# MIRROR SYMMETRY FOR TORIC FANO MANIFOLDS VIA SYZ TRANSFORMATIONS

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Abstract. We construct and apply Strominger-Yau-Zaslow mirror transformations to understand the geometry of the mirror symmetry between toric Fano manifolds and Landau-Ginzburg models.

Contents



## 1. Introduction

Mirror symmetry has been extended to the non-Calabi-Yau setting, notably to Fano manifolds, by the works of Givental [18], [19], [20], Kontsevich [28] and Hori-Vafa [27]. If $\bar{X}$ is a Fano manifold, then its mirror is conjectured to be a pair $(Y, W)$, where $Y$ is a non-compact Kähler manifold and $W : Y \to \mathbb{C}$ is a holomorphic Morse function. In the physics literature, the pair $(Y, W)$ is called a *Landau-Ginzburg model*, and $W$ is called the *superpotential* of the model. One of the very first mathematical predictions of this mirror symmetry is that there should be an isomorphism between the small quantum cohomology ring $QH^*(\bar{X})$ of $\bar{X}$ and the Jacobian ring $Jac(W)$ of the function $W$. This has been verified (at least) for toric Fano manifolds by the works of Batyrev [10], Givental [20] and many others. A version of the *Homological Mirror Symmetry Conjecture* has also been formulated by Kontsevich [28], which again has been checked in many cases [39], [44], [7], [8], [1], [2]. However, no direct geometric explanation for the mirror symmetry phenomenon for Fano manifolds had been given, until the works of Cho-Oh [12], which showed that, when $\bar{X}$ is a toric Fano manifold, the superpotential $W$ can be computed in terms of the counting of Maslov index two holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers.

On the other hand, the celebrated *Strominger-Yau-Zaslow (SYZ) Conjecture* [43] suggested that mirror symmetry for Calabi-Yau manifolds should be understood as a *T-duality*, i.e. dualizing special Lagrangian torus fibrations, modified with suitable quantum corrections. This will explain the geometry underlying mirror symmetry [35]. Recently, Gross and Siebert [23] made a breakthrough in the study of this conjecture, after earlier works of Fukaya [14] and Kontsevich-Soibelman [29]. It is expected that their program will finally provide a very explicit and geometric way to see how mirror symmetry works for *both* Calabi-Yau and non-Calabi-Yau manifolds (more precisely, for varieties with effective anticanonical class). On the other hand, in [6], Auroux started his program which is aimed at understanding mirror symmetry in the non-Calabi-Yau setting by applying the SYZ approach. More precisely, he studied the mirror symmetry between a general compact Kähler manifold equipped with an anticanonical divisor and a Landau-Ginzburg model, and investigated how the superpotential can be computed in terms holomorphic discs counting on the compact Kähler manifold. In particular, this includes the mirror symmetry for toric Fano manifolds as a special case.

In this paper, we shall again follow the SYZ philosophy and study the mirror symmetry phenomenon for toric Fano manifolds by using T-duality. The main point of this work, which is also the crucial difference between this and previous works, is that, explicit transformations, which we call *SYZ mirror transformations*, are constructed and used to understand the results (e.g. $QH^*(\bar{X}) \cong Jac(W)$) implied by mirror symmetry. From this perspective, this paper may be regarded as a sequel to the second author's work [30], where semi-flat SYZ mirror transformations (i.e. fiberwise real Fourier-Mukai transforms) were used to study mirror symmetry for semi-flat Calabi-Yau manifolds. While in that case, quantum corrections do not arise because the Lagrangian torus fibrations are smooth (i.e. they are fiber bundles), we will have to deal with quantum corrections in the toric Fano case.

However, we shall emphasize that the quantum corrections which arise in the toric Fano case are only due to contributions from the anticanonical toric divisor (the toric boundary); correspondingly, the Lagrangian torus fibrations do *not* have proper singular fibers (i.e. singular fibers which are contained in the complement of the anticanonical divisor), so that their bases are affine manifolds with boundaries but *without singularities*. This is simpler than the general non-Calabi-Yau case treated by Gross-Siebert [23] and Auroux [6], where further quantum corrections could arise, due to the fact that general Lagrangian torus fibrations do admit proper singular fibers, so that their bases are affine manifolds with *both* boundaries and singularities. Hence, the toric Fano case is in-between the semi-flat case, which corresponds to nonsingular affine manifolds without boundary, and the general case. In particular, in the toric Fano case, we do not need to worry about wall-crossing phenomena, and this is one of the reasons why we can construct the SYZ mirror transformations explicitly as fiberwise Fourier-type transforms, much like what was done in the semi-flat case [30]. (Another major reason is that holomorphic discs in toric manifolds with boundary in Lagrangian torus fibers are completely classified by Cho-Oh [12].) It is interesting to generalize the results here to non-toric settings, but, certainly, much work needs to be done before we can see how SYZ mirror transformations are constructed and used in the general case. For more detailed discussions of mirror symmetry and the wall-crossing phenomena in non-toric situations , we refer the reader to the works of Gross-Siebert [23] and Auroux [6].

What follows is an outline of our main results. We will focus on one half of the mirror symmetry between a complex $n$-dimensional toric Fano manifold $\bar{X}$ and the mirror Landau-Ginzburg model $(Y, W)$, namely, the correspondence between the symplectic geometry (A-model) of $\bar{X}$ and the complex geometry (B-model) of $(Y, W)$.

To describe our results, let us fix some notations first. Let $N \cong \mathbb{Z}^n$ be a lattice and $M = N^\vee = \mathrm{Hom}(N, \mathbb{Z})$ the dual lattice. Also let $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$, $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$, and denote by $\langle \cdot, \cdot \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ the dual pairing. Let $\bar{X}$ be a toric Fano manifold, i.e. a smooth projective toric variety such that the anticanonical line bundle $K_{\bar{X}}$ is ample. Let $v_1, \ldots, v_d \in N$ be the primitive generators of the 1-dimensional cones of the fan $\Sigma$ defining $\bar{X}$. Then a polytope $\bar{P} \subset M_\mathbb{R}$ defined by the inequalities

$$\langle x, v_i \rangle \geq \lambda_i, \quad i = 1, \ldots, d,$$

and with normal fan $\Sigma$, associates a Kähler structure $\omega_{\bar{X}}$ to $\bar{X}$. Physicists [27] predicted that the mirror of $(\bar{X}, \omega_{\bar{X}})$ is given by the pair $(Y, W)$, where $Y$, which we call *Hori-Vafa's mirror manifold*, is biholomorphic to the non-compact Kähler manifold $(\mathbb{C}^*)^n$, and $W : Y \to \mathbb{C}$ is the Laurent polynomial

$$e^{\lambda_1} z^{v_1} + \ldots + e^{\lambda_d} z^{v_d},$$

where $z_1, \ldots, z_n$ are the standard complex coordinates of $Y \cong (\mathbb{C}^*)^n$ and $z^v$ denotes the monomial $z_1^{v^1} \ldots z_n^{v^n}$ if $v = (v^1, \ldots, v^n) \in N \cong \mathbb{Z}^n$.

The symplectic manifold $(\bar{X}, \omega_{\bar{X}})$ admits a Hamiltonian action by the torus $T_N = N_\mathbb{R}/N$, and the corresponding moment map $\mu_{\bar{X}} : \bar{X} \to \bar{P}$ is naturally a Lagrangian torus fibration. While this fibration is singular (with collapsed fibers) along $\partial\bar{P}$, the restriction to the open dense $T_N$-orbit $X \subset \bar{X}$ is a Lagrangian torus bundle

$$\mu = \mu_{\bar{X}}|_X : X \to P,$$

where $P = \bar{P} \setminus \partial\bar{P}$ is the interior of the polytope $\bar{P}$.[1] Applying T-duality and the *semi-flat SYZ mirror transformation* (see Definition 3.2) to this torus bundle, we can, as suggested by the SYZ philosophy, obtain the mirror manifold $Y$ (see Proposition 3.1 and Proposition 3.2).[2] However, we are not going to get the superpotential $W : Y \to \mathbb{C}$ because we have ignored the anticanonical toric divisor $D_\infty = \bigcup_{i=1}^d D_i = \bar{X} \setminus X$, and hence quantum corrections. Here, for $i = 1, \ldots, d$, $D_i$ denotes the toric prime divisor which corresponds to $v_i \in N$. To recapture the quantum corrections, we consider the (trivial) $\mathbb{Z}^n$-cover

$$\pi : LX = X \times N \to X$$

[1] $\mu : X \to P$ is a special Lagrangian fibration if we equip $X$ with the standard holomorphic volume form on $(\mathbb{C}^*)^n$, so that $X$ becomes an almost Calabi-Yau manifold. See Definition 2.1 and Lemma 4.1 in Auroux [6].

[2] In fact, we prove that T-duality gives a bounded domain in the Hori-Vafa mirror manifold. This result also appeared in Auroux's paper ([6], Proposition 4.2).

and various functions on it.[3] Let $\mathcal{K}(\bar{X}) \subset H^2(\bar{X}, \mathbb{R})$ be the Kähler cone of $\bar{X}$. For each $q = (q_1, \ldots, q_l) \in \mathcal{K}(\bar{X})$ (here $l = d - n =$ Picard number of $\bar{X}$), we define a $T_N$-invariant function $\Phi_q : LX \to \mathbb{R}$ in terms of the counting of Maslov index two holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers of $\mu : X \to P$ (see Definition 2.1 and Remark 2.2). If we further assume that $\bar{X}$ is a product of projective spaces, then this family of functions $\{\Phi_q\}_{q \in \mathcal{K}(\bar{X})} \subset C^\infty(LX)$ can be used to compute the small quantum cohomology ring $QH^*(\bar{X})$ of $\bar{X}$ as follows (see Section 2 for details).

**Proposition 1.1.**

1. *The logarithmic derivatives of $\Phi_q$, with respect to $q_a$, for $a = 1, \ldots, l$, are given by*
$$q_a \frac{\partial \Phi_q}{\partial q_a} = \Phi_q \star \Psi_{n+a}.$$
*Here, for each $i = 1, \ldots, d$, the function $\Psi_i : LX \to \mathbb{R}$ is defined, in terms of the counting of Maslov index two holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers which intersect the toric prime divisor $D_i$ at an interior point (see the statement of Proposition 2.1 and the subsequent discussion), and $\star$ denotes a convolution product of functions on $LX$ with respect to the lattice $N$.*
2. *We have a natural isomorphism of $\mathbb{C}$-algebras*
$$QH^*(\bar{X}) \cong \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L}, \tag{1.1}$$
*where $\mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]$ is the polynomial algebra generated by $\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm}$ with respect to the convolution product $\star$, and $\mathcal{L}$ is the ideal generated by linear relations that are defined by the linear equivalence among the toric divisors $D_1, \ldots, D_d$, provided that $\bar{X}$ is a product of projective spaces.*

The proof of the above isomorphism (1.1) given in Subsection 2.1 will be combinatorial in nature and is done by a simple computation of certain Gromov-Witten invariants. While this result may follow easily from known results in the literature, we choose to include an elementary proof to make this paper more self-contained. Our proof relies on the assumption that $\bar{X}$ is a product of projective spaces. However, the more important reason for us to impose such a strong assumption is that, when $\bar{X}$ is a product of projective spaces, there is a better way to understand the geometry underlying the isomorphism (1.1) by using *tropical geometry*. A brief explanation is now in order. More details can be found in Subsection 2.2.

Suppose that $\bar{X}$ is a product of projective spaces. We first define a tropical analog of the small quantum cohomology ring of $\bar{X}$, call it $QH^*_{trop}(\bar{X})$. The results of Mikhalkin [32] and Nishinou-Siebert [38] provided a one-to-one correspondence between those holomorphic curves in $\bar{X}$ which have contribution to the quantum product in $QH^*(\bar{X})$ and those tropical curves in $N_\mathbb{R}$ which have contribution to the tropical quantum product in $QH^*_{trop}(\bar{X})$. From this follows the natural isomorphism
$$QH^*(\bar{X}) \cong QH^*_{trop}(\bar{X}).$$

[3] In the expository paper [11], we interpreted LX as a finite dimensional subspace of the free loop space $\mathcal{L}\bar{X}$ of $\bar{X}$.

Next comes a simple but crucial observation: *each tropical curve which contributes to the tropical quantum product in $QH^*_{trop}(\bar{X})$ can be obtained by gluing tropical discs.*[4] Now, making use of the fundamental results of Cho and Oh [12] on the classification of holomorphic discs in toric Fano manifolds, we get a one-to-one correspondence between the relevant tropical discs and the Maslov index two holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers of $\mu : X \to P$. The latter were used to define the functions $\Psi_i$'s. So we naturally have another canonical isomorphism

$$QH^*_{trop}(\bar{X}) \cong \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L}.$$

Hence, by factoring through the tropical quantum cohomology ring $QH^*_{trop}(\bar{X})$ and using the correspondence between symplectic geometry (holomorphic curves and discs) of $\bar{X}$ and tropical geometry (tropical curves and discs) of $N_{\mathbb{R}}$, we obtain a more conceptual and geometric understanding of the isomorphism (1.1). This is in line with the general philosophy advocated in the Gross-Siebert program [23].

Notice that all these can only be done for products of projective spaces because, as is well known, tropical geometry cannot be used to count curves which have irreducible components mapping to the toric boundary divisor, and if $\bar{X}$ is not a product of projective spaces, those curves *do* contribute to $QH^*(\bar{X})$ (see Example 3 in Section 4). This is the main reason why we confine ourselves to the case of products of projective spaces, although the isomorphism (1.1) holds for all toric Fano manifolds (see Remark 2.3).

Now we come to the upshot of this paper, namely, we can explicitly construct and apply SYZ mirror transformations to understand the mirror symmetry between $\bar{X}$ and $(Y, W)$. We shall define the SYZ mirror transformation $\mathcal{F}$ for the toric Fano manifold $\bar{X}$ as *a combination of the semi-flat SYZ mirror transformation and taking fiberwise Fourier series* (see Definition 3.3 for the precise definition). Our first result says that the SYZ mirror transformation of $\Phi_q$ is precisely the exponential of the superpotential $W$, i.e. $\mathcal{F}(\Phi_q) = \exp(W)$. Then, by proving that the SYZ mirror transformation $\mathcal{F}(\Psi_i)$ of the function $\Psi_i$ is nothing but the monomial $e^{\lambda_i} z^{v_i}$, for $i = 1, \ldots, d$, we show that $\mathcal{F}$ exhibits a natural and canonical isomorphism between the small quantum cohomology ring $QH^*(\bar{X})$ and the Jacobian ring $Jac(W)$, which takes the quantum product $*$ (which can now, by Proposition 1.1, be realized as the convolution product $\star$) to the ordinary product of Laurent polynomials, just as what classical Fourier series do. This is our main result (see Section 3):

**Theorem 1.1.**

1. *The SYZ mirror transformation of the function $\Phi_q \in C^\infty(LX)$, defined in terms of the counting of Maslov index two holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers, is the exponential of the superpotential $W$ on the mirror manifold $Y$, i.e.*

$$\mathcal{F}(\Phi_q) = e^W.$$

*Furthermore, we can incorporate the symplectic structure $\omega_X = \omega_{\bar{X}}|_X$ on $X$ to give the holomorphic volume form on the Landau-Ginzburg model $(Y, W)$*

[4]More recently, Gross [22] generalized this idea further to give a tropical interpretation of the *big* quantum cohomology of $\mathbb{P}^2$.

*through the SYZ mirror transformation $\mathcal{F}$, in the sense that,*

$$\mathcal{F}(\Phi_q e^{\sqrt{-1}\omega_X}) = e^W \Omega_Y.$$

2. *The SYZ mirror transformation gives a canonical isomorphism of $\mathbb{C}$-algebras*

$$\mathcal{F} : QH^*(\bar{X}) \xrightarrow{\cong} Jac(W),$$

*provided that $\bar{X}$ is a product of projective spaces.*

Here we view $\Phi_q e^{\sqrt{-1}\omega_X}$ as the *symplectic structure corrected by Maslov index two holomorphic discs*,[5] and $e^W \Omega_Y$ as the holomorphic volume form of the Landau-Ginzburg model $(Y, W)$.

As mentioned at the beginning, the existence of an isomorphism $QH^*(\bar{X}) \cong Jac(W)$ is not a new result, and was established before by the works of Batyrev [10] and Givental [20]. However, we shall emphasize that the key point here is that there is an isomorphism which is realized by an explicit Fourier-type transformation, namely, the SYZ mirror transformation $\mathcal{F}$. This hopefully provides a more conceptual understanding of what is going on.

In [15] (Section 5), Fukaya-Oh-Ohta-Ono studied the isomorphism $QH^*(\bar{X}) \cong Jac(W)$ from the point of view of Lagrangian Floer theory. They worked over the Novikov ring, instead of $\mathbb{C}$, and gave a proof (Theorem 1.9) of this isomorphism (over the Novikov ring) for all toric Fano manifolds basing on Batyrev's formulas for presentations of the small quantum cohomology rings of toric manifolds and Givental's mirror theorem [20]. Their proof was also combinatorial in nature, but they claimed that a geometric proof would appear in a sequel of [15]. A brief history and a more detailed discussion of the proof of the isomorphism were also contained in Remark 1.10 of [15]. See also the sequel [16].

The rest of this paper is organized as follows. In the next section, we define the family of functions $\{\Phi_q\}_{q \in \mathcal{K}(\bar{X})}$ in terms of the counting of Maslov index two holomorphic discs and give a combinatorial proof Proposition 1.1, which is followed by a discussion of the role played by tropical geometry. The heart of this paper is Section 3, where we construct explicitly the SYZ mirror transformation $\mathcal{F}$ for a toric Fano manifold $\bar{X}$ and show that it indeed transforms the symplectic structure of $\bar{X}$ to the complex structure of $(Y, W)$, and vice versa. This is the first part of Theorem 1.1. We then move on to prove the second part, which shows how the SYZ mirror transformation $\mathcal{F}$ can realize the isomorphism $QH^*(\bar{X}) \cong Jac(W)$. Section 4 contains some examples. We conclude with some discussions in the final section.

