)} (h_\theta(G) - h_{\theta \circ \psi^{-1}}(\widetilde{R})) = 0.$$

The first equality in (3.12) follows from (3.10).

Let $H = \psi^{-1}(\pi_1^{-1} X)$, $\Gamma = G|_H$ and $\pi = \pi_1 \circ \psi|_H$. Then (H, Γ) be a zero-dimensional TDS and $\pi : (H, \Gamma) \to (X, T)$ be a factor map. Applying Proposition 3.12 to the factor map $\pi : (H, \Gamma) \to (X, T)$, we obtain

$$
\begin{aligned}
\sup_{x \in X} h_{\mathrm{top}}^{UC}(\Gamma, \pi^{-1}(x)) \ &= \ \sup_{\mu \in M(H,\Gamma)} (h_\mu(\Gamma) - h_{\mu \circ \pi^{-1}}(T)) \\
&\leq \ \sup_{\mu \in M(W,G)} (h_\mu(\Gamma) - h_{\mu \circ \pi^{-1}}(T)) \\
&= \ \sup_{\mu \in M(W,G)} (h_\mu(\Gamma) - h_{\mu \circ \psi^{-1}}(T) + h_{\mu \circ \psi^{-1}}(T) - h_{\mu \circ \pi^{-1}}(T)) \\
&\leq \ \sup_{\mu \in M(W,G)} (h_\mu(\Gamma) - h_{\mu \circ \psi^{-1}}(T)) + \sup_{\nu \in (\widetilde{Z}, \widetilde{R})} (h_\nu(\widetilde{R}) - h_{\nu \circ \pi_1^{-1}}(R)) \\
&= \ \sup_{\widetilde{z} \in \widetilde{Z}} h_{\mathrm{top}}^{UC}(G, \psi^{-1}(\widetilde{z})) + \sup_{z \in Z} h_{\mathrm{top}}^{UC}(\widetilde{R}, \pi_1^{-1}(z)) \\
&= \ 0 \qquad \qquad (\text{ by } (3.12), (3.11)).
\end{aligned}
$$

This shows $\sup_{x \in X} h_{\mathrm{top}}^{UC}(\Gamma, \pi^{-1}(x)) = 0$. $\qquad\qquad\qquad\qquad\qquad\square$

Proof of Theorem 1.2(ii). By Lemma 3.13, there exists a factor map $\pi : (Y, S) \to (X, T)$ such that (Y, S) is zero dimensional and $\sup_{x \in X} h_{\mathrm{top}}^{UC}(S, \pi^{-1}(x)) = 0$. By Theorem 3.11, we have that for any $F \subset Y$,

$$(3.13) \qquad\qquad\qquad h_{\mathrm{top}}^B(S, F) = h_{\mathrm{top}}^B(T, \pi(F)).$$

Let Z be an analytic subset of X. Then $\pi^{-1}(Z)$ is also an analytic set of Y (cf. Federer [11, 2.2.10]). By (3.13) and Theorem 3.6,

$$
\begin{aligned}
h_{\mathrm{top}}^B(T, Z) = h_{\mathrm{top}}^B(S, \pi^{-1}(Z)) &= \sup\{h_{\mathrm{top}}^B(S, E) : \ E \subseteq \pi^{-1}(Z), \ E \text{ is compact}\} \\
&= \sup\{h_{\mathrm{top}}^B(T, \pi(E)) : \ E \subseteq \pi^{-1}(Z), \ E \text{ is compact}\} \\
&\leq \sup\{h_{\mathrm{top}}^B(T, K) : \ K \subseteq Z, \ K \text{ is compact}\}.
\end{aligned}
$$

The reverse inequality is trivial, so

$$h_{\mathrm{top}}^B(T, Z) = \sup\{h_{\mathrm{top}}^B(T, K) : \ K \subseteq Z, \ K \text{ is compact}\}.$$

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.14. For an invertible TDS (X, T), Lindenstrauss and Weiss [21] introduced the mean dimension $mdim(X, T)$ (an idea suggested by Gromov). It is well known that for an invertible TDS (X, T), if $h_{\text{top}}(T) < \infty$ or the topological dimension of X is finite, then $mdim(X, T) = 0$ (see [21, Definition 2.6 and Theorem 4.2]).

