NON-UNIQUENESS OF ERGODIC MEASURES WITH FULL HAUSDORFF DIMENSION ON A GATZOURAS-LALLEY CARPET

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Abstract. In this note, we show that on certain Gatzouras-Lalley carpet, there exist more than one ergodic measures with full Hausdorff dimension. This gives a negative answer to a conjecture of Gatzouras and Peres in [8].

1. Introduction

The problem we are interested in is the uniqueness of ergodic invariant measures on non-conformal repellers with full Hausdorff dimension (see [7, 3] for a survey). For $C^{1+\alpha}$ conformal repellers, the existence and the uniqueness of an ergodic measure with full dimension follows from Bowen's equation together with the classical thermodynamic formalism [17].

For non-conformal repellers much less is known. The problem of existence of an ergodic measure with full dimension is solved for the class of Lalley-Gatzouras carpets and its nonlinear version [6, 11, 12]. In [8], Gatzouras and Peres conjectured that such a measure is unique. However, in this note, we show that this may fail on linear Lalley-Gatzouras carpets. Such a phenomenon is known for some other examples of self-affine sets constructed by Käenmäki and Vilppolainen [5].

To construct our example, let (X, σ_X) and (Y, σ_Y) be one-sided full shifts over finite alphabets $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $\pi : X \to Y$ be a 1-block factor map, i.e., there is a map $\widetilde{\pi} : \mathcal{A} \to \mathcal{B}$ such that

$$\pi(x) = (\widetilde{\pi}(x_i))_{i=1}^{\infty}, \quad x = (x_i)_{i=1}^{\infty} \in X.$$

Let $\phi : X \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ be two positive functions which are constants over the cylinders of first generation of X and Y respectively, i.e.,

$$\phi(x) = \phi(x_1), \quad \psi(y) = \psi(y_1)$$

for each $x = (x_i)_{i=1}^{\infty} \in X$ and $y = (y_i)_{i=1}^{\infty} \in Y$. Furthermore, assume that $\phi(x) > \psi(\pi(x))$ for all $x \in X$.

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Define

$$P(\phi,\psi)=\sup\left\{\frac{h_\mu(\sigma_X)-h_{\mu\circ\pi^{-1}}(\sigma_Y)}{\int\phi\,d\mu}+\frac{h_{\mu\circ\pi^{-1}}(\sigma_Y)}{\int\psi\circ\pi\,d\mu}\right\}, \tag{1.1}$$

where the supremum is taken over the collection $M(X,\sigma_X)$ of all σ_X-invariant Borel probability measures on X. Here $h_\mu(\sigma_X)$ stands for the measure-theoretic entropy of σ_X with respect to μ (cf. [17, 19]). Since the entropy maps $\mu\mapsto h_\mu(\sigma_X)$ and $\mu\mapsto h_{\mu\circ\pi^{-1}}(\sigma_Y)$ are upper semi-continuous, the supremum is attained on $M(X,\sigma_X)$. Moreover, since $\phi(x)$ and $\psi(y)$ only depend on the first coordinates of x and y, the supermum can be only attained at Bernoulli measures in $M(X,\sigma_X)$ [1].

In the next section, we construct an example to show that in the above general setting, there may have two different Bernoulli measures in $M(X,\sigma_X)$ attaining the supermum in (1.1), which leads to a counter-example to Gatzouras and Peres conjecture on Lalley-Gatzouras carpets (see Section 3).

In fact, Gatzouras and Peres raised the wider conjecture claiming that if f is a smooth expanding map, then any compact invariant set K which satisfies specification carries a unique ergodic invariant measure μ of full dimension. Moreover, μ is mixing for f. This conjecture was proved to be true in some special cases, e.g., as we said when f is a conformal $C^{1+\alpha}$ map on smooth Riemanian manifolds [8], and also when f is a linear diagonal endomorphism on the d-torus [4]. In particular, it is true for Bedford-McMullen self-affine carpets and sponges [2, 14, 9] and some sofic self-affine sets [18, 20, 15].

The same kind of questions have been studied on horseshoes. It is proved in [13] that for nonlinear horseshoes there may be no ergodic measure of full dimension, while such a measure exists for linear horseshoes [1], but may be not unique [16].