## 2. Maslov index two holomorphic discs and $QH^*(\bar{X})$

In the first part of this section, we define the functions $\Phi_q$, $q \in \mathcal{K}(\bar{X})$, and $\Psi_1, \ldots, \Psi_d$ on $LX$ in terms of the counting of Maslov index two holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers of the moment map $\mu : X \to P$, and show how they can be used to compute the small quantum cohomology ring

[5] In [11], we rewrote the function $\Phi_q$ as $\exp(\Psi_1 + \ldots + \Psi_d)$, so that $\Phi_q e^{\sqrt{-1}\omega_X} = \exp(\sqrt{-1}\omega_X + \Psi_1 + \ldots + \Psi_d)$ and the formula in part 1. of Theorem 1.1 becomes $\mathcal{F}(e^{\sqrt{-1}\omega_X + \Psi_1 + \ldots + \Psi_d}) = e^W \Omega_Y$. May be it is more appropriate to call $\sqrt{-1}\omega_X + \Psi_1 + \ldots + \Psi_d \in \Omega^2(LX) \oplus \Omega^0(LX)$ the symplectic structure corrected by Maslox index two holomorphic discs.

$QH^*(\bar{X})$ in the case when $\bar{X}$ is a product of projective spaces. In particular, we demonstrate how the quantum product can be realized as a convolution product (part 2. of Proposition 1.1). In the second part, we explain the geometry of these results by using tropical geometry.

2.1. **Computing $QH^*(\bar{X})$ in terms of functions on $LX$.** Recall that the primitive generators of the 1-dimensional cones of the fan $\Sigma$ defining the toric Fano manifold $\bar{X}$ are denoted by $v_1, \ldots, v_d \in N$. Without loss of generality, we can assume that $v_1 = e_1, \ldots, v_n = e_n$ is the standard basis of $N \cong \mathbb{Z}^n$. The map

$$\partial : \mathbb{Z}^d \to N, \ (k_1, \ldots, k_d) \mapsto \sum_{i=1}^{d} k_i v_i$$

is surjective since $\bar{X}$ is compact. Let $K$ be the kernel of $\partial$, so that the sequence

$$0 \longrightarrow K \xrightarrow{\iota} \mathbb{Z}^d \xrightarrow{\partial} N \longrightarrow 0 \tag{2.1}$$

is exact (see, for example, Appendix 1 in the book of Guillemin [24]).

Now consider the Kähler cone $\mathcal{K}(\bar{X}) \subset H^2(\bar{X}, \mathbb{R})$ of $\bar{X}$, and let $q_1, \ldots, q_l \in \mathbb{R}_{>0}$ ($l = d - n$) be the coordinates of $\mathcal{K}(\bar{X})$. For each $q = (q_1, \ldots, q_l) \in \mathcal{K}(\bar{X})$, we choose $\bar{P}$ to be the polytope defined by

$$\bar{P} = \{x \in M_{\mathbb{R}} : \langle x, v_i \rangle \geq \lambda_i, \quad i = 1, \ldots, d\}$$

with $\lambda_i = 0$, for $i = 1, \ldots, n$, and $\lambda_{n+a} = \log q_a$, for $a = 1, \ldots, l$. This associates a Kähler structure $\omega_{\bar{X}}$ to $\bar{X}$.

**Remark 2.1.** *Let $\bar{P}$ be the polytope defined by the inequalities*

$$\langle x, v_i \rangle \geq \lambda_i, \quad i = 1, \ldots, d.$$

*Also let*

$$Q_1 = (Q_{11}, \ldots, Q_{d1}), \ldots, Q_l = (Q_{1l}, \ldots, Q_{dl}) \in \mathbb{Z}^d$$

*be a $\mathbb{Z}$-basis of $K$. Then the coordinates $q = (q_1, \ldots, q_l) \in \mathcal{K}(\bar{X})$ of the Kähler cone are given by $q_a = e^{-r_a}$, where*

$$r_a = -\sum_{i=1}^{d} Q_{ia} \lambda_i,$$

*for $a = 1, \ldots, l$. Hence, different choices of the $\mathbb{Z}$-basis of $K$ and the constants $\lambda_1, \ldots, \lambda_d$ can give rise to the same Kähler structure parametrized by $q \in \mathcal{K}(\bar{X})$. We choose the $\mathbb{Z}$-basis $\{Q_1, \ldots, Q_l\}$ of $K$ such that $(Q_{n+a,b})_{1 \leq a,b \leq l} = Id_{l \times l}$, and the constants $\lambda_1, \ldots, \lambda_d$ such that $\lambda_1 = \ldots = \lambda_n = 0$.*

Recall that $\mu : X \to P$ is the restriction of the moment map $\mu_{\bar{X}} : \bar{X} \to \bar{P}$ to the open dense $T_N$-orbit $X \subset \bar{X}$, where $P$ is the interior of the polytope $\bar{P}$. For a point $x \in P$, we let $L_x = \mu^{-1}(x) \subset X$ be the Lagrangian torus fiber over $x$. Then the groups $H_2(\bar{X}, \mathbb{Z})$, $\pi_2(\bar{X}, L_x)$ and $\pi_1(L_x)$ can be identified canonically with $K$, $\mathbb{Z}^d$ and $N$ respectively, so that the exact sequence (2.1) above coincides with the following exact sequence of homotopy groups associated to the pair $(\bar{X}, L_x)$:

$$0 \longrightarrow H_2(\bar{X}, \mathbb{Z}) \xrightarrow{\iota} \pi_2(\bar{X}, L_x) \xrightarrow{\partial} \pi_1(L_x) \longrightarrow 0.$$

To proceed, we shall recall some of the fundamental results of Cho-Oh [12] on the classification of holomorphic discs in $(\bar{X}, L_x)$:

**Theorem 2.1** (Theorem 5.2 and Theorem 8.1 in Cho-Oh [12]). *$\pi_2(\bar{X}, L_x)$ is generated by $d$ Maslov index two classes $\beta_1, \ldots, \beta_d \in \pi_2(\bar{X}, L_x)$, which are represented by holomorphic discs with boundary in $L_x$. Moreover, given a point $p \in L_x$, then, for each $i = 1, \ldots, d$, there is a unique (up to automorphism of the domain) Maslov index two holomorphic disc $\varphi_i : (D^2, \partial D^2) \to (\bar{X}, L_x)$ in the class $\beta_i$ whose boundary passes through $p$,[6] and the symplectic area of $\varphi_i$ is given by*

$$Area(\varphi_i) = \int_{\beta_i} \omega_{\bar{X}} = \int_{D^2} \varphi_i^* \omega_{\bar{X}} = 2\pi(\langle x, v_i \rangle - \lambda_i). \tag{2.2}$$

Furthermore, for each $i = 1, \ldots, d$, the disc $\varphi_i$ intersects the toric prime divisor $D_i$ at a unique interior point. (We can in fact choose the parametrization of $\varphi_i$ so that $\varphi_i(0) \in D_i$.) Indeed, a result of Cho-Oh (Theorem 5.1 in [12]) says that, the Maslov index of a holomorphic disc $\varphi : (D^2, \partial D^2) \to (\bar{X}, L_x)$ representing a class $\beta \in \pi_2(\bar{X}, L_x)$ is given by twice the algebraic intersection number $\beta \cdot D_\infty$, where $D_\infty = \bigcup_{i=1}^d D_i$ is the toric boundary divisor (see also Auroux [6], Lemma 3.1).

Let $LX$ be the product $X \times N$. We view $LX$ as a (trivial) $\mathbb{Z}^n$-cover over $X$:

$$\pi : LX = X \times N \to X,$$

and we equip $LX$ with the symplectic structure $\pi^*(\omega_X)$, so that it becomes a symplectic manifold. We are now in a position to define $\Phi_q$.

**Definition 2.1.** *Let $q = (q_1, \ldots, q_l) \in \mathcal{K}(\bar{X})$. The function $\Phi_q : LX \to \mathbb{R}$ is defined as follows. For $(p, v) \in LX = X \times N$, let $x = \mu(p) \in P$ and $L_x = \mu^{-1}(x)$ be the Lagrangian torus fiber containing $p$. Denote by*

$$\pi_2^+(\bar{X}, L_x) = \left\{ \sum_{i=1}^d k_i \beta_i \in \pi_2(\bar{X}, L_x) : k_i \in \mathbb{Z}_{\geq 0},\ i = 1, \ldots, d \right\}$$

*the positive cone generated by the Maslov index two classes $\beta_1, \ldots, \beta_d$ which are represented by holomorphic discs with boundary in $L_x$. For $\beta = \sum_{i=1}^d k_i \beta_i \in \pi_2^+(\bar{X}, L_x)$, we denote by $w(\beta)$ the number $k_1! \ldots k_d!$. Then set*

$$\Phi_q(p, v) = \sum_{\beta \in \pi_2^+(\bar{X}, L_x),\ \partial\beta = v} \frac{1}{w(\beta)} e^{-\frac{1}{2\pi} \int_\beta \omega_{\bar{X}}}.$$

**Remark 2.2.**

1. *We say that $\Phi_q$ is defined by the counting of Maslov index two holomorphic discs because of the following: Let $(p, v) \in LX, x = \mu(p) \in P, L_x \subset X$ and $\beta_1, \ldots, \beta_d \in \pi_2(\bar{X}, L_x)$ be as before. For $i = 1, \ldots, d$, let $n_i(p)$ be the (algebraic) number of Maslov index two holomorphic discs $\varphi : (D^2, \partial D^2) \to (\bar{X}, L_x)$ in the class $\beta_i$ whose boundary passes through $p$. This number is well-defined since $\bar{X}$ is toric Fano (see Section 3.1 and Section 4 in Auroux* [6]*). Then we can re-define*

$$\Phi_q(p, v) = \sum_{\beta \in \pi_2^+(\bar{X}, L_x),\ \partial\beta = v} \frac{n_\beta(p)}{w(\beta)} e^{-\frac{1}{2\pi} \int_\beta \omega_{\bar{X}}},$$

[6]Another way to state this result: Let $\mathcal{M}_1(L_x, \beta_i)$ be the moduli space of holomorphic discs in $(\bar{X}, L_x)$ in the class $\beta_i$ and with one boundary marked point. In the toric Fano case, $\mathcal{M}_1(L_x, \beta_i)$ is a smooth compact manifold of real dimension $n$. Let $ev : \mathcal{M}_1(L_x, \beta_i) \to L_x$ be the evaluation map at the boundary marked point. Then we have $ev_*[\mathcal{M}_1(L_x, \beta_i)] = [L_x]$ as $n$-cycles in $L$. See Cho-Oh [12] and Auroux [6] for details.

*where $n_\beta(p) = n_1(p)^{k_1} \ldots n_d(p)^{k_d}$ if $\beta = \sum_{i=1}^d k_i\beta_i$. Defining $\Phi_q$ in this way makes it explicit that $\Phi_q$ carries enumerative meaning. By Theorem 2.1, we have $n_i(p) = 1$, for all $i = 1, \ldots, d$ and for any $p \in X$. So this definition reduces to the one above.*

2. *By definition, $\Phi_q$ is invariant under the $T_N$-action on $X \subset \bar{X}$. Since $X = T^*P/N = P \times \sqrt{-1}T_N$ (and the moment map $\mu : X \to P$ is nothing but the projection to the first factor), we may view $\Phi_q$ as a function on $P \times N$.*
3. *The function $\Phi_q$ is well-defined, i.e. the infinite sum in its definition converges. To see this, notice that, by the symplectic area formula (2.2) of Cho-Oh, we have*

$$\Phi_q(p,v) = \left( \sum_{\substack{k_1,\ldots,k_d \in \mathbb{Z}_{\geq 0}, \\ \sum_{i=1}^d k_i v_i = v}} \frac{q_1^{k_{n+1}} \ldots q_l^{k_d}}{k_1! \ldots k_d!} \right) e^{-\langle x,v\rangle},$$

*and the sum inside the big parentheses is less than $e^{n+q_1+\ldots+q_l}$.*

For $T_N$-invariant functions $f, g : LX \to \mathbb{R}$, we define their *convolution product* $f \star g : LX \to \mathbb{R}$ by

$$(f \star g)(p,v) = \sum_{v_1,v_2 \in N,\ v_1+v_2=v} f(p,v_1)g(p,v_2),$$

for $(p,v) \in LX$. As in the theory of Fourier analysis, for the convolution $f \star g$ to be well-defined, we need some conditions for both $f$ and $g$. We leave this to Subsection 3.2 (see Definition 3.4 and the subsequent discussion). Nevertheless, if one of the functions is nonzero only for finitely many $v \in N$, then the sum in the definition of $\star$ is a finite sum, so it is well-defined. This is the case in the following proposition.

**Proposition 2.1.** *[=part 1. of Proposition 1.1] The logarithmic derivatives of $\Phi_q$, with respect to $q_a$ for $a = 1, \ldots, l$, are given by*

$$q_a \frac{\partial \Phi_q}{\partial q_a} = \Phi_q \star \Psi_{n+a}$$

*where $\Psi_i : LX \to \mathbb{R}$ is defined, for $i = 1, \ldots, d$, by*

$$\Psi_i(p,v) = \begin{cases} e^{-\frac{1}{2\pi}\int_{\beta_i}\omega_{\bar{X}}} & \text{if } v = v_i \\ 0 & \text{if } v \neq v_i, \end{cases}$$

*for $(p,v) \in LX = X \times N$, and with $x = \mu(p) \in P$, $L_x = \mu^{-1}(x)$ and $\beta_1, \ldots, \beta_d \in \pi_2(\bar{X}, L_x)$ as before.*

*Proof.* We will compute $q_l \frac{\partial \Phi_q}{\partial q_l}$. The others are similar. By using Cho-Oh's formula (2.2) and our choice of the polytope $\bar{P}$, we have

$$e^{\langle x,v\rangle}\Phi_q(p,v) = \sum_{\substack{k_1,\ldots,k_d \in \mathbb{Z}_{\geq 0}, \\ \sum_{i=1}^d k_i v_i = v}} \frac{q_1^{k_{n+1}} \ldots q_l^{k_d}}{k_1! \ldots k_d!}.$$

Note that the right-hand-side is independent of $p \in X$. Differentiating both sides with respect to $q_l$ gives

$$\begin{aligned} e^{\langle x,v\rangle}\frac{\partial \Phi_q(p,v)}{\partial q_l} &= \sum_{\substack{k_1,\ldots,k_{d-1}\in\mathbb{Z}_{\geq 0},\ k_d\in\mathbb{Z}_{\geq 1},\\ \sum_{i=1}^d k_i v_i = v}} \frac{q_1^{k_{n+1}}\ldots q_{l-1}^{k_{d-1}} q_l^{k_d-1}}{k_1!\ldots k_{d-1}!(k_d-1)!} \\ &= \sum_{\substack{k_1,\ldots,k_d\in\mathbb{Z}_{\geq 0},\\ \sum_{i=1}^d k_i v_i = v - v_d}} \frac{q_1^{k_{n+1}}\ldots q_l^{k_d}}{k_1!\ldots k_d!} \\ &= e^{\langle x, v-v_d\rangle}\Phi_q(p, v-v_d). \end{aligned}$$

Hence, we obtain

$$q_l\frac{\partial \Phi_q(p,v)}{\partial q_l} = q_l e^{-\langle x,v_d\rangle}\Phi_q(p,v-v_d).$$

Now, by the definition of the convolution product $\star$, we have

$$\Phi_q \star \Psi_d(p,v) = \sum_{v_1,v_2\in N,\ v_1+v_2=v} \Phi_q(p,v_1)\Psi_d(p,v_2) = \Phi_q(p,v-v_d)\Psi_d(p,v_d),$$

and $\Psi_d(p,v_d) = e^{-\frac{1}{2\pi}\int_{\beta_d}\omega_{\bar{X}}} = e^{\lambda_d - \langle x,v_d\rangle} = q_l e^{-\langle x,v_d\rangle}$. The result follows. □

In the previous proposition, we introduce the $T_N$-invariant functions $\Psi_1,\ldots,\Psi_d \in C^\infty(LX)$. Similar to what has been said in Remark 2.2(1), these functions carry enumerative meanings, and we should have defined $\Psi_i(p,v)$, $i = 1,\ldots,d$ in terms of the counting of Maslov index two holomorphic discs in $(\bar{X}, L_{\mu(p)})$ with boundary $v$ which pass through $p$, i.e.

$$\Psi_i(p,v) = \begin{cases} n_i(p)e^{-\frac{1}{2\pi}\int_{\beta_i}\omega_{\bar{X}}} & \text{if } v = v_i \\ 0 & \text{if } v \neq v_i, \end{cases}$$

for $(p,v) \in LX = X \times N$, where $x = \mu(p) \in P$, $L_x = \mu^{-1}(x) \subset X$ and $\beta_1,\ldots,\beta_d \in \pi_2(\bar{X}, L_x)$ are as before. Again, since the number $n_i(p)$ is always equal to one, for any $p \in X$ and for all $i = 1,\ldots,d$, this definition of $\Psi_i$ is the same as the previous one. But we should keep in mind that the function $\Psi_i \in C^\infty(LX)$ encodes the following enumerative information: for each $p \in X$, there is a unique Maslov index two holomorphic disc $\varphi_i$ in the class $\beta_i$ with boundary in the Lagrangian torus fiber $L_{\mu(p)}$ whose boundary passes through $p$ and whose interior intersects the toric prime divisor $D_i$ at one point. In view of this, we put the $d$ functions $\{\Psi_i\}_{i=1}^d$, the $d$ families of Maslov index two holomorphic discs $\{\varphi_i\}_{i=1}^d$ and the $d$ toric prime divisors $\{D_i\}_{i=1}^d$ in one-to-one correspondences:

$$\{\Psi_i\}_{i=1}^d \xleftrightarrow{1-1} \{\varphi_i\}_{i=1}^d \xleftrightarrow{1-1} \{D_i\}_{i=1}^d. \tag{2.3}$$

Through these correspondences, we introduce linear relations in the $d$-dimensional $\mathbb{C}$-vector space spanned by the functions $\Psi_1,\ldots,\Psi_d$ using the linear equivalences among the divisors $D_1,\ldots,D_d$.