In general, one can show that for an invertible TDS (X, T), if $mdim(X, T) = 0$ then (X, T) has a zero dimensional principal extension (Z, R) with R being invertible. Indeed, let (Y, S) be an irrational rotation on the circle. Then $(X \times Y, T \times S)$ admits a nonperiodic minimal factor (Y, S) and $mdim(X \times Y, T \times S) = 0$. Hence $(X \times Y, T \times S)$ has the so called small boundary property [20, Theorem 6.2], which implies the existence of a basis of the topology consisting of sets whose boundaries have measure zero for every invariant measure. With these results it is easy to construct a refining sequence of small-boundary partitions for $(X \times Y, T \times S)$, where the partitions have small boundaries if their boundaries have measure zero for all $\mu \in \mathcal{M}(X \times Y, T \times S)$. Then by a standard construction (see p. 152-153 in [5]), which associates to this sequence a zero dimensional principal extension (Z, R) of $(X \times Y, T \times S)$ with R being invertible. Finally note that $(X \times Y, T \times S)$ is a principal extension of (X, T), we know that (Z, R) is also a zero dimensional principal extension of (X, T) since the composition of two principal extensions is still a principal extension.

Remark 3.15. By Remark 3.14, we may strengthen Theorem 1.2(ii) as follows: Let (X, T) be a TDS with $mdim(X, T) = 0$. Then for any analytic set $Z \subseteq X$,

$$h_{\text{top}}^B(T, Z) = \sup\{h_{\text{top}}^B(T, K) : \ K \subseteq Z, \ K \text{ is compact}\}.$$

4. Variational principle for the packing topological entropy

In this section we prove Theorem 1.3. We first give a lemma.

Lemma 4.1. *Let $Z \subset X$ and $s, \epsilon > 0$. Assume $P_\epsilon^s(Z) = \infty$. Then for any given finite interval $(a, b) \subset \mathbb{R}$ with $a \geq 0$ and any $N \in \mathbb{N}$, there exists a finite disjoint collection $\{\overline{B}_{n_i}(x_i, \epsilon)\}$ such that $x_i \in Z$, $n_i \geq N$ and $\sum_i e^{-n_i s} \in (a, b)$.*

Proof. Take $N_1 > N$ large enough such that $e^{-N_1 s} < b - a$. Since $P_\epsilon^s(Z) = \infty$, we have $P_{N_1, \epsilon}^s(Z) = \infty$. Thus there is a finite disjoint collection $\{\overline{B}_{n_i}(x_i, \epsilon)\}$ such that $x_i \in Z$, $n_i \geq N_1$ and $\sum_i e^{-n_i s} > b$. Since $e^{-n_i s} < b - a$, by discarding elements in this collection one by one until we can have $\sum'_i e^{-n_i s} \in (a, b)$. $\square$

Proof of Theorem 1.3. We divide the proof into two parts:

Part 1. $h_{\text{top}}^P(T, Z) \geq \sup\{\overline{h}_\mu(T) : \mu \in M(X), \ \mu(Z) = 1\}$ *for any Borel set $Z \subseteq X$.*

To see this, let $\mu \in \mathcal{M}(X)$ with $\mu(Z) = 1$ for some Borel set $Z \subseteq X$. We need to show that $h_{\text{top}}^P(T, Z) \geq \overline{h}_\mu(T)$. For this purpose we may assume $\overline{h}_\mu(T) > 0$; otherwise we have nothing to prove. Let $0 < s < \overline{h}_\mu(T)$. Then there exist $\epsilon, \delta > 0$, and a Borel set $A \subset Z$ with $\mu(A) > 0$ such that

$$\overline{h}_\mu(T, x, \epsilon) > s + \delta, \quad \forall \, x \in A,$$

where $\overline{h}_\mu(T, x, \epsilon) := \limsup_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \epsilon))$.