2. An example

Let $M(Y,\sigma_Y)$ denote the collection of all σ_Y-invariant Borel probability measures on Y. Notice that

$$P(\phi,\psi)=\sup_{\nu\in\mathcal{M}(Y,\sigma_Y)}P(\phi,\psi,\nu),$$

where

$$P(\phi,\psi,\nu)=\frac{h_\nu(\sigma_Y)}{\int\psi\,d\nu}+P(\phi,\nu),\quad P(\phi,\nu)=\sup_{\substack{\mu\in\mathcal{M}(X,\sigma_X),\\ \mu\circ\pi^{-1}=\nu}}\frac{h_\mu(\sigma_X)-h_\nu(\sigma_Y)}{\int\phi\,d\mu}$$

for $\nu\in M(Y,\sigma_Y)$. Since $\phi(x)$ and $\psi(y)$ only depend on the first coordinates of x and y, $P(\phi,\psi,\nu)$ can only be maximized at Bernoulli measures ν in $M(Y,\sigma_Y)$.

We make the following assumptions:

[1] As a related result, Luzia [12] proved recently that the supremum in (1.1) always can be attained at ergodic measures when ϕ and ψ are assumed to be general positive Hölder continuous functions.

(1) $\phi(x) \equiv \lambda > 0$ on X for some constant λ;

(2) $\mathcal{B} = \{a, b\}$, $\mathcal{A} = \{1, \cdots, \ell_a, \ell_a + 1, \cdots, \ell_a + \ell_b\}$, $\widetilde{\pi}(\{1, \cdots, \ell_a\}) = \{a\}$, $\widetilde{\pi}(\{\ell_a + 1, \cdots, \ell_a + \ell_b\}) = \{b\}$), where $\ell_a, \ell_b \in \mathbb{N}$.

Then, since due to our assumption we have $\int \phi \, d\mu = \lambda$ for all $\mu \in M(X, \sigma_X)$, the Ledrappier-Walters relativized variational principal [10] yields

$$P(\phi, \nu) = \frac{1}{\lambda}(\log(\ell_a)\nu([a]) + \log(\ell_b)\nu([b])),$$

where $[c] := \{y = (y_i)_{i=1}^{\infty} \in Y : \; y_1 = c\}$ for $c \in \mathcal{B}$. Setting $x = \nu([a])$ and $H(x) = -x\log(x) - (1-x)\log(1-x)$, we thus have for all Bernoulli measures $\nu \in M(Y, \sigma_Y)$,

$$P(\phi, \psi, \nu) = f(x) = \frac{1}{\lambda}(\log(\ell_a/\ell_b)x + \log(\ell_b)) + \frac{H(x)}{\psi_a x + \psi_b(1-x)}, \tag{2.1}$$

where ψ_a and ψ_b stand for the constant values of ψ over $[a]$ and $[b]$ respectively.

A counter-example will appear if we find λ, ℓ_a, ℓ_b, ψ_a, and ψ_b such that f attains its maximum for at least two values of x in $[0, 1]$.

Setting $U = \dfrac{\psi_b}{\lambda}\log(\ell_a/\ell_b)$ and $V = \dfrac{\psi_a - \psi_b}{\psi_b}$, the problem transfers to finding $U \in \mathbb{R}$, $V \in (-1, \infty)$ and $M \geq 0$ such that

$$g(x) = Ux - M + \frac{H(x)}{1 + Vx} \leq 0, \quad \forall \, x \in [0, 1]$$

and $g(x) = 0$ has more than one solution in $[0, 1]$. We can seek for a quadratic polynomial $F(x) = A - B(x - 1/2)^2$ with $A, B > 0$ such that

(i) $F(x) \geq H(x)$ for all $x \in [0, 1]$; and

(ii) the equation $F(x) = H(x)$ has more than one solution in $[0, 1]$.