**Definition 2.2.** *Two linear combinations $\sum_{i=1}^d a_i\Psi_i$ and $\sum_{i=1}^d b_i\Psi_i$, where $a_i, b_i \in \mathbb{C}$, are said to be linearly equivalent, denoted by $\sum_{i=1}^d a_i\Psi_i \sim \sum_{i=1}^d b_i\Psi_i$, if the corresponding divisors $\sum_{i=1}^d a_iD_i$ and $\sum_{i=1}^d b_iD_i$ are linearly equivalent.*

We further define $\Psi_i^{-1} : LX \to \mathbb{R}$, $i = 1, \ldots, d$, by

$$\Psi_i^{-1}(p, v) = \begin{cases} e^{\frac{1}{2\pi}\int_{\beta_i}\omega_X} & \text{if } v = -v_i \\ 0 & \text{if } v \neq -v_i, \end{cases}$$

for $(p, v) \in LX$, so that $\Psi_i^{-1} \star \Psi_i = \mathbb{1}$, where $\mathbb{1} : LX \to \mathbb{R}$ is the function defined by

$$\mathbb{1}(p, v) = \begin{cases} 1 & \text{if } v = 0 \\ 0 & \text{if } v \neq 0. \end{cases}$$

The function $\mathbb{1}$ serves as a multiplicative identity for the convolution product, i.e. $\mathbb{1} \star f = f \star \mathbb{1} = f$ for any $f \in C^\infty(LX)$. Now the second part of Proposition 1.1 says that

**Proposition 2.2** (=part 2. of Proposition 1.1)**.** *We have a natural isomorphism of $\mathbb{C}$-algebras*

$$QH^*(\bar{X}) \cong \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L}, \tag{2.4}$$

*where $\mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]$ is the polynomial algebra generated by $\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}$ with respect to the convolution product $\star$ and $\mathcal{L}$ is the ideal generated by linear equivalences, provided that $\bar{X}$ is a product of projective spaces.*

In the rest of this subsection, we will give an elementary proof of this proposition by simple combinatorial arguments and computation of certain Gromov-Witten invariants.

First of all, each toric prime divisor $D_i$ ($i = 1, \ldots, d$) determines a cohomology class in $H^2(\bar{X}, \mathbb{C})$, which will be, by abuse of notations, also denoted by $D_i$. It is known by the general theory of toric varieties that the cohomology ring $H^*(\bar{X}, \mathbb{C})$ of the compact toric manifold $\bar{X}$ is generated by the classes $D_1, \ldots, D_d$ in $H^2(\bar{X}, \mathbb{C})$ (see, for example, Fulton [17] or Audin [5]). More precisely, there is a presentation of the form:

$$H^*(\bar{X}, \mathbb{C}) = \mathbb{C}[D_1, \ldots, D_d]/(\mathcal{L} + \mathcal{SR}),$$

where $\mathcal{L}$ is the ideal generated by linear equivalences and $\mathcal{SR}$ is the *Stanley-Reisner ideal* generated by *primitive relations* (see Batyrev [9]). Now, by a result of Siebert and Tian (Proposition 2.2 in [41]), $QH^*(\bar{X})$ is also generated by $D_1, \ldots, D_d$ and a presentation of $QH^*(\bar{X})$ is given by replacing each relation in $\mathcal{SR}$ by its quantum counterpart. Denote by $\mathcal{SR}_Q$ the quantum Stanley-Reisner ideal. Then we can rephrase what we said as:

$$QH^*(\bar{X}) = \mathbb{C}[D_1, \ldots, D_d]/(\mathcal{L} + \mathcal{SR}_Q).$$

The computation of $QH^*(\bar{X})$ (as a presentation) therefore reduces to computing the generators of the ideal $\mathcal{SR}_Q$.

Let $\bar{X} = \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_l}$ be a product of projective spaces. The complex dimension of $\bar{X}$ is $n = n_1 + \ldots + n_l$. For $a = 1, \ldots, l$, let $v_{1,a} = e_1, \ldots, v_{n_a,a} = e_{n_a}, v_{n_a+1,a} = -\sum_{j=1}^{n_a} e_j \in N_a$ be the primitive generators of the 1-dimensional cones in the fan of $\mathbb{C}P^{n_a}$, where $\{e_1, \ldots, e_{n_a}\}$ is the standard basis of $N_a \cong \mathbb{Z}^{n_a}$. For $j = 1, \ldots, n_a + 1$, $a = 1, \ldots, l$, we use the same symbol $v_{j,a}$ to denote the vector

$$(0, \ldots, \underbrace{v_{j,a}}_{a\text{-th}}, \ldots, 0) \in N = N_1 \oplus \ldots \oplus N_l,$$

where $v_{j,a}$ sits in the $a$th place. These $d = \sum_{a=1}^{l}(n_a + 1) = n + l$ vectors in $N$ are the primitive generators of the 1-dimensional cones of the fan $\Sigma$ defining $\bar{X}$. In the following, we shall also denote the toric prime divisor, the relative homotopy class, the family of Maslov index two holomorphic discs with boundary in Lagrangian torus fibers and the function on $LX$ corresponding to $v_{j,a}$ by $D_{j,a}$, $\beta_{j,a}$, $\varphi_{j,a}$ and $\Psi_{j,a}$ respectively.

**Lemma 2.1.** *There are exactly $l$ primitive collections given by*

$$\mathfrak{P}_a = \{v_{j,a} : j = 1, \ldots, n_a + 1\},\ a = 1, \ldots, l,$$

*and hence the Stanley-Reisner ideal of $\bar{X} = \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_l}$ is given by*

$$\mathcal{SR} = \langle D_{1,a} \cup \ldots \cup D_{n_a+1,a} : a = 1 \ldots, l\rangle.$$

*Proof.* Let $\mathfrak{P}$ be any primitive collection. By definition, $\mathfrak{P}$ is a collection of primitive generators of 1-dimensional cones of the fan $\Sigma$ defining $\bar{X}$ such that for any $v \in \mathfrak{P}$, $\mathfrak{P} \setminus \{v\}$ generates a $(|\mathfrak{P}| - 1)$-dimensional cone in $\Sigma$, while $\mathfrak{P}$ itself does not generate a $|\mathfrak{P}|$-dimensional cone in $\Sigma$. Suppose that $\mathfrak{P} \not\subset \mathfrak{P}_a$ for any $a$. For each $a$, choose $v \in \mathfrak{P} \setminus (\mathfrak{P} \cap \mathfrak{P}_a)$. By definition, $\mathfrak{P} \setminus \{v\}$ generates a cone in $\Sigma$. But all the cones in $\Sigma$ are direct sums of cones in the fans of the factors. So, in particular, $\mathfrak{P} \cap \mathfrak{P}_a$, whenever it's nonempty, will generate a cone in the fan of $\mathbb{C}P^{n_a}$. Since $\mathfrak{P} = \bigsqcup_{a=1}^{l} \mathfrak{P} \cap \mathfrak{P}_a$, this implies that the set $\mathfrak{P}$ itself generates a cone, which is impossible. We therefore conclude that $\mathfrak{P}$ must be contained in, and hence equal to one of the $\mathfrak{P}_a$'s. $\square$

Hence, to compute the quantum Stanley-Reisner ideal $\mathcal{SR}_Q$, we must compute the expression $D_{1,a} * \ldots * D_{n_a+1,a}$, for $a = 1, \ldots, l$, where $*$ denotes the small quantum product of $QH^*(\bar{X})$. Before doing this, we shall recall the definitions and properties of the relevant Gromov-Witten invariants and the small quantum product for $\bar{X} = \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_l}$ as follows.

For $\delta \in H_2(\bar{X}, \mathbb{Z})$, let $\overline{\mathcal{M}}_{0,m}(\bar{X}, \delta)$ be the moduli space of genus 0 stable maps with $m$ marked points and class $\delta$. Since $\bar{X}$ is convex (i.e. for all maps $\varphi : \mathbb{C}P^1 \to \bar{X}$, $H^1(\mathbb{C}P^1, \varphi^* T\bar{X}) = 0$), the moduli space $\overline{\mathcal{M}}_{0,m}(\bar{X}, \delta)$, if nonempty, is a variety of pure complex dimension $\dim_{\mathbb{C}}(\bar{X}) + c_1(\bar{X}) \cdot \delta + m - 3$ (see, for example, the book [4], p.3). For $k = 1, \ldots, m$, let $ev_k : \overline{\mathcal{M}}_{0,m}(\bar{X}, \delta) \to \bar{X}$ be the evaluation map at the $k$th marked point, and let $\pi : \overline{\mathcal{M}}_{0,m}(\bar{X}, \delta) \to \overline{\mathcal{M}}_{0,m}$ be the forgetful map, where $\overline{\mathcal{M}}_{0,m}$ denotes the Deligne-Mumford moduli space of genus 0 stable curves with $m$ marked points. Then, given cohomology classes $A \in H^*(\overline{\mathcal{M}}_{0,m}, \mathbb{Q})$ and $\gamma_1, \ldots, \gamma_m \in H^*(\bar{X}, \mathbb{Q})$, the Gromov-Witten invariant is defined by

$$GW_{0,m}^{\bar{X},\delta}(A; \gamma_1, \ldots, \gamma_m) = \int_{[\overline{\mathcal{M}}_{0,m}(\bar{X},\delta)]} \pi^*(A) \wedge ev_1^*(\gamma_1) \wedge \ldots \wedge ev_m^*(\gamma_m),$$

where $[\overline{\mathcal{M}}_{0,m}(\bar{X}, \delta)]$ denotes the fundamental class of $\overline{\mathcal{M}}_{0,m}(\bar{X}, \delta)$. Let $*$ be the small quantum product of $QH^*(\bar{X})$. Then it is not hard to show that, for any classes $\gamma_1, \ldots, \gamma_r \in H^*(\bar{X}, \mathbb{Q})$, the expression $\gamma_1 * \ldots * \gamma_r$ can be computed by the formula

$$\gamma_1 * \ldots * \gamma_r = \sum_{\delta \in H_2(\bar{X}, \mathbb{Z})} \sum_i GW_{0,r+1}^{\bar{X},\delta}(\mathrm{PD(pt)}; \gamma_1, \ldots, \gamma_r, t_i) t^i q^\delta,$$

where $\{t_i\}$ is a basis of $H^*(\bar{X}, \mathbb{Q})$, $\{t^i\}$ denotes the dual basis of $\{t_i\}$ with respect to the Poincaré pairing, and $\mathrm{PD(pt)} \in H^{2m-6}(\overline{\mathcal{M}}_{0,m}, \mathbb{Q})$ denotes the Poincaré dual of a point in $\overline{\mathcal{M}}_{0,m}$ (see, e.g. formula (1.4) in Spielberg [42]). Moreover, since $\bar{X}$ is homogeneous of the form $G/P$, where $G$ is a Lie group and $P$ is a parabolic subgroup, the Gromov-Witten invariants are enumerative, in the sense that $GW^{\bar{X},\delta}_{0,r+1}(\mathrm{PD(pt)}; \gamma_1, \ldots, \gamma_r, t_i)$ is equal to the number of holomorphic maps $\varphi : (\mathbb{C}P^1; x_1, \ldots, x_r, x_{r+1}) \to \bar{X}$ with $\varphi_*([\mathbb{C}P^1]) = \delta$ such that $(\mathbb{C}P^1; x_1, \ldots, x_r, x_{r+1})$ is a given point in $\overline{\mathcal{M}}_{0,r+1}$, $\varphi(x_k) \in \Gamma_k$, for $k = 1, \ldots, r$, and $\varphi(x_{r+1}) \in T_i$, where $\Gamma_1, \ldots, \Gamma_r, T_i$ are representatives of cycles Poincaré duals to the classes $\gamma_1, \ldots, \gamma_r, t_i$ respectively (see [4], p.12).

We shall now use the above facts to compute $D_{1,a} * \ldots * D_{n_a+1,a}$, which is given by the formula

$$D_{1,a} * \ldots * D_{n_a+1,a} = \sum_{\delta \in H_2(\bar{X},\mathbb{Z})} \sum_i GW^{\bar{X},\delta}_{0,n_a+2}(\mathrm{PD(pt)}; D_{1,a}, \ldots, D_{n_a+1,a}, t_i) t^i q^\delta.$$

First of all, since $H_2(\bar{X}, \mathbb{Z})$ is the kernel of the boundary map $\partial : \pi_2(\bar{X}, L_x) = \mathbb{Z}^d \to \pi_1(L_x) = N$, a homology class $\delta \in H_2(\bar{X}, \mathbb{Z})$ can be represented by a $d$-tuple of integers

$$\delta = (c_{1,1}, \ldots, c_{n_1+1,1}, \ldots, c_{1,b}, \ldots, c_{n_b+1,b}, \ldots, c_{1,l}, \ldots, c_{n_l+1,l}) \in \mathbb{Z}^d$$

satisfying $\sum_{b=1}^{l} \sum_{j=1}^{n_b+1} c_{j,b} v_{j,b} = 0 \in N$. Then we have $c_1(\bar{X}) \cdot \delta = \sum_{b=1}^{l} \sum_{j=1}^{n_b+1} c_{j,b}$. For the Gromov-Witten invariant $GW^{\bar{X},\delta}_{0,n_a+2}(\mathrm{PD(pt)}; D_{1,a}, \ldots, D_{n_a+1,a}, t_i)$ to be nonzero, $\delta$ must be represented by irreducible holomorphic curves which pass through all the divisors $D_{1,a}, \ldots, D_{n_a+1,a}$. This implies that $c_{j,a} \geq 1$, for $j = 1, \ldots, n_a + 1$, and moreover, $\delta$ lies in the cone of effective classes $H_2^{\mathrm{eff}}(\bar{X}, \mathbb{Z}) \subset H_2(\bar{X}, \mathbb{Z})$. By Theorem 2.15 of Batyrev [9], $H_2^{\mathrm{eff}}(\bar{X}, \mathbb{Z})$ is given by the kernel of the restriction of the boundary map $\partial|_{\mathbb{Z}^d_{\geq 0}} : \mathbb{Z}^d_{\geq 0} \to N$. So we must also have $c_{j,b} \geq 0$ for all $j$ and $b$, and we conclude that

$$c_1(\bar{X}) \cdot \delta = \sum_{b=1}^{l} \sum_{j=1}^{n_b+1} c_{j,b} \geq n_a + 1.$$

By dimension counting, $GW^{\bar{X},\delta}_{0,n_a+2}(\mathrm{PD(pt)}; D_{1,a}, \ldots, D_{n_a+1,a}, t_i) \neq 0$ only when

$$2(\dim_{\mathbb{C}}(\bar{X}) + c_1(\bar{X}) \cdot \delta + (n_a + 2) - 3) = 2((n_a + 2) - 3) + 2(n_a + 1) + \deg(t_i).$$

The above inequality then implies that $\deg(t_i) \geq 2\dim(\bar{X})$. We therefore must have $t_i = \mathrm{PD(pt)} \in H^{2\dim(\bar{X})}(\bar{X}, \mathbb{Q})$ and $\delta \in H_2(\bar{X}, \mathbb{Z})$ is represented by the $d$-tuple of integers $\delta_a := (c_{1,1}, \ldots, c_{n_l+1,l}) \in \mathbb{Z}^d$, where

$$c_{j,b} = \begin{cases} 1 & \text{if } b = a \text{ and } j = 1, \ldots, n_a + 1 \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $\delta = \delta_a$ is the pullback of the class of a line in the factor $\mathbb{C}P^{n_a}$. Hence,

$$D_{1,a} * \ldots * D_{n_a+1,a} = GW^{\bar{X},\delta_a}_{0,n_a+2}(\mathrm{PD(pt)}; D_{1,a}, \ldots, D_{n_a+1,a}, \mathrm{PD(pt)}) q^{\delta_a}.$$

By Theorem 9.3 in Batyrev [10] (see also Siebert [40], section 4), the Gromov-Witten invariant on the right-hand-side is equal to 1. Geometrically, this means that, for any given point $p \in X \subset \bar{X}$, there is a unique holomorphic map $\varphi_a : (\mathbb{C}P^1; x_1, \ldots, x_{n_a+2}) \to \bar{X}$ with class $\delta_a$, $\varphi_a(x_j) \in D_{j,a}$, for $j = 1, \ldots, n_a + 1$, $\varphi_a(x_{n_a+2}) = p$ and such that $(\mathbb{C}P^1; x_1, \ldots, x_{n_a+2})$ is a given configuration in $\overline{\mathcal{M}}_{0,n_a+2}$. Also note that, for $a = 1, \ldots, l$, $q^{\delta_a} = \exp(-\frac{1}{2\pi}\int_{\delta_a}\omega_{\bar{X}}) = e^{-r_a} = q_a$, where $(q_1, \ldots, q_l)$ are the coordinates of the Kähler cone $\mathcal{K}(\bar{X})$. Thus, we have the following lemma.

**Lemma 2.2.** *For $a = 1, \ldots, l$, we have*

$$D_{1,a} * \ldots * D_{n_a+1,a} = q_a.$$

*Hence, the quantum Stanley-Reisner ideal of $\bar{X} = \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_l}$ is given by*

$$\mathcal{SR}_Q = \langle D_{1,a} * \ldots * D_{n_a+1,a} - q_a : a = 1 \ldots, l \rangle,$$

*and the quantum cohomology ring of $\bar{X}$ has a presentation given by*

$$QH^*(\bar{X}) = \frac{\mathbb{C}[D_{1,1}, \ldots, D_{n_1+1,1}, \ldots, D_{1,l}, \ldots, D_{n_l+1,l}]}{\left\langle D_{j,a} - D_{n_a+1,a} : j=1,\ldots,n_a,\ a=1,\ldots,l \right\rangle + \left\langle \prod_{j=1}^{n_a+1} D_{j,a} - q_a : a=1,\ldots,l \right\rangle}.$$

Proposition 2.2 now follows from a simple combinatorial argument:

*Proof of Proposition 2.2.* For a general toric Fano manifold $\bar{X}$, recall, from Remark 2.1, that we have chosen a $\mathbb{Z}$-basis

$$Q_1 = (Q_{11}, \ldots, Q_{d1}), \ldots, Q_l = (Q_{1l}, \ldots, Q_{dl}) \in \mathbb{Z}^d$$

of $K = H_2(\bar{X}, \mathbb{Z})$ such that $(Q_{n+a,b})_{1 \leq a,b \leq l} = \mathrm{Id}_{l \times l}$. So, by Cho-Oh's symplectic area formula (2.2), for $a = 1, \ldots, l$, we have

$$\begin{aligned} \Big(\sum_{i=1}^n Q_{ia}\int_{\beta_i}\omega_{\bar{X}}\Big) + \int_{\beta_{n+a}}\omega_{\bar{X}} &= 2\pi\Big(\sum_{i=1}^n Q_{ia}(\langle x, v_i\rangle - \lambda_i)\Big) + 2\pi(\langle x, v_{n+a}\rangle - \lambda_{n+a}) \\ &= 2\pi\langle x, \sum_{i=1}^n Q_{ia}v_i + v_{n+a}\rangle - 2\pi(\sum_{i=1}^a Q_{ia}\lambda_a + \lambda_{n+a}) \\ &= 2\pi r_a. \end{aligned}$$

Then, by the definition of the convolution product of functions on $LX$, we have

$$\begin{aligned} \Psi_1^{Q_{1a}} \star \ldots \star \Psi_n^{Q_{na}} \star \Psi_{n+a}(x, v) &= \begin{cases} e^{-\frac{1}{2\pi}(\sum_{i=1}^n Q_{ia}\int_{\beta_i}\omega_{\bar{X}}) - \frac{1}{2\pi}\int_{\beta_{n+a}}\omega_{\bar{X}}} & \text{if } v = 0 \\ 0 & \text{if } v \neq 0 \end{cases} \\ &= \begin{cases} e^{-r_a} & \text{if } v = 0 \\ 0 & \text{if } v \neq 0 \end{cases} \\ &= q_a \mathbb{1}, \end{aligned}$$

or $\Psi_{n+a} = q_a(\Psi_1^{-1})^{Q_{1a}} \star \ldots \star (\Psi_n^{-1})^{Q_{na}}$, for $a = 1, \ldots, l$.