Next we show that $\mathcal{P}_{\epsilon/5}^s(Z) = \infty$, which implies that $h_{\text{top}}^P(T, Z) \geq h_{\text{top}}^P(T, Z, \epsilon/5) \geq s$. To achieve this, it suffices to show that $P_{\epsilon/5}^s(E) = \infty$ for any Borel $E \subset A$ with $\mu(E) > 0$. Fix such a set E. Define

$$E_n = \{x \in E : \ \mu(B_n(x, \epsilon)) < e^{-n(s+\delta)}\}, \quad n \in \mathbb{N}.$$

Since $E \subset A$, we have $\bigcup_{n=N}^\infty E_n = E$ for each $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$. Then $\mu(\bigcup_{n=N}^\infty E_n) = \mu(E)$, and hence there exists $n \geq N$ such that

$$\mu(E_n) \geq \frac{1}{n(n+1)} \mu(E).$$

Fix such n and consider the family $\{B_n(x, \epsilon/5) : \ x \in E_n\}$. By Lemma 3.3 (in which we use d_n instead of d), there exists a finite pairwise disjoint family $\{B_n(x_i, \epsilon/5)\}$ with $x_i \in E_n$ such that

$$\bigcup_i B_n(x_i, \epsilon) \supset \bigcup_{x \in E_n} B_n(x, \epsilon/5) \supset E_n.$$

Hence

$$
\begin{aligned}
P_{N, \epsilon/5}^s(E) &\geq P_{N, \epsilon/5}^s(E_n) \geq \sum_i e^{-ns} \geq e^{n\delta} \sum_i e^{-n(s+\delta)} \\
&\geq e^{n\delta} \sum_i \mu(B_n(x_i, \epsilon)) \geq e^{n\delta} \mu(E_n) \geq \frac{e^{n\delta}}{n(n+1)} \mu(E).
\end{aligned}
$$

Since $\frac{e^{n\delta}}{n(n+1)} \to \infty$ as $n \to \infty$, letting $N \to \infty$ we obtain that $P_{\epsilon/5}^s(E) = \infty$.

Part 2. Let $Z \subseteq X$ be analytic with $h_{\text{top}}^P(T, Z) > 0$. For any $0 < s < h_{\text{top}}^P(T, Z)$, there exists a compact set $K \subseteq Z$ and $\mu \in M(K)$ such that $\overline{h}_\mu(T) \geq s$.

Since Z is analytic, there exists a continuous surjective map $\phi : \ \mathcal{N} \to Z$. Let $\Gamma_{n_1, n_2, \ldots, n_p}$ be the set of $(m_1, m_2, \ldots) \in \mathcal{N}$ such that $m_1 \leq n_1, m_2 \leq n_2, \ldots, m_p \leq n_p$ and let $Z_{n_1, \ldots, n_p}$ be the image of $\Gamma_{n_1, \ldots, n_p}$ under ϕ.

Take $\epsilon > 0$ small enough so that $0 < s < h_{\text{top}}^P(T, Z, \epsilon)$. Take $t \in (s, h_{\text{top}}^P(T, Z, \epsilon))$. We are going to construct inductively a sequence of finite sets $(K_i)_{i=1}^\infty$ and a sequence of finite measures $(\mu_i)_{i=1}^\infty$ so that $K_i \subset Z$ and μ_i is supported on K_i for each i.

Together with these two sequences, we construct also a sequence of integers (n_i), a sequence of positive numbers (γ_i) and a sequence of integer-valued functions $(m_i : K_i \to \mathbb{N})$. The method of our construction is inspired somehow by the work of Joyce and Preiss [16] on packing measures.

The construction is divided into several small steps:

Step 1. Construct K_1 and μ_1, as well as $m_1(\cdot)$, n_1 and γ_1.

Note that $\mathcal{P}_\epsilon^t(Z) = \infty$. Let
$$H = \bigcup \{G \subset X : \; G \text{ is open}, \; \mathcal{P}_\epsilon^t(Z \cap G) = 0\}.$$
Then $\mathcal{P}_\epsilon^t(Z \cap H) = 0$ by the separability of X. Let $Z' = Z \backslash H = Z \cap (X \backslash H)$. For any open set $G \subset X$, either $Z' \cap G = \emptyset$, or $\mathcal{P}_\epsilon^t(Z' \cap G) > 0$. To see this, assume $\mathcal{P}_\epsilon^t(Z' \cap G) = 0$ for an open set G; then $\mathcal{P}_\epsilon^t(Z \cap G) \le \mathcal{P}_\epsilon^t(G \cap Z') + \mathcal{P}_\epsilon^t(Z \cap H) = 0$, implying $G \subset H$ and hence $Z' \cap G = \emptyset$.