Due to the common symmetry properties of F and H with respect to $x = 1/2$ and the concavity of these functions, this will be the case if we make sure that the curvature of F at $1/2$ is larger than that of H at $1/2$ and $\inf_{x \in [0,1]}(F(x) - H(x)) = 0$. Recalling that the curvature of a smooth function $h(x)$ being given by

$$\mathcal{K}_h(x) = \frac{|h''(x)|}{(1 + (h'(x))^2)^{3/2}},$$

we have $\mathcal{K}_H(1/2) = 4$ and $\mathcal{K}_F(1/2) = 2B$. Thus we get the following necessary and sufficient condition to guarantee that (i)-(ii) hold:

$$B > 2, \quad A = \max_{0 \leq x \leq 1}(B(x - 1/2)^2 + H(x)). \tag{2.2}$$

Now take a pair of numbers A, B so that (2.2) holds. Then the identity

$$-(Ux - M)(1 + Vx) = A - B(x - 1/2)^2$$

yields

$$\begin{cases} UV = B, \\ MV - U = B, \\ M = A - B/4. \end{cases}$$

This forces

$$(A - B/4)V^2 - BV - B = 0.$$

The positive root of the above equation is

$$V = \frac{2B + 4\sqrt{AB}}{4A - B}. \tag{2.3}$$

Then, using the equality $UV = B$ yields

$$U = \sqrt{AB} - B/2. \tag{2.4}$$

Next take

$$\psi_b = 1, \ \psi_a - \psi_b = V, \tag{2.5}$$

and take positive integers ℓ_a, ℓ_b such that

$$\log(\ell_a/\ell_b) > \frac{1+V}{V} B. \tag{2.6}$$

In the end, take λ such that

$$\frac{\log(\ell_a/\ell_b)}{\lambda} = U, \ i.e., \ \lambda = \frac{\log(\ell_a/\ell_b)}{U} = \log(\ell_a/\ell_b)\frac{V}{B}. \tag{2.7}$$

According to (2.6)-(2.7), $\lambda > 1 + V$ and thus $\phi \equiv \lambda > \max(\psi) = \max(\psi_a, \psi_b)$.

Then for the above constructed $\lambda, \ \ell_a, \ \ell_b, \psi_a$, and ψ_b, the function $f(x)$ defined in (2.1) attains its supremum at two different points x in $[0, 1]$. This yields an example that the supermum in (1.1) is attained at two different Bernoulli measures in $M(X, \sigma_X)$.

In the end, we provide a more concrete example for $\lambda, \ \ell_a, \ \ell_b, \psi_a$, and ψ_b.

Example 2.1. *Set*

$$B = 3 \log 2 \approx 2.07944$$

and

$$A = \log 3 - \frac{7}{12} \log 2 \approx 0.69427643.$$

One can check that (1.1) *holds for such* A *and* B*. Indeed, the supermum in defining* A *is attained at* $x = 1/3$*. Then*

$$U = \sqrt{AB} - B/2 \approx 0.16182292, \quad V = \frac{2B + 4\sqrt{AB}}{4A - B} \approx 12.8501046.$$

Take

$$\psi_a = 1 + V \approx 13.8501046, \quad \psi_b = 1$$

and

$$\ell_a = 150, \quad \ell_b = 1, \quad \lambda = \log(\ell_a/\ell_b) \cdot \frac{V}{B} \approx 30.9636922.$$

3. Application to Gatzouras-Lalley carpets

Let $\lambda,\ \ell_a,\ \ell_b, \psi_a$, and ψ_b be constructed as in Example 2.1. Notice that

$$3\exp(-\lambda)\ell_a < 3 \cdot e^{-30} \cdot 150 < 1, \quad \exp(-\psi_a) + \exp(-\psi_b) < 2e^{-1} < 1.$$

Then we can build a Gatzouras-Lalley carpet in the unit square as the attractor K of the IFS $\{S_{a,r} : 1 \leq r \leq \ell_a\} \cup \{S_{b,s} : 1 \leq s \leq \ell_b\}$, where

$$\begin{cases} S_{a,r}(x,y) = (\exp(-\lambda)x, \exp(-\psi_a)y) + (2r\exp(-\lambda), 0),\ 1 \leq r \leq \ell_a, \\ S_{b,s}(x,y) = (\exp(-\lambda)x, \exp(-\psi_b)y) + (2r\exp(-\lambda), 1 - \exp(-\psi_b)),\ 1 \leq s \leq \ell_b. \end{cases}$$

Gatzouras and Lalley [6] proved that the Hausdorff dimension of K is equal to $P(\phi, \psi)$, which is attained by some Bernoulli measure on X. The previous section shows that such a measure is not unique; in our example there are exactly two such measures.

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