Suppose that the following condition: $Q_{ia} \geq 0$ for $i = 1, \ldots, n$, $a = 1, \ldots, l$, and for each $i = 1, \ldots, n$, there exists $1 \leq a \leq l$ such that $Q_{ia} > 0$, is satisfied, which is the case when $\bar{X}$ is a product of projective spaces. Then the inclusion

$$\mathbb{C}[\Psi_1, \ldots, \Psi_n, \Psi_{n+1}, \ldots, \Psi_d] \hookrightarrow \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]$$

is an isomorphism. Consider the surjective map

$$\rho : \mathbb{C}[D_1, \ldots, D_d] \to \mathbb{C}[\Psi_1, \ldots, \Psi_d]$$

defined by mapping $D_i$ to $\Psi_i$ for $i = 1, \ldots, d$. This map is not injective because there are nontrivial relations in $\mathbb{C}[\Psi_1, \ldots, \Psi_d]$ generated by the relations

$$\Psi_1^{Q_{1a}} \star \ldots \star \Psi_n^{Q_{na}} \star \Psi_{n+a} - q_a \mathbb{1} = 0, \ a = 1, \ldots, l.$$

By Lemma 2.2, the kernel of $\rho$ is exactly given by the ideal $\mathcal{SR}_Q$ when $\bar{X}$ is a product of projective spaces. Thus, we have an isomorphism

$$\mathbb{C}[D_1, \ldots, D_d]/\mathcal{SR}_Q \xrightarrow{\cong} \mathbb{C}[\Psi_1, \ldots, \Psi_d].$$

Since $(\mathbb{C}[D_1, \ldots, D_d]/\mathcal{SR}_Q)/\mathcal{L} = \mathbb{C}[D_1, \ldots, D_d]/(\mathcal{L} + \mathcal{SR}_Q) = QH^*(\bar{X})$, we obtain the desired isomorphism

$$QH^*(\bar{X}) \cong \mathbb{C}[\Psi_1, \ldots, \Psi_d]/\mathcal{L} \cong \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L},$$

provided that $\bar{X}$ is a product of projective spaces. □

**Remark 2.3.**

1. *In* [10], *Theorem 5.3, Batyrev gave a "formula" for the quantum Stanley-Reisner ideal $\mathcal{SR}_Q$ for any compact toric Kähler manifolds, using his own definition of the small quantum product (which is different from the usual one because Batyrev counted only holomorphic maps from $\mathbb{C}P^1$). By Givental's mirror theorem* [20], *Batyrev's formula is true, using the usual definition of the small quantum product, for all toric Fano manifolds. Our proof of Lemma 2.2 is nothing but a simple verification of Batyrev's formula in the case of products of projective spaces, without using Givental's mirror theorem.*
2. *In any event, Batyrev's formula in* [10] *for a presentation of the small quantum cohomology ring $QH^*(\bar{X})$ of a toric Fano manifold $\bar{X}$ is correct. In the same paper, Batyrev also proved that $QH^*(\bar{X})$ is canonically isomorphic to the Jacobian ring $Jac(W)$, where $W$ is the superpotential mirror to $\bar{X}$ (Theorem 8.4 in* [10]*). Now, by Theorem 3.3 in Subsection 3.3, the inverse SYZ transformation $\mathcal{F}^{-1}$ gives a canonical isomorphism $\mathcal{F}^{-1} : Jac(W) \xrightarrow{\cong} \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L}$. Then, the composition map $QH^*(\bar{X}) \to \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L}$, which maps $D_i$ to $\Psi_i$, for $i = 1, \ldots, d$, is an isomorphism. This proves Proposition 2.2 all toric Fano manifolds. We choose not to use this proof because all the geometry is then hid by the use of Givental's mirror theorem.*

2.2. **The role of tropical geometry.** While our proof of the isomorphism (2.4) in Proposition 2.2 is combinatorial in nature, the best way to understand the geometry behind it is through the correspondence between holomorphic curves and discs in $\bar{X}$ and their tropical counterparts in $N_{\mathbb{R}}$. Indeed, this is the main reason why we confine ourselves to the case of products of projective spaces. Our first task is to define a tropical analog $QH^*_{trop}(\bar{X})$ of the small quantum cohomology ring of $\bar{X}$, when $\bar{X}$ is a product of projective spaces. For this, we shall recall some notions in tropical geometry. We will not state the precise definitions, for which we refer the reader to Mikhalkin [32], [33], [34] and Nishinou-Siebert [38].

A *genus 0 tropical curve with $m$ marked points* is a connected tree $\Gamma$ with exactly $m$ unbounded edges (also called leaves) and each bounded edge is assigned a positive length. Let $\overline{\mathcal{M}}_{0,m}^{trop}$ be the moduli space of genus 0 tropical curves with $m$ marked points (modulo isomorphisms). The combinatorial types of $\Gamma$ partition

$\overline{\mathcal{M}}_{0,m}^{trop}$ into disjoint subsets, each of which has the structure of a polyhedral cone $\mathbb{R}^e_{>0}$ (where $e$ is the number of bounded edges in $\Gamma$). There is a distinguished point in $\overline{\mathcal{M}}_{0,m}^{trop}$ corresponding to the (unique) tree $\Gamma_m$ with exactly one ($m$-valent) vertex $V$, $m$ unbounded edges $E_1, \ldots, E_m$ and *no* bounded edges. See Figure 2.1 below. We will fix this point in $\overline{\mathcal{M}}_{0,m}^{trop}$; this is analog to fixing a point in $\overline{\mathcal{M}}_{0,m}$.

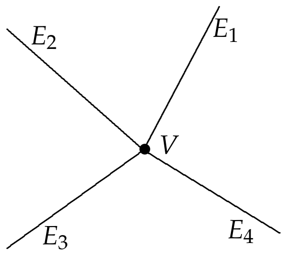


Figure 2.1: $\Gamma_4 \in \overline{\mathcal{M}}_{0,4}^{trop}$.

Let $\Sigma$ be the fan defining $\bar{X} = \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_l}$, and denote by $\Sigma[1] = \{v_{1,1}, \ldots, v_{n_1+1,1}, \ldots, v_{1,a}, \ldots, v_{n_a+1,a}, \ldots, v_{1,l}, \ldots, v_{n_l+1,l}\} \subset N$ the set of primitive generators of 1-dimensional cones in $\Sigma$. Let $h : \Gamma_m \to N_{\mathbb{R}}$ be a continuous embedding such that, for each $k = 1, \ldots, m$, $h(E_k) = h(V) + \mathbb{R}_{\geq 0} v(E_k)$ for some $v(E_k) \in \Sigma[1]$, and the following *balancing condition* is satisfied:

$$\sum_{k=1}^{m} v(E_k) = 0.$$

Then the tuple $(\Gamma_m; E_1, \ldots, E_m; h)$ is a *parameterized $m$-marked, genus 0 tropical curve in $\bar{X}$*. The *degree* of $(\Gamma_m; E_1, \ldots, E_m; h)$ is the $d$-tuple of integers $\delta(h) = (c_{1,1}, \ldots, c_{n_1+1,1}, \ldots, c_{1,a}, \ldots, c_{n_a+1,a}, \ldots, c_{1,l}, \ldots, c_{n_l+1,l}) \in \mathbb{Z}^d$, where

$$c_{j,a} = \begin{cases} 1 & \text{if } v_{j,a} \in \{v(E_1), \ldots, v(E_m)\} \\ 0 & \text{otherwise.} \end{cases}$$

By the balancing condition, we have $\sum_{a=1}^{l} \sum_{j=1}^{n_a+1} c_{j,a} v_{j,a} = 0$, i.e. $\delta(h)$ lies in the kernel of $\partial : \mathbb{Z}^d \to N$, and so $\delta(h) \in H_2(\bar{X}, \mathbb{Z})$.

We want to consider the tropical counterpart, denoted by

$$TGW_{0,n_a+2}^{\bar{X},\delta}(\mathrm{PD}(\mathrm{pt}); D_{1,a}, \ldots, D_{n_a+1,a}, t_i),$$

of the Gromov-Witten invariant $GW_{0,n_a+2}^{\bar{X},\delta}(\mathrm{PD}(\mathrm{pt}); D_{1,a}, \ldots, D_{n_a+1,a}, t_i)$. [7] Since a general definition is not available, we introduce a tentative definition as follows.

**Definition 2.3.** *We define $TGW_{0,n_a+2}^{\bar{X},\delta}(PD(pt); D_{1,a}, \ldots, D_{n_a+1,a}, t_i)$ to be the number of parameterized $(n_a+1)$-marked, genus 0 tropical curves of the form $(\Gamma_{n_a+1}; E_1, \ldots, E_{n_a+1}; h)$ with $\delta(h) = \delta$ such that $h(E_j) = h(V) + \mathbb{R}_{\geq 0} v_{j,a}$, for $j = 1, \ldots, n_a+1$, and $h(V) \in Log(T_i)$, where $T_i$ is a cycle Poincaré dual to $t_i$, whenever this number is finite. We set $TGW_{0,n_a+2}^{\bar{X},\delta}(PD(pt); D_{1,a}, \ldots, D_{n_a+1,a}, t_i)$ to be 0 if this number is infinite. Here, $Log : X \to N_{\mathbb{R}}$ is the map, after identifying $X$ with $(\mathbb{C}^*)^n$, defined by $Log(w_1, \ldots, w_n) = (\log|w_1|, \ldots, \log|w_n|)$, for $(w_1, \ldots, w_n) \in X$.*

[7]"TGW" stands for "tropical Gromov-Witten".

We then define the *tropical small quantum cohomology ring* $QH^*_{trop}(\bar{X})$ of $\bar{X} = \mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_l}$ as a presentation:

$$QH^*_{trop}(\bar{X}) = \mathbb{C}[D_{1,1}, \ldots, D_{n_1+1,1}, \ldots, D_{1,l} \ldots, D_{n_l+1,l}]/(\mathcal{L} + \mathcal{SR}_Q^{trop}),$$

where $\mathcal{SR}_Q^{trop}$ is the tropical version of the quantum Stanley-Reisner ideal, defined to be the ideal generated by the relations

$$D_{1,a} *_T \ldots *_T D_{n_a+1,a} = \sum_{\delta \in H_2(\bar{X},\mathbb{Z})} \sum_i TGW_{0,n_a+2}^{\bar{X},\delta}(\mathrm{PD}(\mathrm{pt}); D_{1,a}, \ldots, D_{n_a+1,a}, t_i) t^i q^\delta,$$

for $a = 1, \ldots, l$. Here $*_T$ denotes the product in $QH^*_{trop}(\bar{X})$, which we call the tropical small quantum product. It is not hard to see that, as in the holomorphic case, we have

$$TGW_{0,n_a+2}^{\bar{X},\delta}(\mathrm{PD}(\mathrm{pt}); D_{1,a}, \ldots, D_{n_a+1,a}, t_i) = \begin{cases} 1 & \text{if } t_i = \mathrm{PD}(\mathrm{pt}) \text{ and } \delta = \delta_a \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, as a special case of the *correspondence theorem* of Mikhalkin [32] and Nishinou-Siebert [38], we have: For a given point $p \in X$, let $\xi := \mathrm{Log}(p) \in N_\mathbb{R}$. Then the unique holomorphic curve $\varphi_a : (\mathbb{C}P^1; x_1, \ldots, x_{n_a+2}) \to \bar{X}$ with class $\delta_a$, $\varphi_a(x_j) \in D_{j,a}$, for $j = 1, \ldots, n_a + 1$, $\varphi_a(x_{n_a+2}) = p$ and such that $(\mathbb{C}P^1; x_1, \ldots, x_{n_a+2})$ is a given configuration in $\overline{\mathcal{M}}_{0,n_a+2}$, is corresponding to the unique parameterized $(n_a + 1)$-marked tropical curve $(\Gamma_{n_a+1}; E_1, \ldots, E_{n_a+1}; h_a)$ of genus 0 and degree $\delta_a$ such that $h_a(V) = \xi$ and $h_a(E_j) = \xi + \mathbb{R}_{\geq 0} v_{j,a}$, for $j = 1, \ldots, n_a + 1$. It follows that

$$\mathcal{SR}_Q^{trop} = \langle D_{1,a} *_T \ldots *_T D_{n_a+1,a} - q_a : a = 1 \ldots, l \rangle,$$

and there is a canonical isomorphism

$$QH^*(\bar{X}) \cong QH^*_{trop}(\bar{X}). \tag{2.5}$$

**Remark 2.4.** *All these arguments and definitions rely, in an essential way, on the fact that $\bar{X}$ is a product of projective spaces, so that Gromov-Witten invariants are enumerative and all the (irreducible) holomorphic curves, which contribute to $QH^*(\bar{X})$, are not mapped into the toric boundary divisor $D_\infty$. Remember that tropical geometry cannot be used to count nodal curves or curves with irreducible components mapping into $D_\infty$.*

Next, we take a look at tropical discs. Consider the point $\Gamma_1 \in \overline{\mathcal{M}}_{0,1}^{trop}$. This is nothing but a half line, consisting of an unbounded edge $E$ emanating from a univalent vertex $V$. See Figure 2.2 below.

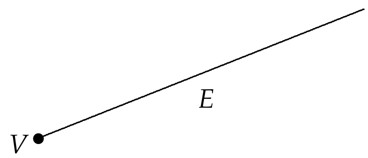


Figure 2.2: $\Gamma_1 \in \overline{\mathcal{M}}_{0,1}^{trop}$.

A *parameterized Maslov index two tropical disc in* $\bar{X}$ is a tuple $(\Gamma_1, E, h)$, where $h : \Gamma_1 \to N_\mathbb{R}$ is an embedding such that $h(E) = h(V) + \mathbb{R}_{\geq 0} v$ for some $v \in \Sigma[1]$.[8] For

[8]For precise definitions of general tropical discs (with higher Maslov indices), we refer the reader to Nishinou [36]; see also the recent work of Gross [22].

any given point $\xi \in N_{\mathbb{R}}$, it is obvious that, there is a unique parameterized Maslov index two tropical disc $(\Gamma_1, E, h_{j,a})$ such that $h_{j,a}(V) = \xi$ and $h_{j,a}(E) = \xi + \mathbb{R}_{\geq 0} v_{j,a}$, for any $v_{j,a} \in \Sigma[1]$. Comparing this to the result (Theorem 2.1) of Cho-Oh on the classification of Maslov index two holomorphic discs in $\bar{X}$ with boundary in the Lagrangian torus fiber $L_\xi := \mathrm{Log}^{-1}(\xi) \subset X$, we get a one-to-one correspondence between the families of Maslov index two holomorphic discs in $(\bar{X}, L_\xi)$ and the parameterized Maslov index two tropical discs $(\Gamma_1, E, h)$ in $\bar{X}$ such that $h(V) = \xi$. We have the holomorphic disc $\varphi_{j,a} : (D^2, \partial D^2) \to (\bar{X}, L_\xi)$ corresponding to the tropical disc $(\Gamma_1, E, h_{j,a})$.[9] Then, by (2.3), we also get a one-to-one correspondence between the parameterized Maslov index two tropical discs $(\Gamma_1, E, h_{j,a})$ in $\bar{X}$ and the functions $\Psi_{j,a} : L_X \to \mathbb{R}$:

$$\{\varphi_{j,a}\} \overset{1-1}{\longleftrightarrow} \{(\Gamma_1, E, h_{j,a})\} \overset{1-1}{\longleftrightarrow} \{\Psi_{j,a}\}.$$

Now, while the canonical isomorphism

$$QH^*_{trop}(\bar{X}) \cong \mathbb{C}[\Psi^{\pm 1}_{1,1}, \ldots, \Psi^{\pm 1}_{n_1,1}, \ldots, \Psi^{\pm 1}_{1,l}, \ldots, \Psi^{\pm 1}_{n_l,l}]/\mathcal{L} \tag{2.6}$$

follows from the same simple combinatorial argument in the proof of Proposition 2.2, the geometry underlying it is exhibited by a simple but crucial observation, which we formulate as the following proposition.

**Proposition 2.3.** *Let $\xi \in N_{\mathbb{R}}$, then the unique parameterized $(n_a+1)$-marked, genus 0 tropical curve $(\Gamma_{n_a+1}; E_1, \ldots, E_{n_a+1}; h_a)$ such that $h_a(V) = \xi$ and $h_a(E_j) = \xi + \mathbb{R}_{\geq 0} v_{j,a}$, for $j = 1, \ldots, n_a+1$, is obtained by gluing the $n_a+1$ parameterized Maslov index two tropical discs $(\Gamma_1, E, h_{1,a}), \ldots, (\Gamma_1, E, h_{n_a+1,a})$ with $h_{j,a}(V) = \xi$, for $j = 1, \ldots, n_a+1$, in the following sense: The map $h : (\Gamma_{n_a+1}; E_1, \ldots, E_{n_a+1}) \to N_{\mathbb{R}}$ defined by $h|_{E_j} = h_{j,a}|_E$, for $j = 1, \ldots, n_a+1$, gives a parameterized $(n_a+1)$-marked, genus 0 tropical curve, which coincides with $(\Gamma_{n_a+1}; E_1, \ldots, E_{n_a+1}; h_a)$.*

*Proof.* Since $\sum_{j=1}^{n_a+1} v_{j,a} = 0$, the balancing condition at $V \in \Gamma_{n_a+1}$ is automatically satisfied. So $h$ defines a parameterized $(n_a+1)$-marked, genus 0 tropical curve, which satisfies the same conditions as $(\Gamma_{n_a+1}; E_1, \ldots, E_{n_a+1}; h_a)$. ☐

For example, in the case of $\bar{X} = \mathbb{C}P^2$, this can be seen in Figure 2.3 below.