Note that $\mathcal{P}_\epsilon^t(Z') = \mathcal{P}_\epsilon^t(Z) = \infty$ (because $\mathcal{P}_\epsilon^t(Z) \le \mathcal{P}_\epsilon^t(Z') + \mathcal{P}_\epsilon^t(Z \cap H) = \mathcal{P}_\epsilon^t(Z')$). It follows $\mathcal{P}_\epsilon^s(Z') = \infty$. By Lemma 4.1, we can find a finite set $K_1 \subset Z'$, an integer-valued function $m_1(x)$ on K_1 such that the collection $\{\overline{B}_{m_1(x)}(x, \epsilon)\}_{x \in K_1}$ is disjoint and
$$\sum_{x \in K_1} e^{-m_1(x)s} \in (1, 2).$$
Define $\mu_1 = \sum_{x \in K_1} e^{-m_1(x)s} \delta_x$, where δ_x denotes the Dirac measure at x. Take a small $\gamma_1 > 0$ such that for any function $z : K_1 \to X$ with $d(x, z(x)) \le \gamma_1$, we have for each $x \in K_1$,
$$(4.1) \quad \left(\overline{B}(z(x), \gamma_1) \cup \overline{B}_{m_1(x)}(z(x), \epsilon) \right) \cap \left(\bigcup_{y \in K_1 \backslash \{x\}} \overline{B}(z(y), \gamma_1) \cup \overline{B}_{m_1(y)}(z(y), \epsilon) \right) = \emptyset.$$

Here and afterwards, $\overline{B}(x, \epsilon)$ denotes the closed ball $\{y \in X : \; d(x, y) \le \epsilon\}$. Since $K_1 \subset Z'$, $\mathcal{P}_\epsilon^t(Z \cap B(x, \gamma_1/4)) \ge \mathcal{P}_\epsilon^t(Z' \cap B(x, \gamma_1/4)) > 0$ for each $x \in K_1$. Therefore we can pick a large $n_1 \in \mathbb{N}$ so that $Z_{n_1} \supset K_1$ and $\mathcal{P}_\epsilon^t(Z_{n_1} \cap B(x, \gamma_1/4)) > 0$ for each $x \in K_1$.

Step 2. Construct K_2 and μ_2, as well as $m_2(\cdot)$, n_2 and γ_2.

By (4.1), the family of balls $\{\overline{B}(x, \gamma_1)\}_{x \in K_1}$, are pairwise disjoint. For each $x \in K_1$, since $\mathcal{P}_\epsilon^t(Z_{n_1} \cap B(x, \gamma_1/4)) > 0$, we can construct as Step 1, a finite set
$$E_2(x) \subset Z_{n_1} \cap B(x, \gamma_1/4)$$
and an integer-valued function
$$m_2 : E_2(x) \to \mathbb{N} \cap [\max\{m_1(y) : \; y \in K_1\}, \infty)$$
such that

(2-a) $\mathcal{P}_\epsilon^t(Z_{n_1} \cap G) > 0$ for each open set G with $G \cap E_2(x) \neq \emptyset$;

(2-b) The elements in $\{\overline{B}_{m_2(y)}(y, \epsilon)\}_{y \in E_2(x)}$ are disjoint, and

$$\mu_1(\{x\}) < \sum_{y \in E_2(x)} e^{-m_2(y)s} < (1 + 2^{-2})\mu_1(\{x\}).$$

To see it, we fix $x \in K_1$. Denote $F = Z_{n_1} \cap B(x, \gamma_1/4)$. Let

$$H_x := \bigcup\{G \subset X : G \text{ is open } \mathcal{P}_\epsilon^t(F \cap G) = 0\}.$$

Set $F' = F \backslash H_x$. Then as in Step 1, we can show that $\mathcal{P}_\epsilon^t(F') = \mathcal{P}_\epsilon^t(F) > 0$ and furthermore, $\mathcal{P}_\epsilon^t(F' \cap G) > 0$ for any open set G with $G \cap F' \neq \emptyset$. Note that $\mathcal{P}_\epsilon^s(F') = \infty$ (since $s < t$), by Lemma 4.1, we can find a finite set $E_2(x) \subset F'$ and a map $m_2 : E_2(x) \to \mathbb{N} \cap [\max\{m_1(y) : y \in K_1\}, \infty)$ so that (2-b) holds. Observe that if a open set G satisfies $G \cap E_2(x) \neq \emptyset$, then $G \cap F' \neq \emptyset$, and hence $\mathcal{P}_\epsilon^t(Z_{n_1} \cap G) \geq \mathcal{P}_\epsilon^t(F' \cap G) > 0$. Thus (2-a) holds.

Since the family $\{\overline{B}(x, \gamma_1)\}_{x \in K_1}$ is disjoint, $E_2(x) \cap E_2(x') = \emptyset$ for different $x, x' \in K_1$. Define $K_2 = \bigcup_{x \in K_1} E_2(x)$ and

$$\mu_2 = \sum_{y \in K_2} e^{-m_2(y)s} \delta_y.$$

By (4.1) and (2-b), the elements in $\{\overline{B}_{m_2(y)}(y, \epsilon)\}_{y \in K_2}$ are pairwise disjoint. Hence we can take $0 < \gamma_2 < \gamma_1/4$ such that for any function $z : K_2 \to X$ with $d(x, z(x)) < \gamma_2$ for $x \in K_2$, we have

$$(4.2) \quad \left(\overline{B}(z(x), \gamma_2) \cup \overline{B}_{m_2(x)}(z(x), \epsilon)\right) \cap \left(\bigcup_{y \in K_2 \backslash \{x\}} \overline{B}(z(y), \gamma_2) \cup \overline{B}_{m_2(y)}(z(y), \epsilon)\right) = \emptyset$$

for each $x \in K_2$. Choose a large $n_2 \in \mathbb{N}$ such that $Z_{n_1, n_2} \supset K_2$ and $\mathcal{P}_\epsilon^t(Z_{n_1, n_2} \cap B(x, \gamma_2/4)) > 0$ for each $x \in K_2$.

Step 3. Assume that K_i, μ_i, $m_i(\cdot)$, n_i and γ_i have been constructed for $i = 1, \ldots, p$. In particular, assume that for any function $z : K_p \to X$ with $d(x, z(x)) < \gamma_p$ for $x \in K_p$, we have

$$(4.3) \quad \left(\overline{B}(z(x), \gamma_p) \cup \overline{B}_{m_p(x)}(z(x), \epsilon)\right) \cap \left(\bigcup_{y \in K_p \backslash \{x\}} \overline{B}(z(y), \gamma_p) \cup \overline{B}_{m_p(y)}(z(y), \epsilon)\right) = \emptyset$$

for each $x \in K_p$; and $Z_{n_1, \ldots, n_p} \supset K_p$ and $\mathcal{P}_\epsilon^t(Z_{n_1, \ldots, n_p} \cap B(x, \gamma_p/4)) > 0$ for each $x \in K_p$. We construct below each term of them for $i = p + 1$ in a way similar to Step 2.

Note that the elements in $\{\overline{B}(x, \gamma_p)\}_{x \in K_p}$ are pairwise disjoint. For each $x \in K_p$, since $\mathcal{P}_\epsilon^t(Z_{n_1, \ldots, n_p} \cap B(x, \gamma_p/4)) > 0$, we can construct as Step 2, a finite set

$$E_{p+1}(x) \subset Z_{n_1, \ldots, n_p} \cap B(x, \gamma_p/4)$$

and an integer-valued function

$$m_{p+1}: \ E_{p+1}(x) \to \mathbb{N} \cap [\max\{m_p(y): \ y \in K_p\}, \infty)$$

such that

(3-a) $\mathcal{P}_\epsilon^t(Z_{n_1,\dots,n_p} \cap G) > 0$ for each open set G with $G \cap E_{p+1}(x) \neq \emptyset$; and