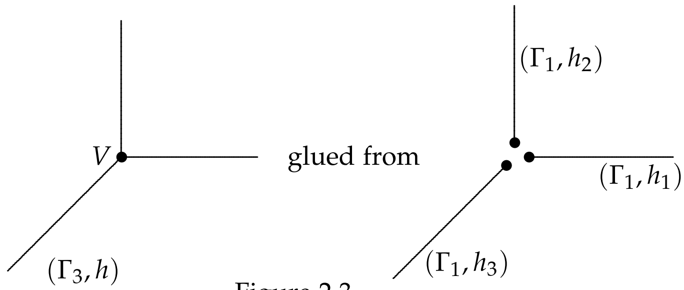


Figure 2.3

The functions $\Psi_{j,a}$'s could have been defined by counting parameterized Maslov index two tropical discs, instead of counting Maslov index two holomorphic discs.

[9]This correspondence also holds for other toric manifolds, not just for products of projective spaces.

So the above proposition indeed gives a geometric reason to explain why the relation

$$D_{1,a} *_T \ldots *_T D_{n_a+1,a} = q_a$$

in $QH^*_{trop}(\bar{X})$ should coincide with the relation

$$\Psi_{1,a} \star \ldots \star \Psi_{n_a+1,a} = q_a \mathbb{1}$$

in $\mathbb{C}[\Psi^{\pm 1}_{1,1}, \ldots, \Psi^{\pm 1}_{n_1,1}, \ldots, \Psi^{\pm 1}_{1,l}, \ldots, \Psi^{\pm 1}_{n_l,l}]/\mathcal{L}$. The convolution product $\star$ may then be thought of as a way to encode the gluing of tropical discs.

We summarize what we have said as follows: In the case of products of projective spaces, we factor the isomorphism $QH^*(\bar{X}) \cong \mathbb{C}[\Psi^{\pm 1}_1, \ldots, \Psi^{\pm 1}_n]/\mathcal{L}$ in Proposition 2.2 into two isomorphisms (2.5) and (2.6). The first one comes from the correspondence between holomorphic curves in $\bar{X}$ which contribute to $QH^*(\bar{X})$ and tropical curves in $N_{\mathbb{R}}$ which contribute to $QH^*_{trop}(\bar{X})$. The second isomorphism is due to, on the one hand, the fact that each tropical curve which contributes to $QH^*_{trop}(\bar{X})$ can be obtained by gluing Maslov index two tropical discs, and, on the other hand, the correspondence between these tropical discs in $N_{\mathbb{R}}$ and Maslov index two holomorphic discs in $\bar{X}$ with boundary on Lagrangian torus fibers. See Figure 2.4 below.

$$\begin{tikzcd}
QH^*(\bar{X}) \arrow[dd, leftrightarrow] \arrow[dr, leftrightarrow, "\text{Prop 2.2}"] & & \\
& \mathbb{C}[\Psi^{\pm 1}_1, \ldots, \Psi^{\pm 1}_n]/\mathcal{L} \arrow[r, leftrightarrow, "\mathcal{F}"] & Jac(W) \\
QH^*_{trop}(\bar{X}) \arrow[ur, leftrightarrow, "\text{Prop 2.3}"'] & &
\end{tikzcd}$$

Figure 2.4

Here $\mathcal{F}$ denotes the SYZ mirror transformation for $\bar{X}$, which is the subject of Section 3.

## 3. SYZ MIRROR TRANSFORMATIONS

In this section, we first derive Hori-Vafa's mirror manifold using semi-flat SYZ mirror transformations. Then we introduce the main character in this paper: the SYZ mirror transformations for toric Fano manifolds, and prove our main result.

3.1. **Derivation of Hori-Vafa's mirror manifold by T-duality.** Recall that we have an exact sequence (2.1):

$$0 \longrightarrow K \xrightarrow{\iota} \mathbb{Z}^d \xrightarrow{\partial} N \longrightarrow 0, \tag{3.1}$$

and we denote by

$$Q_1 = (Q_{11}, \ldots, Q_{d1}), \ldots, Q_l = (Q_{1l}, \ldots, Q_{dl}) \in \mathbb{Z}^d$$

a $\mathbb{Z}$-basis of $K$. The mirror manifold of $\bar{X}$, derived by Hori and Vafa in [27] using physical arguments, is the complex submanifold

$$Y_{HV} = \Big\{ (Z_1, \ldots, Z_d) \in \mathbb{C}^d : \prod_{i=1}^{d} Z_i^{Q_{ia}} = q_a,\ a = 1, \ldots, l \Big\},$$

in $\mathbb{C}^d$, where $q_a = e^{-r_a} = \exp(-\sum_{i=1}^d Q_{ia}\lambda_i)$, for $a = 1, \ldots, l$. As a complex manifold, $Y_{HV}$ is biholomorphic to the algebraic torus $(\mathbb{C}^*)^n$. By our choice of the $\mathbb{Z}$-basis $Q_1, \ldots, Q_l$ of $K$ in Remark 2.1, $Y_{HV}$ can also be written as

$$Y_{HV} = \left\{(Z_1, \ldots, Z_d) \in \mathbb{C}^d : Z_1^{Q_{1a}} \ldots Z_n^{Q_{na}} Z_{n+a} = q_a,\ a = 1, \ldots, l\right\}.$$

Note that, in fact, $Y_{HV} \subset (\mathbb{C}^*)^d$. In terms of these coordinates, Hori and Vafa predicted that the superpotential $W : Y_{HV} \to \mathbb{C}$ is given by

$$\begin{aligned} W &= Z_1 + \ldots + Z_d \\ &= Z_1 + \ldots + Z_n + \frac{q_1}{Z_1^{Q_{11}} \ldots Z_n^{Q_{n1}}} + \ldots + \frac{q_l}{Z_1^{Q_{1l}} \ldots Z_n^{Q_{nl}}}. \end{aligned}$$

The goal of this subsection is to show that the SYZ mirror manifold $Y_{SYZ}$, which is obtained by applying T-duality to the open dense orbit $X \subset \bar{X}$, is contained in Hori-Vafa's manifold $Y_{HV}$ as a bounded open subset. The result itself is not new, and can be found, for example, in Auroux [6], Proposition 4.2. For the sake of completeness, we give a self-contained proof, which will show how T-duality, i.e. fiberwise dualizing torus bundles, transforms the *symplectic quotient space* $X$ into the *complex subspace* $Y_{SYZ}$.

We shall first briefly recall the constructions of $\bar{X}$ and $X$ as symplectic quotients. For more details, we refer the reader to Appendix 1 in Guillemin [24].

From the above exact sequence (3.1), we get an exact sequence of real tori

$$1 \longrightarrow T_K \xrightarrow{\iota} T^d \xrightarrow{\partial} T_N \longrightarrow 1, \tag{3.2}$$

where $T^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ and we denote by $K_\mathbb{R}$ and $T_K$ the real vector space $K \otimes_\mathbb{Z} \mathbb{R}$ and the torus $K_\mathbb{R}/K$ respectively. Considering their Lie algebras and dualizing give another exact sequence

$$0 \longrightarrow M_\mathbb{R} \xrightarrow{\check{\partial}} (\mathbb{R}^d)^\vee \xrightarrow{\check{\iota}} K_\mathbb{R}^\vee \longrightarrow 0. \tag{3.3}$$

Denote by $W_1, \ldots, W_d \in \mathbb{C}$ the complex coordinates on $\mathbb{C}^d$. The standard diagonal action of $T^d$ on $\mathbb{C}^d$ is Hamiltonian with respect to the standard symplectic form $\frac{\sqrt{-1}}{2}\sum_{i=1}^d dW_i \wedge d\bar{W}_i$ and the moment map $h : \mathbb{C}^d \to (\mathbb{R}^d)^\vee$ is given by

$$h(W_1, \ldots, W_d) = \frac{1}{2}(|W_1|^2, \ldots, |W_d|^2).$$

Restricting to $T_K$, we get a Hamiltonian action of $T_K$ on $\mathbb{C}^d$ with moment map $h_K = \check{\iota} \circ h : \mathbb{C}^d \to \check{K}_\mathbb{R}$. In terms of the $\mathbb{Z}$-basis $\{Q_1, \ldots, Q_l\}$ of $K$, the map $\check{\iota} : (\mathbb{R}^d)^\vee \to K_\mathbb{R}^\vee$ is given by

$$\check{\iota}(X_1, \ldots, X_d) = \left(\sum_{i=1}^d Q_{i1}X_i, \ldots, \sum_{i=1}^d Q_{il}X_i\right), \tag{3.4}$$

for $(X_1, \ldots, X_d) \in (\mathbb{R}^d)^\vee$, in the coordinates associated to the dual basis $\check{Q}_1, \ldots, \check{Q}_l$ of $K^\vee = \mathrm{Hom}(K, \mathbb{Z})$. The moment map $h_K : \mathbb{C}^d \to K_\mathbb{R}^\vee$ can thus be written as

$$h_K(W_1, \ldots, W_d) = \frac{1}{2}\left(\sum_{i=1}^d Q_{i1}|W_i|^2, \ldots, \sum_{i=1}^d Q_{il}|W_i|^2\right) \in K_\mathbb{R}^\vee.$$

In these coordinates, $r = (r_1, \ldots, r_l) = -\check{\iota}(\lambda_1, \ldots, \lambda_d)$ is an element in $K^\vee_\mathbb{R} = H^2(\bar{X}, \mathbb{R})$, and $\bar{X}$ and $X$ are given by the symplectic quotients

$$\bar{X} = h_K^{-1}(r)/T_K \text{ and } X = (h_K^{-1}(r) \cap (\mathbb{C}^*)^d)/T_K$$

respectively.

In the above process, the image of $h_K^{-1}(r)$ under the map $h : \mathbb{C}^d \to (\mathbb{R}^d)^\vee$ lies inside the affine linear subspace $M_\mathbb{R}(r) = \{(X_1, \ldots, X_d) \in (\mathbb{R}^d)^\vee : \check{\iota}(X_1, \ldots, X_d) = r\}$, i.e. a translate of $M_\mathbb{R}$. In fact, $h(h_K^{-1}(r)) = \check{\iota}^{-1}(r) \cap \{(X_1, \ldots, X_d) \in (\mathbb{R}^d)^\vee : X_i \geq 0, \text{ for } i = 1, \ldots, d\}$ is the polytope $\bar{P} \subset M_\mathbb{R}(r)$, and $h(h_K^{-1}(r) \cap (\mathbb{C}^*)^d) = \check{\iota}^{-1}(r) \cap \{(X_1, \ldots, X_d) \in (\mathbb{R}^d)^\vee : X_i > 0, \text{ for } i = 1, \ldots, d\}$ is the interior $P$ of $\bar{P}$. Now, restricting $h$ to $h_K^{-1}(r) \cap (\mathbb{C}^*)^d$ gives a $T^d$-bundle $h : h_K^{-1}(r) \cap (\mathbb{C}^*)^d \to P$ (which is trivial), and $X$ is obtained by taking the quotient of this $T^d$-bundle fiberwise by $T_K$. Hence, $X$ is naturally a $T_N$-bundle over $P$, which can be written as

$$X = T^*P/N = P \times \sqrt{-1}T_N$$

(cf. Abreu [3]).[10] The reduced symplectic form $\omega_X = \omega_{\bar{X}}|_X$ is the standard symplectic form

$$\omega_X = \sum_{j=1}^n dx_j \wedge du_j$$

where $x_1, \ldots, x_n \in \mathbb{R}$ and $u_1, \ldots, u_n \in \mathbb{R}/2\pi\mathbb{Z}$ are respectively the coordinates on $P \subset M_\mathbb{R}(r)$ and $T_N$. In other words, the $x_j$'s and $u_j$'s are *symplectic coordinates* (i.e. action-angle coordinates). And the moment map is given by the projection to $P$

$$\mu : X \to P.$$

We define the *SYZ mirror manifold* by T-duality as follows.

**Definition 3.1.** *The SYZ mirror manifold $Y_{SYZ}$ is defined as the total space of the $T_M$-bundle, where $T_M = M_\mathbb{R}/M = (T_N)^\vee$, which is obtained by fiberwise dualizing the $T_N$-bundle $\mu : X \to P$.*

In other words, we have

$$Y_{SYZ} = TP/M = P \times \sqrt{-1}T_M \subset M_\mathbb{R}(r) \times \sqrt{-1}T_M.$$

$Y_{SYZ}$ has a natural complex structure, which is induced from the one on $M_\mathbb{R}(r) \times \sqrt{-1}T_M \cong (\mathbb{C}^*)^n$. We let $z_j = \exp(-x_j - \sqrt{-1}y_j)$, $j = 1, \ldots, n$, be the *complex coordinates* on $M_\mathbb{R}(r) \times \sqrt{-1}T_M \cong (\mathbb{C}^*)^n$ and restricted to $Y_{SYZ}$, where $y_1, \ldots, y_n \in \mathbb{R}/2\pi\mathbb{Z}$ are the coordinates on $T_M = (T_N)^\vee$ dual to $u_1, \ldots, u_n$. We also let $\Omega_{Y_{SYZ}}$ be the following nowhere vanishing holomorphic $n$-form on $Y_{SYZ}$:

$$\Omega_{Y_{SYZ}} = \bigwedge_{j=1}^n (-dx_j - \sqrt{-1}dy_j) = \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{dz_n},$$

and denote by

$$\nu : Y_{SYZ} \to P$$

the torus fibration dual to $\mu : X \to P$.

[10] We have, by abuse of notations, used $N$ to denote the family of lattices $P \times \sqrt{-1}N$ over $P$. Similarly, we denote by $M$ the family of lattices $P \times \sqrt{-1}M$ below.

**Proposition 3.1.** *The SYZ mirror manifold $Y_{SYZ}$ is contained in Hori-Vafa's mirror manifold $Y_{HV}$ as an open complex submanifold. More precisely, $Y_{SYZ}$ is the bounded domain $\{(Z_1,\ldots,Z_d)\in Y_{SYZ}:|Z_i|<1,\ i=1,\ldots,d\}$ inside $Y_{HV}$.*

*Proof.* Dualizing the sequence (3.2), we get

$$1\longrightarrow T_M\xrightarrow{\check{\partial}}(T^d)^\vee\xrightarrow{\check{\iota}}(T_K)^\vee\longrightarrow 1,$$

while we also have the sequence (3.3)

$$0\longrightarrow M_{\mathbb{R}}\xrightarrow{\check{\partial}}(\mathbb{R}^d)^\vee\xrightarrow{\check{\iota}}K_{\mathbb{R}}^\vee\longrightarrow 0.$$

Let $T_i=X_i+\sqrt{-1}Y_i\in\mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$, $i=1,\ldots,d$, be the complex coordinates on $(\mathbb{R}^d)^\vee\times\sqrt{-1}(T^d)^\vee\cong(\mathbb{C}^*)^d$. If we let $Z_i=e^{-T_i}\in\mathbb{C}^*$, $i=1,\ldots,d$, then, by the definition of $Y_{HV}$ and by (3.4), we can identify $Y_{HV}$ with the following complex submanifold in $(\mathbb{C}^*)^d$:

$$\check{\iota}^{-1}(r\in K_{\mathbb{R}}^\vee)\cap\check{\iota}^{-1}(1\in(T_K)^\vee)\subset(\mathbb{R}^d)^\vee\times\sqrt{-1}(T^d)^\vee=(\mathbb{C}^*)^d.$$

Hence,

$$Y_{HV}=M_{\mathbb{R}}(r)\times\sqrt{-1}T_M\cong(\mathbb{C}^*)^n$$

as complex submanifolds in $(\mathbb{C}^*)^d$. Since $Y_{SYZ}=TP/M=P\times\sqrt{-1}T_M$, $Y_{SYZ}$ is a complex submanifold in $Y_{HV}$. In fact, as $P=\check{\iota}^{-1}(r)\cap\{(X_1,\ldots,X_d)\in(\mathbb{R}^d)^*: X_i>0,i=1,\ldots,d\}$, we have

$$Y_{SYZ}=\{(Z_1,\ldots,Z_d)\in Y_{HV}:|Z_i|<1,\ \text{for } i=1,\ldots,d\}.$$

So $Y_{SYZ}$ is a bounded domain in $Y_{HV}$. □

We remark that, in terms of the complex coordinates $z_j=\exp(-x_j-\sqrt{-1}y_j)$, $j=1,\ldots,n$, on $Y_{HV}=M_{\mathbb{R}}(r)\times\sqrt{-1}T_M\cong(\mathbb{C}^*)^n$, the embedding $\check{\partial}:Y_{HV}\hookrightarrow(\mathbb{C}^*)^d$ is given by

$$\check{\partial}(z_1,\ldots,z_n)=(e^{\lambda_1}z^{v_1},\ldots,e^{\lambda_d}z^{v_d}),$$

where $z^v$ denotes the monomial $z_1^{v^1}\ldots z_n^{v^n}$ if $v=(v^1,\ldots,v^n)\in N=\mathbb{Z}^n$. So the coordinates $Z_i$'s and $z_i$'s are related by

$$Z_i=e^{\lambda_i}z^{v_i},$$

for $i=1,\ldots,d$, and the SYZ mirror manifold $Y_{SYZ}$ is given by the bounded domain

$$Y_{SYZ}=\{(z_1,\ldots,z_n)\in Y_{HV}=(\mathbb{C}^*)^n:|e^{\lambda_i}z^{v_i}|<1,\ i=1,\ldots,d\}.$$

Now, the superpotential $W:Y_{SYZ}\to\mathbb{C}$ (or $W:Y_{HV}\to\mathbb{C}$) is of the form

$$W=e^{\lambda_1}z^{v_1}+\ldots+e^{\lambda_d}z^{v_d}.$$

From the above proposition, the SYZ mirror manifold $Y_{SYZ}$ is strictly *smaller* than Hori-Vafa's mirror manifold $Y_{HV}$. This issue was discussed in Hori-Vafa [27], Section 3 and Auroux [6], Section 4.2, and may be resolved by a process called *renormalization*. We refer the interested reader to those references for the details. In this paper, we shall always be (except in this subsection) looking at the SYZ mirror manifold, and the letter $Y$ will also be used exclusively to denote the SYZ mirror manifold.

3.2. **SYZ mirror transformations as fiberwise Fourier transforms.** In this subsection, we first give a brief review of semi-flat SYZ mirror transformations (for details, see Hitchin [26], Leung-Yau-Zaslow [31] and Leung [30]). Then we introduce the SYZ mirror transformations for toric Fano manifolds, and prove part 1. of Theorem 1.1.