(3-b) $\{\overline{B}_{m_{p+1}(y)}(y, \epsilon)\}_{y \in E_{p+1}(x)}$ are disjoint and satisfy

$$\mu_p(\{x\}) < \sum_{y \in E_{p+1}(x)} e^{-m_{p+1}(y)s} < (1 + 2^{-p-1})\mu_p(\{x\}).$$

Clearly $E_{p+1}(x) \cap E_{p+1}(x') = \emptyset$ for different $x, x' \in K_p$. Define $K_{p+1} = \bigcup_{x \in K_p} E_{p+1}(x)$ and

$$\mu_{p+1} = \sum_{y \in K_{p+1}} e^{-m_{p+1}(y)s}\delta_y.$$

By (4.3) and (3-b), $\{\overline{B}_{m_{p+1}(y)}(y, \epsilon)\}_{y \in K_{p+1}}$ are disjoint. Hence we can take $0 < \gamma_{p+1} < \gamma_p/4$ such that for any function $z: \ K_{p+1} \to X$ with $d(x, z(x)) < \gamma_{p+1}$, we have for each $x \in K_{p+1}$,

(4.4)
$$\left(\overline{B}(z(x), \gamma_{p+1}) \cup \overline{B}_{m_{p+1}(x)}(z(x), \epsilon)\right) \cap \left(\bigcup_{y \in K_{p+1} \setminus \{x\}} \overline{B}(z(y), \gamma_{p+1}) \cup \overline{B}_{m_{p+1}(y)}(z(y), \epsilon)\right) = \emptyset.$$

Choose a large $n_{p+1} \in \mathbb{N}$ such that $Z_{n_1,\dots,n_{p+1}} \supset K_{p+1}$ and

$$\mathcal{P}_\epsilon^t(Z_{n_1,\dots,n_{p+1}} \cap B(x, \gamma_{p+1}/4)) > 0$$

for each $x \in K_{p+1}$.

As in the above steps, we can construct by induction the sequences (K_i), (μ_i), $(m_i(\cdot))$, (n_i) and (γ_i). We summarize some of their basic properties as follows:

(a) For each i, the family $\mathcal{F}_i := \{\overline{B}(x, \gamma_i): \ x \in K_i\}$ is disjoint. Each element in $\mathcal{F}_{i+1}$ is a subset of $\overline{B}(x, \gamma_i/2)$ for some $x \in K_i$.

(b) For each $x \in K_i$ and $z \in \overline{B}(x, \gamma_i)$,

$$\overline{B}_{m_i(x)}(z, \epsilon) \cap \bigcup_{y \in K_i \setminus \{x\}} \overline{B}(y, \gamma_i) = \emptyset \text{ and}$$

$$\mu_i(\overline{B}(x, \gamma_i)) = e^{-m_i(x)s} \leq \sum_{y \in E_{i+1}(x)} e^{-m_{i+1}(y)s} \leq (1 + 2^{-i-1})\mu_i(\overline{B}(x, \gamma_i)),$$

where $E_{i+1}(x) = B(x, \gamma_i) \cap K_{i+1}$.

The second part in (b) implies,

$$\mu_i(F_i) \leq \mu_{i+1}(F_i) = \sum_{F \in \mathcal{F}_{i+1}: \ F \subset F_i} \mu_{i+1}(F) \leq (1 + 2^{-i-1})\mu_i(F_i), \qquad F_i \in \mathcal{F}_i$$

Using the above inequalities repeatedly, we have for any $j > i$,

$$(4.5) \qquad \mu_i(F_i) \leq \mu_j(F_i) \leq \prod_{n=i+1}^{j} (1 + 2^{-n})\mu_i(F_i) \leq C\mu_i(F_i), \quad \forall F_i \in \mathcal{F}_i,$$

where $C := \prod_{n=1}^{\infty}(1 + 2^{-n}) < \infty$.

Let $\tilde{\mu}$ be a limit point of (μ_i) in the weak-star topology. Let

$$K = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i \geq n} K_i}.$$

Then μ is supported on K. Furthermore

$$K = \bigcap_{n=1}^{\infty} \overline{\bigcup_{i \geq n} K_i} \subset \bigcap_{p=1}^{\infty} \overline{Z_{n_1,\ldots,n_p}}.$$

However by the continuity of ϕ, we can show that $\bigcap_{p=1}^{\infty} Z_{n_1,\ldots,n_p} = \bigcap_{p=1}^{\infty} \overline{Z_{n_1,\ldots,n_p}}$ by applying Cantor's diagonal argument. Hence K is a compact subset of Z.