To begin with, recall that the dual torus $T_M = (T_N)^\vee$ of $T_N$ can be interpreted as the moduli space of flat $U(1)$-connections on the trivial line bundle $T_N \times \mathbb{C} \to T_N$. In more explicit terms, a point $y = (y_1, \ldots, y_n) \in M_\mathbb{R} \cong \mathbb{R}^n$ corresponds to the flat $U(1)$-connection $\nabla_y = d + \frac{\sqrt{-1}}{2}\sum_{j=1}^n y_j du_j$ on the trivial line bundle $T_N \times \mathbb{C} \to T_N$. The *holonomy* of this connection is given, in our convention, by the map

$$\mathrm{hol}_{\nabla_y} : N \to U(1),\ v \mapsto e^{-\sqrt{-1}\langle y,v\rangle}.$$

$\nabla_y$ is gauge equivalent to the trivial connection $d$ if and only if $(y_1, \ldots, y_n) \in (2\pi\mathbb{Z})^n = M$. So, in the following, we will regard $(y_1, \ldots, y_n) \in T_M \cong \mathbb{R}^n/(2\pi\mathbb{Z})^n$. Moreover, this construction gives all flat $U(1)$-connections on $T_N \times \mathbb{C} \to T_N$ up to unitary gauge transformations. The universal $U(1)$-bundle $\mathcal{P}$, i.e. the Poincaré line bundle, is the trivial line bundle over the product $T_N \times T_M$ equipped with the $U(1)$-connection $d + \frac{\sqrt{-1}}{2}\sum_{j=1}^n (y_j du_j - u_j dy_j)$, where $u_1, \ldots, u_n \in \mathbb{R}/2\pi\mathbb{Z}$ are the coordinates on $T_N \cong \mathbb{R}^d/(2\pi\mathbb{Z})^d$. The curvature of this connection is given by the two-form

$$F = \sqrt{-1}\sum_{j=1}^n dy_j \wedge du_j \in \Omega^2(T_N \times T_M).$$

From this perspective, the SYZ mirror manifold $Y$ is the moduli space of pairs $(L_x, \nabla_y)$, where $L_x$ ($x \in P$) is a Lagrangian torus fiber of $\mu : X \to P$ and $\nabla_y$ is a flat $U(1)$-connection on the trivial line bundle $L_x \times \mathbb{C} \to L_x$. The construction of the mirror manifold in this way is originally advocated in the SYZ Conjecture [43] (cf. Hitchin [26] and Sections 2 and 4 in Auroux [6]).

Now recall that we have the dual torus bundles $\mu : X \to P$ and $\nu : Y \to P$. Consider their fiber product $X \times_P Y = P \times \sqrt{-1}(T_N \times T_M)$.

$$\begin{array}{ccc} X \times_P Y & \xrightarrow{\pi_Y} & Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \nu} \\ X & \xrightarrow{\mu} & P \end{array}$$

By abuse of notations, we still use $F$ to denote the fiberwise universal curvature two-form $\sqrt{-1}\sum_{j=1}^n dy_j \wedge du_j \in \Omega^2(X \times_P Y)$.

**Definition 3.2.** *The semi-flat SYZ mirror transformation* $\mathcal{F}^{sf} : \Omega^*(X) \to \Omega^*(Y)$ *is defined by*

$$\begin{aligned} \mathcal{F}^{sf}(\alpha) &= (-2\pi\sqrt{-1})^{-n}\pi_{Y,*}(\pi_X^*(\alpha) \wedge e^{\sqrt{-1}F}) \\ &= (-2\pi\sqrt{-1})^{-n}\int_{T_N} \pi_X^*(\alpha) \wedge e^{\sqrt{-1}F}, \end{aligned}$$

*where* $\pi_X : X \times_P Y \to X$ *and* $\pi_Y : X \times_P Y \to Y$ *are the two natural projections.*

The key point is that, the semi-flat SYZ mirror transformation $\mathcal{F}^{\mathrm{sf}}$ transforms the (exponential of $\sqrt{-1}$ times the) symplectic structure $\omega_X = \sum_{j=1}^n dx_j \wedge du_j$ on $X$ to the holomorphic $n$-form $\Omega_Y = \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}$ on $Y$, where $z_j = \exp(-x_j - \sqrt{-1}y_j)$, $j = 1, \ldots, n$. This is probably well-known and implicitly contained in the literature, but we include a proof here because we cannot find a suitable reference.

**Proposition 3.2.** *We have*

$$\mathcal{F}^{sf}(e^{\sqrt{-1}\omega_X}) = \Omega_Y.$$

*Moreover, if we define the inverse SYZ transformation* $(\mathcal{F}^{sf})^{-1} : \Omega^*(Y) \to \Omega^*(X)$ *by*

$$\begin{aligned} (\mathcal{F}^{sf})^{-1}(\alpha) &= (-2\pi\sqrt{-1})^{-n}\pi_{X,*}(\pi_Y^*(\alpha) \wedge e^{-\sqrt{-1}F}) \\ &= (-2\pi\sqrt{-1})^{-n}\int_{T_M} \pi_Y^*(\alpha) \wedge e^{-\sqrt{-1}F}, \end{aligned}$$

*then we also have*

$$(\mathcal{F}^{sf})^{-1}(\Omega_Y) = e^{\sqrt{-1}\omega_X}.$$

*Proof.* The proof is by straightforward computations.

$$\begin{aligned} \mathcal{F}^{\mathrm{sf}}(e^{\sqrt{-1}\omega_X}) &= (-2\pi\sqrt{-1})^{-n}\int_{T_N} \pi_X^*(e^{\sqrt{-1}\omega_X}) \wedge e^{\sqrt{-1}F} \\ &= (-2\pi\sqrt{-1})^{-n}\int_{T_N} e^{\sqrt{-1}\sum_{j=1}^n (dx_j + \sqrt{-1}dy_j)\wedge du_j} \\ &= (-2\pi\sqrt{-1})^{-n}\int_{T_N} \bigwedge_{j=1}^n \left(1 + \sqrt{-1}(dx_j + \sqrt{-1}dy_j) \wedge du_j\right) \\ &= (2\pi)^{-n}\int_{T_N} \left(\bigwedge_{j=1}^n (-dx_j - \sqrt{-1}dy_j)\right) \wedge du_1 \wedge \ldots \wedge du_n \\ &= \Omega_Y, \end{aligned}$$

where we have $\int_{T_N} du_1 \wedge \ldots \wedge du_n = (2\pi)^n$ for the last equality. On the other hand,

$$\begin{aligned} (\mathcal{F}^{\mathrm{sf}})^{-1}(\Omega_Y) &= (-2\pi\sqrt{-1})^{-n}\int_{T_M} \pi_Y^*(\Omega_Y) \wedge e^{-\sqrt{-1}F} \\ &= (-2\pi\sqrt{-1})^{-n}\int_{T_M} \left(\bigwedge_{j=1}^n (-dx_j - \sqrt{-1}dy_j)\right) \wedge e^{\sum_{j=1}^n dy_j\wedge du_j} \\ &= (2\pi\sqrt{-1})^{-n}\int_{T_M} \bigwedge_{j=1}^n \left((dx_j + \sqrt{-1}dy_j) \wedge e^{dy_j\wedge du_j}\right) \\ &= (2\pi\sqrt{-1})^{-n}\int_{T_M} \bigwedge_{j=1}^n \left(dx_j + \sqrt{-1}dy_j - dx_j \wedge du_j \wedge dy_j\right) \\ &= (2\pi)^{-n}\int_{T_M} \bigwedge_{j=1}^n \left(1 + \sqrt{-1}dx_j \wedge du_j\right) \wedge dy_j \\ &= (2\pi)^{-n}\int_{T_M} \bigwedge_{j=1}^n \left(e^{\sqrt{-1}dx_j\wedge du_j} \wedge dy_j\right) \end{aligned}$$

$$\begin{aligned} &= (2\pi)^{-n} \int_{T_M} e^{\sqrt{-1}\sum_{j=1}^n dx_j \wedge du_j} \wedge dy_1 \wedge \ldots \wedge dy_n \\ &= e^{\sqrt{-1}\omega_X}, \end{aligned}$$

where we again have $\int_{T_M} dy_1 \wedge \ldots \wedge dy_n = (2\pi)^n$ in the last step. □

One can also apply the semi-flat SYZ mirror transformations to other geometric structures and objects. For details, see Leung [30].

The semi-flat SYZ mirror transformation $\mathcal{F}^{\mathrm{sf}}$ can transform the symplectic structure $\omega_X$ on $X$ to the holomorphic $n$-form $\Omega_Y$ on $Y$. However, as we mentioned in the introduction, we are not going to obtain the superpotential $W : Y \to \mathbb{C}$ in this way because we have ignored the toric boundary divisor $\bar{X} \setminus X = D_\infty = \bigcup_{i=1}^d D_i$. Indeed, it is the toric boundary divisor $D_\infty$ which gives rise to the quantum corrections in the A-model of $\bar{X}$. More precisely, these quantum corrections are due to the existence of holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers which have intersections with the divisor $D_\infty$. To restore this information, our way out is to look at the (trivial) $\mathbb{Z}^n$-cover

$$\pi : LX = X \times N \to X.$$

Recall that we equip $LX$ with the symplectic structure $\pi^*(\omega_X)$; we will confuse the notations and use $\omega_X$ to denote either the symplectic structure on $X$ or that on $LX$. We will further abuse the notations by using $\mu$ to denote the fibration

$$\mu : LX \to P,$$

which is the composition of the map $\pi : LX \to X$ with $\mu : X \to P$.

We are now ready to define the SYZ mirror transformation $\mathcal{F}$ for the toric Fano manifold $\bar{X}$. It will be constructed as *a combination of the semi-flat SYZ transformation $\mathcal{F}^{sf}$ and taking fiberwise Fourier series*.

Analog to the semi-flat case, consider the fiber product

$$LX \times_P Y = P \times N \times \sqrt{-1}(T_N \times T_M)$$

of the maps $\mu : LX \to P$ and $\nu : Y \to P$.

$$\begin{array}{ccc} LX \times_P Y & \xrightarrow{\pi_Y} & Y \\ \big\downarrow \pi_{LX} & & \big\downarrow \nu \\ LX & \xrightarrow{\mu} & P \end{array}$$

Note that we have a covering map $LX \times_P Y \to X \times_P Y$. Pulling back $F \in \Omega^2(X \times_P Y)$ to $LX \times_P Y$ by this covering map, we get the fiberwise universal curvature two-form

$$F = \sqrt{-1} \sum_{j=1}^n dy_j \wedge du_j \in \Omega^2(LX \times_P Y).$$

We further define the *holonomy function* hol : $LX \times_P Y \to U(1)$ as follows. For $(p, v) \in LX$ and $z = (z_1, \ldots, z_n) \in Y$ such that $\mu(p) = \nu(z) =: x \in P$, we let

$x = (x_1, \ldots, x_n)$, and write $z_j = \exp(-x_j - \sqrt{-1}y_j)$, so that $y = (y_1, \ldots, y_n) \in (L_x)^\vee := \nu^{-1}(x) \subset Y$. Then we set

$$\mathrm{hol}(p, v, z) := \mathrm{hol}_{\nabla_y}(v) = e^{-\sqrt{-1}\langle y, v\rangle},$$

where $\nabla_y$ is the flat $U(1)$-connection on the trivial line bundle $L_x \times \mathbb{C} \to L_x$ over $L_x := \mu^{-1}(x)$ corresponding to the point $y \in (L_x)^\vee$.

**Definition 3.3.** *The SYZ mirror transformation $\mathcal{F} : \Omega^*(LX) \to \Omega^*(Y)$ for the toric Fano manifold $\bar{X}$ is defined by*

$$\begin{aligned} \mathcal{F}(\alpha) &= (-2\pi\sqrt{-1})^{-n}\pi_{Y,*}(\pi_{LX}^*(\alpha) \wedge e^{\sqrt{-1}F} hol) \\ &= (-2\pi\sqrt{-1})^{-n}\int_{N\times T_N} \pi_{LX}^*(\alpha) \wedge e^{\sqrt{-1}F} hol, \end{aligned}$$

*where $\pi_{LX} : LX \times_P Y \to LX$ and $\pi_Y : LX \times_P Y \to Y$ are the two natural projections.*

Before stating the basic properties of $\mathcal{F}$, we introduce the class of functions on $LX$ relevant to our applications.

**Definition 3.4.** *A $T_N$-invariant function $f : LX \to \mathbb{C}$ is said to be admissible if for any $(p, v) \in LX = X \times N$,*

$$f(p, v) = f_v e^{-\langle x, v\rangle},$$

*where $x = \mu(p) \in P$ and $f_v \in \mathbb{C}$ is a constant, and the fiberwise Fourier series*

$$\widehat{f} := \sum_{v\in N} f_v e^{-\langle x,v\rangle} hol_{\nabla_y}(v) = \sum_{v\in N} f_v z^v,$$

*where $z^v = \exp(\langle -x - \sqrt{-1}y, v\rangle)$, is convergent and analytic, as a function on $Y$. We denote by $\mathcal{A}(LX) \subset C^\infty(LX)$ set of all admissible functions on $LX$.*

Examples of admissible functions on $LX$ include those $T_N$-invariant functions which are not identically zero on $X \times \{v\} \subset LX$ for only finitely many $v \in N$. In particular, the functions $\Psi_1, \ldots, \Psi_d$ are all in $\mathcal{A}(LX)$. We will see (in the proof of Theorem 3.2) shortly that $\Phi_q$ is also admissible.

Now, for functions $f, g \in \mathcal{A}(LX)$, we define their *convolution product* $f \star g : LX \to \mathbb{C}$, as before, by

$$(f \star g)(p, v) = \sum_{v_1, v_2 \in N,\ v_1+v_2=v} f(p, v_1) g(p, v_2).$$

That the right-hand-side is convergent can be seen as follows. By definition, $f, g \in \mathcal{A}(LX)$ implies that for any $p \in X$ and any $v_1, v_2 \in N$,

$$f(p, v_1) = f_{v_1} e^{-\langle x, v_1\rangle},\ g(p, v_2) = g_{v_2} e^{-\langle x, v_2\rangle},$$

where $x = \mu(p)$ and $f_{v_1}, g_{v_2} \in \mathbb{C}$ are constants; also, the series $\widehat{f} = \sum_{v_1\in N} f_{v_1} z^{v_1}$ and $\widehat{g} = \sum_{v_2\in N} g_{v_2} z^{v_2}$ are convergent and analytic. Then their product, given by

$$\widehat{f} \cdot \widehat{g} = \left(\sum_{v_1\in N} f_{v_1} z^{v_1}\right)\left(\sum_{v_2\in N} g_{v_2} z^{v_2}\right) = \sum_{v\in N}\left(\sum_{\substack{v_1,v_2\in N,\\ v_1+v_2=v}} f_{v_1} g_{v_2}\right) z^v,$$

is also analytic. This shows that the convolution product $f \star g$ is well defined and gives another admissible function on $LX$. Hence, the $\mathbb{C}$-vector space $\mathcal{A}(LX)$, together with the convolution product $\star$, forms a $\mathbb{C}$-algebra.

Let $\mathcal{O}(Y)$ be the $\mathbb{C}$-algebra of holomorphic functions on $Y$. Recall that $Y = TP/M = P \times \sqrt{-1}T_M$. For $\phi \in \mathcal{O}(Y)$, the restriction of $\phi$ to a fiber $(L_x)^\vee = \nu^{-1}(x) \cong T_M$ gives a $C^\infty$ function $\phi_x : T_M \to \mathbb{C}$ on the torus $T_M$. For $v \in N$, the $v$-th Fourier coefficient of $\phi_x$ is given by

$$\widehat{\phi}_x(v) = \int_{T_M} \phi_x(y) e^{\sqrt{-1}\langle y,v\rangle} dy_1 \wedge \ldots \wedge dy_n.$$

Then, we define a function $\widehat{\phi} : LX \to \mathbb{C}$ on $LX$ by

$$\widehat{\phi}(p,v) = \widehat{\phi}_x(v),$$

where $x = \mu(p) \in P$. $\widehat{\phi}$ is clearly admissible. We call the process, $\phi \in \mathcal{O}(Y) \mapsto \widehat{\phi} \in \mathcal{A}(LX)$, taking *fiberwise Fourier coefficients*. The following lemma follows from the standard theory of Fourier analysis on tori (see, for example, Edwards [13]).

**Lemma 3.1.** *Taking fiberwise Fourier series, i.e. the map*

$$\mathcal{A}(LX) \to \mathcal{O}(Y), \quad f \mapsto \widehat{f}$$

*is an isomorphism of $\mathbb{C}$-algebras, where we equip $\mathcal{A}(LX)$ with the convolution product and $\mathcal{O}(Y)$ with the ordinary product of functions. The inverse is given by taking fiberwise Fourier coefficients. In particular, $\widehat{\widehat{f}} = f$ for any $f \in \mathcal{A}(LX)$.*

The basic properties of the SYZ mirror transformation $\mathcal{F}$ are summarized in the following theorem.

**Theorem 3.1.** *Let $\mathcal{A}(LX)e^{\sqrt{-1}\omega_X} := \{fe^{\sqrt{-1}\omega_X} : f \in \mathcal{A}(LX)\} \subset \Omega^*(LX)$ and $\mathcal{O}(Y)\Omega_Y := \{\phi\Omega_Y : \phi \in \mathcal{O}(Y)\} \subset \Omega^*(Y)$.*

*(i) For any admissible function $f \in \mathcal{A}(LX)$,*

$$\mathcal{F}(fe^{\sqrt{-1}\omega_X}) = \widehat{f}\Omega_Y \in \mathcal{O}(Y)\Omega_Y.$$

*(ii) If we define the inverse SYZ mirror transformation $\mathcal{F}^{-1} : \Omega^*(Y) \to \Omega^*(LX)$ by*

$$\begin{aligned}\mathcal{F}^{-1}(\alpha) &= (-2\pi\sqrt{-1})^{-n}\pi_{LX,*}(\pi_Y^*(\alpha) \wedge e^{-\sqrt{-1}F} hol^{-1}) \\ &= (-2\pi\sqrt{-1})^{-n}\int_{T_M} \pi_Y^*(\alpha) \wedge e^{-\sqrt{-1}F} hol^{-1},\end{aligned}$$

*where $hol^{-1} : LX \times_P Y \to \mathbb{C}$ is the function defined by $hol^{-1}(p,v,z) = 1/hol(p,v,z) = e^{\sqrt{-1}\langle y,v\rangle}$, for any $(p,v,z) \in L_X \times_P Y$, then*

$$\mathcal{F}^{-1}(\phi\Omega_Y) = \widehat{\phi}e^{\sqrt{-1}\omega_X} \in \mathcal{A}(LX)e^{\sqrt{-1}\omega_X},$$

*for any $\phi \in \mathcal{O}(Y)$.*

*(iii) The restriction map $\mathcal{F} : \mathcal{A}(LX)e^{\sqrt{-1}\omega_X} \to \mathcal{O}(Y)\Omega_Y$ is a bijection with inverse $\mathcal{F}^{-1} : \mathcal{O}(Y)\Omega_Y \to \mathcal{A}(LX)e^{\sqrt{-1}\omega_X}$, i.e. we have*

$$\mathcal{F}^{-1} \circ \mathcal{F} = Id_{\mathcal{A}(LX)e^{\sqrt{-1}\omega_X}}, \ \mathcal{F} \circ \mathcal{F}^{-1} = Id_{\mathcal{O}(Y)\Omega_Y}.$$

*This shows that the SYZ mirror transformation $\mathcal{F}$ has the inversion property.*

*Proof.* Let $f \in \mathcal{A}(LX)$. Then, for any $v \in N$, $f(p,v) = f_v e^{-\langle x,v\rangle}$ for some constant $f_v \in \mathbb{C}$. By observing that both functions $\pi_{LX}^*(f)$ and hol are $T_N$-invariant functions on $LX \times_P Y$, we have

$$\begin{aligned}
\mathcal{F}(fe^{\sqrt{-1}\omega_X}) &= (-2\pi\sqrt{-1})^{-n}\int_{N\times T_N}\pi_{LX}^*(fe^{\sqrt{-1}\omega_X})\wedge e^{\sqrt{-1}F}\mathrm{hol}\\
&= (-2\pi\sqrt{-1})^{-n}\sum_{v\in N}\pi_{LX}^*(f)\cdot\mathrm{hol}\int_{T_N}\pi_{LX}^*(e^{\sqrt{-1}\omega_X})\wedge e^{\sqrt{-1}F}\\
&= (-2\pi\sqrt{-1})^{-n}\left(\sum_{v\in N}\pi_{LX}^*(f)\cdot\mathrm{hol}\right)\left(\int_{T_N}\pi_X^*(e^{\sqrt{-1}\omega_X})\wedge e^{\sqrt{-1}F}\right).
\end{aligned}$$

The last equality is due to the fact that the forms $\pi_{LX}^*(e^{\sqrt{-1}\omega_X}) = \pi_X^*(e^{\sqrt{-1}\omega_X})$ and $e^F$ are independent of $v \in N$. By Proposition 3.2, the second factor is given by

$$\int_{T_N}\pi_X^*(e^{\sqrt{-1}\omega_X})\wedge e^{\sqrt{-1}F} = (-2\pi\sqrt{-1})^n\mathcal{F}^{\mathrm{sf}}(e^{\sqrt{-1}\omega_X}) = (-2\pi\sqrt{-1})^n\Omega_Y,$$

while the first factor is the function on $Y$ given, for $x = (x_1,\ldots,x_n) \in P$ and $y = (y_1,\ldots,y_n) \in T_M$, by

$$\begin{aligned}
\left(\sum_{v\in N}\pi_{LX}^*(f)\cdot\mathrm{hol}\right)(x,y) &= \sum_{v\in N} f_v e^{-\langle x,v\rangle}e^{-\sqrt{-1}\langle y,v\rangle}\\
&= \sum_{v\in N} f_v z^v\\
&= \widehat{f}(z),
\end{aligned}$$

where $z = (z_1,\ldots,z_n) = (\exp(-x_1-\sqrt{-1}y_1),\ldots,\exp(-x_n-\sqrt{-1}y_n)) \in Y$. Hence $\mathcal{F}(fe^{\sqrt{-1}\omega_X}) = \widehat{f}\Omega_Y \in \mathcal{O}(Y)\Omega_Y$. This proves (i).

For (ii), expand $\phi \in \mathcal{O}(Y)$ into a fiberwise Fourier series

$$\phi(z) = \sum_{w\in N}\widehat{\phi}_x(w)e^{-\sqrt{-1}\langle y,w\rangle},$$

where $x, y, z$ are as before. Then

$$\begin{aligned}
\mathcal{F}^{-1}(\phi\Omega_Y) &= (-2\pi\sqrt{-1})^{-n}\int_{T_M}\pi_Y^*(\phi\Omega_Y)\wedge e^{-\sqrt{-1}F}\mathrm{hol}^{-1}\\
&= (-2\pi\sqrt{-1})^{-n}\sum_{w\in N}\left(\widehat{\phi}_x(w)\int_{T_M}e^{\sqrt{-1}\langle y,v-w\rangle}\pi_Y^*(\Omega_Y)\wedge e^{-\sqrt{-1}F}\right).
\end{aligned}$$

Here comes the key observation: If $v - w \neq 0 \in N$, then, using (the proof of) the second part of Proposition 3.2, we have

$$\begin{aligned}
&\int_{T_M}e^{\sqrt{-1}\langle y,v-w\rangle}\pi_Y^*(\Omega_Y)\wedge e^{-\sqrt{-1}F}\\
=\ &\int_{T_M}e^{\sqrt{-1}\langle y,v-w\rangle}\left(\bigwedge_{j=1}^n(-dx_j-\sqrt{-1}dy_j)\right)\wedge e^{\sum_{j=1}^n dy_j\wedge du_j}\\
=\ &(-\sqrt{-1})^n e^{\sqrt{-1}\omega_X}\int_{T_M}e^{\sqrt{-1}\langle y,v-w\rangle}dy_1\wedge\ldots\wedge dy_n\\
=\ &0.
\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{F}^{-1}(\phi\Omega_Y) &= (-2\pi\sqrt{-1})^{-n}\widehat{\phi}_x(v)\int_{T_M}\pi_Y^*(\Omega_Y)\wedge e^{-\sqrt{-1}F}\\ &= \widehat{\phi}\Big((\mathcal{F}^{\mathrm{sf}})^{-1}(\Omega_Y)\Big)=\widehat{\phi}e^{\omega_X}\in\mathcal{A}(LX)e^{\omega_X},\end{aligned}$$

again by Proposition 3.2.

(iii) follows from (i), (ii) and Lemma 3.1. □

We will, again by abuse of notations, also use $\mathcal{F}:\mathcal{A}(LX)\to\mathcal{O}(Y)$ to denote the process of taking fiberwise Fourier series: $\mathcal{F}(f):=\widehat{f}$ for $f\in\mathcal{A}(LX)$. Similarly, we use $\mathcal{F}^{-1}:\mathcal{O}(Y)\to\mathcal{A}(LX)$ to denote the process of taking fiberwise Fourier coefficients: $\mathcal{F}^{-1}(\phi):=\widehat{\phi}$ for $\phi\in\mathcal{O}(Y)$. To which meanings of the symbols $\mathcal{F}$ and $\mathcal{F}^{-1}$ are we referring will be clear from the context.

We can now prove the first part of Theorem 1.1, as a corollary of Theorem 3.1.

**Theorem 3.2** (=part 1. of Theorem 1.1)**.** *The SYZ mirror transformation of the function $\Phi_q\in C^\infty(LX)$, defined in terms of the counting of Maslov index two holomorphic discs in $\bar{X}$ with boundary in Lagrangian torus fibers, is the exponential of the superpotential $W$ on the mirror manifold $Y$, i.e.*

$$\mathcal{F}(\Phi_q)=e^W.$$

*Conversely, we have*

$$\mathcal{F}^{-1}(e^W)=\Phi_q.$$

*Furthermore, we can incorporate the symplectic structure $\omega_X=\omega_{\bar{X}}|_X$ on $X$ to give the holomorphic volume form on the Landau-Ginzburg model $(Y,W)$ through the SYZ mirror transformation $\mathcal{F}$, and vice versa, in the following sense:*

$$\mathcal{F}(\Phi_q e^{\sqrt{-1}\omega_X})=e^W\Omega_Y,\ \mathcal{F}^{-1}(e^W\Omega_Y)=\Phi_q e^{\sqrt{-1}\omega_X}.$$

*Proof.* By Theorem 3.1, we only need to show that $\Phi_q\in C^\infty(LX)$ is admissible and $\mathcal{F}(\Phi_q)=\widehat{\Phi}_q=e^W\in\mathcal{O}(Y)$. Recall that, for $(p,v)\in LX=X\times N$ and $x=\mu(p)\in P$,

$$\Phi_q(p,v)=\sum_{\beta\in\pi_2^+(\bar{X},L_x),\ \partial\beta=v}\frac{1}{w(\beta)}e^{-\frac{1}{2\pi}\int_\beta\omega_{\bar{X}}}.$$

For $\beta\in\pi_2^+(\bar{X},L_x)$ with $\partial\beta=v$, by the symplectic area formula (2.2) of Cho-Oh, we have $\int_\beta\omega_{\bar{X}}=2\pi\langle x,v\rangle+\text{const}$. So $\Phi_q(p,v)$ is of the form $\text{const}\cdot e^{-\langle x,v\rangle}$. Now,

$$\begin{aligned}\sum_{v\in N}\Phi_q(p,v)\mathrm{hol}_{\nabla_y}(v) &= \sum_{v\in N}\left(\sum_{\beta\in\pi_2^+(\bar{X},L_x),\ \partial\beta=v}\frac{1}{w(\beta)}e^{-\frac{1}{2\pi}\int_\beta\omega_{\bar{X}}}\right)e^{-\sqrt{-1}\langle y,v\rangle}\\ &= \sum_{k_1,\ldots,k_d\in\mathbb{Z}_{\geq0}}\frac{1}{k_1!\ldots k_d!}e^{-\sum_{i=1}^d k_i(\langle x,v_i\rangle-\lambda_i)}e^{-\sum_{i=1}^d k_i\sqrt{-1}\langle y,v_i\rangle}\\ &= \prod_{i=1}^d\left(\sum_{k_i=0}^\infty\frac{1}{k_i!}\left(e^{\lambda_i-\langle x+\sqrt{-1}y,v_i\rangle}\right)^{k_i}\right)\\ &= \prod_{i=1}^d\exp(e^{\lambda_i}z^{v_i})=e^W.\end{aligned}$$

This shows that $\Phi_q$ is admissible and $\widehat{\Phi}_q = e^W$. $\square$

The form $\Phi_q e^{\sqrt{-1}\omega_X} \in \Omega^*(LX)$ can be viewed as *the symplectic structure modified by quantum corrections from Maslov index two holomorphic discs in $\bar{X}$ with boundaries on Lagrangian torus fibers*. That we call $e^W\Omega_Y$ the holomorphic volume form of the Landau-Ginzburg model $(Y, W)$ can be justified in several ways. For instance, in the theory of singularities, one studies the complex oscillating integrals

$$I = \int_\Gamma e^{\frac{1}{\hbar}W}\Omega_Y,$$

where $\Gamma$ is some real $n$-dimensional cycle in $Y$ constructed by the Morse theory of the function Re($W$). These integrals are reminiscent of the periods of holomorphic volume forms on Calabi-Yau manifolds, and they satisfy certain Picard-Fuchs equations (see, for example, Givental [21]). Hence, one may think of $e^W\Omega_Y$ as playing the same role as the holomorphic volume form on a Calabi-Yau manifold.

3.3. **Quantum cohomology vs. Jacobian ring.** The purpose of this subsection is to give a proof of the second part of Theorem 1.1. Before that, let us recall the definition of the Jacobian ring $Jac(W)$. Recall that the SYZ mirror manifold $Y$ is given by the bounded domain

$$Y = \{(z_1, \ldots, z_n) \in (\mathbb{C}^*)^n : |e^{\lambda_i} z^{v_i}| < 1,\ i = 1, \ldots, d\},$$

in $(\mathbb{C}^*)^n$, and the superpotential $W : Y \to \mathbb{C}$ is the Laurent polynomial

$$W = e^{\lambda_1} z^{v_1} + \ldots + e^{\lambda_d} z^{v_d},$$

where, as before, $z^v$ denotes the monomial $z_1^{v^1} \ldots z_n^{v^n}$ if $v = (v^1, \ldots, v^n) \in N = \mathbb{Z}^n$. Let $\mathbb{C}[Y] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be the $\mathbb{C}$-algebra of Laurent polynomials restricted to $Y$. Then the Jacobian ring $Jac(W)$ of $W$ is defined as the quotient of $\mathbb{C}[Y]$ by the ideal generated by the logarithmic derivatives of $W$:

$$\begin{aligned} Jac(W) &= \mathbb{C}[Y] \Big/ \Big\langle z_j \frac{\partial W}{\partial z_j} : j = 1, \ldots, n \Big\rangle \\ &= \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \Big/ \Big\langle z_j \frac{\partial W}{\partial z_j} : j = 1, \ldots, n \Big\rangle. \end{aligned}$$

The second part of Theorem 1.1 is now an almost immediate corollary of Proposition 2.2 and Theorem 3.3.

**Theorem 3.3.** *The SYZ mirror transformation $\mathcal{F}$ gives an isomorphism*

$$\mathcal{F} : \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L} \to Jac(W)$$

*of $\mathbb{C}$-algebras. Hence, $\mathcal{F}$ induces a natural isomorphism of $\mathbb{C}$-algebras between the small quantum cohomology ring of $\bar{X}$ and the Jacobian ring of $W$:*

$$\mathcal{F} : QH^*(\bar{X}) \xrightarrow{\cong} Jac(W),$$

*provided that $\bar{X}$ is a product of projective spaces.*

*Proof.* The functions $\Psi_1, \Psi_1^{-1}, \ldots, \Psi_n, \Psi_n^{-1}$ are all admissible, so $\mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]$ is a subalgebra of $\mathcal{A}(LX)$. It is easy to see that, for $i = 1, \ldots, d$, the SYZ mirror transformation $\mathcal{F}(\Psi_i) = \widehat{\Psi}_i$ of $\Psi_i$ is nothing but the monomial $e^{\lambda_i} z^{v_i}$. By our choice of the polytope $\bar{P} \subset M_{\mathbb{R}}$, $v_1 = e_1, \ldots, v_n = e_n$ is the standard basis of $N = \mathbb{Z}^n$ and $\lambda_1 = \ldots = \lambda_n = 0$. Hence,

$$\mathcal{F}(\Psi_i) = z_i,$$

for $i = 1, \ldots, n$, and the induced map

$$\mathcal{F} : \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}] \to \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$$

is an isomorphism of $\mathbb{C}$-algebras. Now, notice that

$$z_j \frac{\partial W}{\partial z_j} = \sum_{i=1}^{d} z_j \frac{\partial}{\partial z_j}(e^{\lambda_i} z_1^{v_i^1} \ldots z_n^{v_i^n}) = \sum_{i=1}^{d} v_i^j e^{\lambda_i} z_1^{v_i^1} \ldots z_n^{v_i^n} = \sum_{i=1}^{d} v_i^j e^{\lambda_i} z^{v_i},$$

for $j = 1, \ldots, n$. The inverse SYZ transformation of $z_j \frac{\partial W}{\partial z_j}$ is thus given by

$$\mathcal{F}^{-1}(z_j \frac{\partial W}{\partial z_j}) = \widehat{\sum_{i=1}^{d} v_i^j e^{\lambda_i} z^{v_i}} = \sum_{i=1}^{d} v_i^j \Psi_i.$$

Thus,

$$\mathcal{F}^{-1}\Big(\Big\langle z_j \frac{\partial W}{\partial z_j} : j = 1, \ldots, n \Big\rangle\Big) = \mathcal{L},$$

is the ideal in $\mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]$ generated by linear equivalences. The result follows. □

## 4. Examples

In this section, we give some examples to illustrate our results.

**Example 1.** $\bar{X} = \mathbb{C}P^2$**.** In this case, $N = \mathbb{Z}^2$. The primitive generators of the 1-dimensional cones of the fan $\Sigma$ defining $\mathbb{C}P^2$ are given by $v_1 = (1,0), v_2 = (0,1), v_3 = (-1,-1) \in N$, and the polytope $\bar{P} \subset M_{\mathbb{R}} \cong \mathbb{R}^2$ we chose is defined by the inequalities

$$x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq t,$$

where $t > 0$. See Figure 4.1 below.

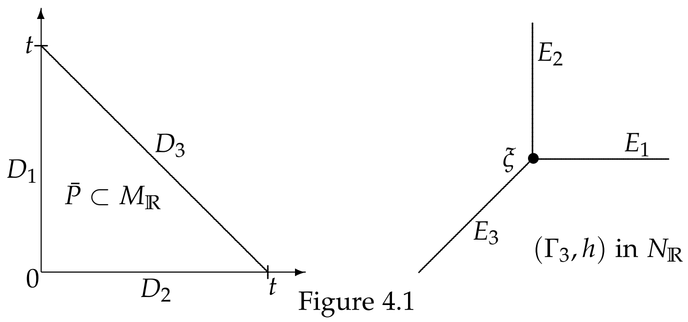


Figure 4.1

The mirror manifold $Y$ is given by

$$\begin{aligned} Y &= \{(Z_1, Z_2, Z_3) \in \mathbb{C}^3 : Z_1 Z_2 Z_3 = q, |Z_i| < 1,\ i = 1,2,3\} \\ &= \{(z_1, z_2) \in (\mathbb{C}^*)^2 : |z_1| < 1, |z_2| < 1, |\frac{q}{z_1 z_2}| < 1\}, \end{aligned}$$

where $q = e^{-t}$ is the Kähler parameter, and, the superpotential $W : Y \to \mathbb{C}$ can be written, in two ways, as

$$W = Z_1 + Z_2 + Z_3 = z_1 + z_2 + \frac{q}{z_1 z_2}.$$

In terms of the coordinates $Z_1, Z_2, Z_3$, the Jacobian ring $Jac(W)$ is given by

$$\begin{aligned} Jac(W) &= \mathbb{C}[Z_1, Z_2, Z_3]/\langle Z_1 - Z_3, Z_2 - Z_3, Z_1Z_2Z_3 - q\rangle \\ &\cong \mathbb{C}[Z]/\langle Z^3 - q\rangle. \end{aligned}$$

There are three toric prime divisors $D_1, D_2, D_3$, which are corresponding to the three admissible functions $\Psi_1, \Psi_2, \Psi_3 : LX \to \mathbb{R}$ defined by

$$\begin{aligned} \Psi_1(p,v) &= \begin{cases} e^{-x_1} & \text{if } v = (1,0) \\ 0 & \text{otherwise,} \end{cases} \\ \Psi_2(p,v) &= \begin{cases} e^{-x_2} & \text{if } v = (0,1) \\ 0 & \text{otherwise,} \end{cases} \\ \Psi_3(p,v) &= \begin{cases} e^{-(t-x_1-x_2)} & \text{if } v = (-1,-1) \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for $(p, v) \in LX$ and where $x = \mu(p) \in P$, respectively. The small quantum cohomology ring of $\mathbb{C}P^2$ has the following presentation:

$$\begin{aligned} QH^*(\mathbb{C}P^2) &= \mathbb{C}[D_1, D_2, D_3]/\langle D_1 - D_3, D_2 - D_3, D_1 * D_2 * D_3 - q\rangle \\ &\cong \mathbb{C}[H]/\langle H^3 - q\rangle, \end{aligned}$$

where $H \in H^2(\mathbb{C}P^2, \mathbb{C})$ is the hyperplane class. Quantum corrections appear only in one relation, namely,

$$D_1 * D_2 * D_3 = q.$$

Fix a point $p \in X$. Then the quantum correction is due to the unique holomorphic curve $\varphi : (\mathbb{C}P^1; x_1, x_2, x_3, x_4) \to \mathbb{C}P^2$ of degree 1 (i.e. a line) with 4 marked points such that $\varphi(x_4) = p$ and $\varphi(x_i) \in D_i$ for $i = 1, 2, 3$. The parameterized 3-marked, genus 0 tropical curve corresponding to this line is $(\Gamma_3; E_1, E_2, E_3; h)$, which is glued from three half lines emanating from the point $\xi = \mathrm{Log}(p) \in N_{\mathbb{R}}$ in the directions $v_1$, $v_2$ and $v_3$. See Figure 4.1 above. These half lines are the parameterized Maslov index two tropical discs $(\Gamma_1, h_i)$, where $h_i(V) = \xi$ an $h_i(E) = \xi + \mathbb{R}_{\geq 0} v_i$, for $i = 1, 2, 3$ (see Figure 2.3). They are corresponding to the Maslov index two holomorphic discs $\varphi_1, \varphi_2, \varphi_3 : (D^2, \partial D^2) \to (\mathbb{C}P^2, L_{\mu(p)})$ which pass through $p$ and intersect the corresponding toric divisors $D_1, D_2, D_3$ respectively.