On the other hand, by (4.5),

$$e^{-m_i(x)s} = \mu_i(\overline{B}(x, \gamma_i)) \leq \tilde{\mu}(B(x, \gamma_i)) \leq C\mu_i(\overline{B}(x, \gamma_i)) = Ce^{-m_i(x)s}, \quad \forall x \in K_i.$$

In particular, $1 \leq \sum_{x \in K_1} \mu_1(B(x, \gamma_1)) \leq \tilde{\mu}(K) \leq \sum_{x \in K_1} C\mu_1(B(x, \gamma_1)) \leq 2C$. Note that $K \subset \bigcup_{x \in K_i} \overline{B}(x, \gamma_i/2)$. By the first part of (b), for each $x \in K_i$ and $z \in \overline{B}(x, \gamma_i)$,

$$\tilde{\mu}(\overline{B}_{m_i(x)}(z, \epsilon)) \leq \tilde{\mu}(\overline{B}(x, \gamma_i/2)) \leq Ce^{-m_i(x)s}.$$

For each $z \in K$ and $i \in N$, $z \in \overline{B}(x, \gamma_i/2)$ for some $x \in K_i$. Hence

$$\tilde{\mu}(B_{m_i(x)}(z, \epsilon)) \leq Ce^{-m_i(x)s}.$$

Define $\mu = \tilde{\mu}/\tilde{\mu}(K)$. Then $\mu \in M(K)$, and for each $z \in K$, there exists a sequence $k_i \uparrow \infty$ such that $\mu(B_{k_i}(z, \epsilon)) \leq Ce^{-k_i s}/\tilde{\mu}(K)$. It follows that $\overline{h}_\mu(T) \geq s$. $\qquad \square$

5. Main notation and conventions

For the reader's convenience, we summarize in Table 1 the main notation and typographical conventions used in this paper.

TABLE 1. Main notation and conventions

(X,T)	A topological dynamical system (Sect. 1)
$M(X)$	Set of all Borel probability measures on X
$M(X,T),\ E(X,T)$	Set of T-invariant (resp. ergodic) Borel probability measures on X
d_n	n-th Bowen's metric (cf. (1.1))
$B(x,\epsilon),\ \overline{B}(x,\epsilon)$	Open (resp. closed) ball in (X,d) centered at x of radius ϵ
$B_n(X,\epsilon),\ \overline{B}_n(x,\epsilon)$	Open (resp. closed) ball in (X,d_n) centered at x of radius ϵ
$\overline{h}_\mu(T),\ \underline{h}_\mu(T)$	Measure-theoretic upper (resp. lower) entropy of T with respect to $\mu \in M(X)$ (Sect. 1)
$h_{\text{top}}^{UC}(T,Z)$	Upper capacity topological entropy of Z (Sect. 2)
$h_{\text{top}}^{B}(T,Z)$	Bowen's topological entropy of of Z (Sect. 2)
$h_{\text{top}}^{P}(T,Z)$	Packing topological entropy of Z (Sect. 2)
$h_{\text{top}}(T)$	Topological entropy of T (Sect. 2)
$\mathcal{M}_{N,\epsilon}^s(Z),\ \mathcal{M}_\epsilon^s(Z),\ \mathcal{M}^s(Z)$	(Sect. 2)
$\mathcal{W}_{N,\epsilon}^s(Z),\ \mathcal{W}_\epsilon^s(Z),\ \mathcal{W}^s(Z)$	(Sect. 2)
$P_{N,\epsilon}^s(Z),\ P_\epsilon^s(Z),\ \mathcal{P}_\epsilon^s(Z)$	(Sect. 2)
$\mathcal{M}_N^s(\mathcal{U},Z),\ \mathcal{M}^s(\mathcal{U},Z)$	(Sect. 2)
$h_{\text{top}}^{B}(T,\mathcal{U},Z)$	(Sect. 2)
$\mathcal{N}$	the set of infinite sequences of natural numbers endowed with product topology.

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