**Example 2.** $\bar{X} = \mathbb{C}P^1 \times \mathbb{C}P^1$**.** The primitive generators of the 1-dimensional cones of the fan $\Sigma$ defining $\mathbb{C}P^1 \times \mathbb{C}P^1$ are given by $v_{1,1} = (1, 0), v_{2,1} = (-1, 0), v_{1,2} = (0, 1), v_{2,2} = (0, -1) \in N = \mathbb{Z}^2$. We choose the polytope $\bar{P} \subset M_{\mathbb{R}} = \mathbb{R}^2$ to be defined by the inequalities

$$0 \leq x_1 \leq t_1,\ 0 \leq x_2 \leq t_2$$

where $t_1, t_2 > 0$. See Figure 4.2 below.

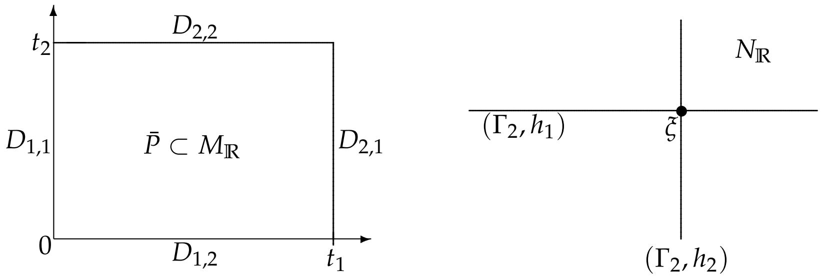


Figure 4.2

The mirror Landau-Ginzburg model $(Y, W)$ consists of

$$\begin{aligned} Y &= \{(Z_{1,1}, Z_{2,1}, Z_{1,2}, Z_{2,2}) \in \mathbb{C}^4 : Z_{1,1}Z_{2,1} = q_1, Z_{1,2}Z_{2,2} = q_2, |Z_{i,j}| < 1, \text{ all } i, j\} \\ &= \{(z_1, z_2) \in (\mathbb{C}^*)^2 : |z_1| < 1, |z_2| < 1, |\frac{q_1}{z_1}| < 1, |\frac{q_2}{z_2}| < 1\}, \end{aligned}$$

where $q_1 = e^{-t_1}$ and $q_2 = e^{-t_2}$ are the Kähler parameters, and

$$W = Z_{1,1} + Z_{2,1} + Z_{1,2} + Z_{2,2} = z_1 + \frac{q_1}{z_1} + z_2 + \frac{q_2}{z_2}.$$

The Jacobian ring $Jac(W)$ is given by

$$\begin{aligned} Jac(W) &= \frac{\mathbb{C}[Z_{1,1}, Z_{2,1}, Z_{1,2}, Z_{2,2}]}{\langle Z_{1,1} - Z_{2,1}, Z_{1,2} - Z_{2,2}, Z_{1,1}Z_{2,1} - q_1, Z_{1,2}Z_{2,2} - q_2\rangle} \\ &\cong \mathbb{C}[Z_1, Z_2]/\langle Z_1^2 - q_1, Z_2^2 - q_2\rangle. \end{aligned}$$

The four toric prime divisors $D_{1,1}, D_{2,1}, D_{1,2}, D_{2,2}$ correspond respectively to the four admissible functions $\Psi_{1,1}, \Psi_{2,1}, \Psi_{1,2}, \Psi_{2,2} : LX \to \mathbb{C}$ defined by

$$\begin{aligned} \Psi_{1,1}(p, v) &= \begin{cases} e^{-x_1} & \text{if } v = (1, 0) \\ 0 & \text{otherwise,} \end{cases} \\ \Psi_{2,1}(p, v) &= \begin{cases} e^{-(t_1 - x_1)} & \text{if } v = (0, -1) \\ 0 & \text{otherwise,} \end{cases} \\ \Psi_{1,2}(p, v) &= \begin{cases} e^{-x_2} & \text{if } v = (0, 1) \\ 0 & \text{otherwise,} \end{cases} \\ \Psi_{2,2}(p, v) &= \begin{cases} e^{-(t_2 - x_2)} & \text{if } v = (0, -1) \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for $(p, v) \in LX$ and where $x = \mu(p) \in P$. The small quantum cohomology ring of $\mathbb{C}P^1 \times \mathbb{C}P^1$ is given by

$$\begin{aligned} QH^*(\mathbb{C}P^1 \times \mathbb{C}P^1) &= \frac{\mathbb{C}[D_{1,1}, D_{2,1}, D_{1,2}, D_{2,2}]}{\langle D_{1,1} - D_{2,1}, D_{1,2} - D_{2,2}, D_{1,1} * D_{2,1} - q_1, D_{1,2} * D_{2,2} - q_2\rangle} \\ &\cong \mathbb{C}[H_1, H_2]/\langle H_1^2 - q_1, H_2^2 - q_2\rangle \end{aligned}$$

where $H_1, H_2 \in H^2(\mathbb{C}P^1 \times \mathbb{C}P^1)$ are the pullbacks of the hyperplane classes in the first and second factors respectively. Quantum corrections appear in two relations:

$$D_{1,1} * D_{2,1} = q_1 \text{ and } D_{1,2} * D_{2,2} = q_2.$$

Let us focus on the first one, as the other one is similar. For any $p \in X$, there are two Maslov index two holomorphic discs $\varphi_{1,1}, \varphi_{2,1} : (D^2, \partial D^2) \to (\mathbb{C}P^1 \times$

$\mathbb{C}P^1, L_{\mu(p)})$ intersecting the corresponding toric divisors. An interesting feature of this example is that, since the sum of the boundaries of the two holomorphic discs is zero as a *chain*, instead of as a class, in $L_{\mu(p)}$, they glue together *directly* to give the unique holomorphic curve $\varphi_1 : (\mathbb{C}P^1 : x_1, x_2, x_3) \to \mathbb{C}P^1 \times \mathbb{C}P^1$ of degree 1 with $\varphi_1(x_1) \in D_{1,1}$, $\varphi_1(x_2) \in D_{2,1}$ and $\varphi_1(x_3) = p$. So the relation $D_{1,1} * D_{2,1} = q_1$ is directly corresponding to $\Psi_{1,1} \star \Psi_{2,1} = q_1 \mathbb{1}$, without going through the corresponding relation in $QH^*_{trop}(\bar{X})$. In other words, we do not need to go to the tropical world to see the geometry of the isomorphism $QH^*(\mathbb{C}P^1 \times \mathbb{C}P^1) \cong \mathbb{C}[\Psi_{1,1}^{\pm 1}, \Psi_{1,2}^{\pm 1}]/\mathcal{L}$ (although in Figure 4.2 above, we have still drawn the tropical lines $h_1$ and $h_2$ passing through $\xi = \mathrm{Log}(p) \in N_{\mathbb{R}}$).

**Example 3. $\bar{X}$ is the toric blowup of $\mathbb{C}P^2$ at one point.** Let $\bar{P} \subset \mathbb{R}^2$ be the polytope defined by the inequalities

$$x_1 \geq 0,\ 0 \leq x_2 \leq t_2,\ x_1 + x_2 \leq t_1 + t_2,$$

where $t_1, t_2 > 0$.

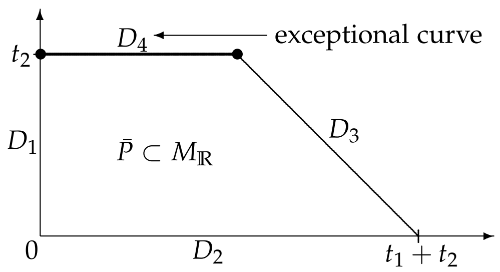


Figure 4.3

The toric Fano manifold $\bar{X}$ corresponding to this trapezoid (see Figure 4.3 above) is the blowup of $\mathbb{C}P^2$ at a $T_N$-fixed point. The primitive generators of the 1-dimensional cones of the fan $\Sigma$ defining $\bar{X}$ are given by $v_1 = (1,0), v_2 = (0,1), v_3 = (-1,-1), v_4 = (0,-1) \in N = \mathbb{Z}^2$. As in the previous examples, we have the mirror manifold

$$\begin{aligned} Y &= \{(Z_1, Z_2, Z_3, Z_4) \in \mathbb{C}^4 : Z_1 Z_3 = q_1 Z_4, Z_2 Z_4 = q_2, |Z_i| < 1 \text{for all } i\} \\ &= \{(z_1, z_2) \in (\mathbb{C}^*)^2 : |z_1| < 1, |z_2| < 1, |\frac{q_1 q_2}{z_1 z_2}| < 1, |\frac{q_2}{z_2}| < 1\}, \end{aligned}$$

and the superpotential

$$W = Z_1 + Z_2 + Z_3 + Z_4 = z_1 + z_2 + \frac{q_1 q_2}{z_1 z_2} + \frac{q_2}{z_2},$$

where $q_1 = e^{-t_1}, q_2 = e^{-t_2}$. The Jacobian ring of $W$ is

$$Jac(W) = \frac{\mathbb{C}[Z_1, Z_2, Z_3, Z_4]}{\langle Z_1 - Z_3 - Z_4, Z_2 - Z_4, Z_1 Z_3 - q_1 Z_4, Z_2 Z_4 - q_2 \rangle}$$

and the small quantum cohomology ring of $\bar{X}$ is given by

$$QH^*(\bar{X}) = \frac{\mathbb{C}[D_1, D_2, D_3, D_4]}{\langle D_1 - D_3 - D_4, D_2 - D_4, D_1 * D_3 - q_1 D_4, D_2 * D_4 - q_2 \rangle}.$$

Obviously, we have an isomorphism $QH^*(\bar{X}) \cong Jac(W)$ and, the isomorphism

$$QH^*(\bar{X}) \cong \mathbb{C}[\Psi_1^{\pm 1}, \Psi_2^{\pm 1}]/\mathcal{L}$$

in Proposition 2.2 still holds, as we have said in Remark 2.3. However, the geometric picture that we have derived in Subsection 2.2 using tropical geometry breaks down. This is because there is a *rigid* holomorphic curve contained in the toric boundary $D_\infty$ which *does* contribute to $QH^*(\bar{X})$. Namely, the quantum relation

$$D_1 * D_3 = q_1 D_4$$

is due to the holomorphic curve $\varphi : \mathbb{C}P^1 \to \bar{X}$ such that $\varphi(\mathbb{C}P^1) \subset D_4$. This curve is exceptional since $D_4^2 = -1$, and thus cannot be deformed to a curve outside the toric boundary. See Figure 4.3 above. Hence, it is *not* corresponding to any tropical curve in $N_\mathbb{R}$. This means that tropical geometry cannot "see" the curve $\varphi$, and it is not clear how one could define the tropical analog of the small quantum cohomology ring in this case.

## 5. Discussions

In this final section, we speculate the possible generalizations of the results of this paper. The discussion will be rather informal.

The proofs of the results in this paper rely heavily on the classification of holomorphic discs in a toric Fano manifold $\bar{X}$ with boundary in Lagrangian torus fibers, and on the explicit nature of toric varieties. Nevertheless, it is still possible to generalize these results, in particular, the construction of SYZ mirror transformations, to non-toric situations. For example, one may consider a *complex flag manifold* $\bar{X}$, where the Gelfand-Cetlin integrable system provide a natural Lagrangian torus fibration structure on $\bar{X}$ (see, for example, Guillemin-Sternberg [25]). The base of this fibration is again an affine manifold with boundary but without singularities. In fact, there is a *toric degeneration* of the complex flag manifold $\bar{X}$ to a toric variety, and the base is nothing but the polytope associated to that toric variety. Furthermore, the classification of holomorphic discs in a complex flag manifold $\bar{X}$ with boundary in Lagrangian torus fibers was recently done by Nishinou-Nohara-Ueda [37], and, at least for the full flag manifolds, there is an isomorphism between the small quantum cohomology ring and the Jacobian ring of the mirror superpotential (cf. Corollary 12.4 in [37]). Hence, one can try to construct the SYZ mirror transformations for a complex flag manifold $\bar{X}$ and prove results like Proposition 1.1 and Theorem 1.1 as in the toric Fano case.

Certainly, the more important (and more ambitious) task is to generalize the constructions of SYZ mirror transformations to the most general situations, where the bases of Lagrangian torus fibrations are affine manifolds with both boundary and singularities. To do this, the first step is to make the construction of the SYZ mirror transformations become a local one. One possible way is the following: Suppose that we have an $n$-dimensional compact Kähler manifold $\bar{X}$, together with an anticanonical divisor $D$. Assume that there is a Lagrangian torus fibration $\mu : \bar{X} \to \bar{B}$, where $\bar{B}$ is a real $n$-dimensional (possibly) singular affine manifold with boundary $\partial\bar{B}$. We should also have $\mu^{-1}(\partial\bar{B}) = D$. Now let $U \subset B := \bar{B} \setminus \partial\bar{B}$ be a small open ball contained in an affine chart of the nonsingular part of $B$, i.e. $\mu^{-1}(b)$ is a nonsingular Lagrangian torus in $\bar{X}$ for any $b \in U$, so that we can identify each fiber $\mu^{-1}(b)$ with $T^n$ and identify $\mu^{-1}(U)$ with $T^*U/\mathbb{Z}^n \cong U \times T^n$.

Let $N \cong \mathbb{Z}^n$ be the fundamental group of any fiber $\mu^{-1}(b)$, and consider the $\mathbb{Z}^n$-cover $L\mu^{-1}(U) = \mu^{-1}(U) \times N$. Locally, the mirror manifold should be given by the dual torus fibration $\nu : U \times (T^n)^\vee \to U$. Denote by $\nu^{-1}(U)$ the local mirror $U \times (T^n)^\vee$. Then we can define the *local SYZ mirror transformation*, as before, through the fiber product $L\mu^{-1}(U) \times_U \nu^{-1}(U)$.

$$\begin{array}{ccc} L\mu^{-1}(U) \times_U \nu^{-1}(U) & \longrightarrow & \nu^{-1}(U) \\ \Big\downarrow & & \Big\downarrow \nu \\ L\mu^{-1}(U) & \xrightarrow{\mu} & U \end{array}$$

Now, fix a reference fiber $L_0 = \mu^{-1}(b_0)$. Given $v \in N$, define a function $\Upsilon_v : L\mu^{-1}U \to \mathbb{R}$ as follows. For any point $p \in \mu^{-1}(U)$, let $L_b = \mu^{-1}(b)$ be the fiber containing $p$, where $b = \mu(p) \in U$. Regard $v$ as an element in $\pi_1(L_b)$. Consider the 2-chain $\gamma$ in $\mu^{-1}(U)$ with boundary in $v \cup L_0$, and define

$$\Upsilon_v(p, v) = \exp(-\frac{1}{2\pi} \int_\gamma \omega_{\mu^{-1}(U)}),$$

where $\omega_{\mu^{-1}(U)} = \omega_{\bar{X}}|_{\mu^{-1}(U)}$ is the restriction of the Kähler form to $\mu^{-1}(U)$. Also set $\Upsilon_v(p, w) = 0$ for any $w \in N \setminus \{v\}$ (cf. the discussion after Lemma 2.7 in Auroux [6]). This is analog to the definitions of the functions $\Psi_1, \ldots, \Psi_d$ in the toric Fano case, and it is easy to see that the local SYZ mirror transformations of these kind of functions give local holomorphic functions on the local mirror $\nu^{-1}(U) = U \times (T^n)^\vee$. We expect that these constructions will be sufficient for the purpose of understanding quantum corrections due to the boundary divisor $D$. However, to take care of the quantum corrections which arise from the proper singular Lagrangian fibers (i.e. singular fibers contained in $X = \mu^{-1}(B)$), one must modify and generalize the constructions of the local SYZ mirror transformations to the case where $U \subset B$ contains singular points. For this, new ideas are needed in order to incorporate the wall-crossing phenomena.

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## References

[1] M. Abouzaid, *Homogeneous coordinate rings and mirror symmetry for toric varieties*, Geom. Topol. 10 (2006), 1097–1157 (math.SG/0511644).

[2] ______, *Morse homology, tropical geometry, and homological mirror symmetry for toric varieties*, Selecta Math. (N.S.) 15 (2009), no. 2, 189–270 (math.SG/0610004).

[3] M. Abreu, *Kähler geometry of toric manifolds in symplectic coordinates*, Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), 1–24, Fields Inst. Commun., 35, Amer. Math. Soc., Providence, RI, 2003 (math.DG/0004122).

[4] *Quantum cohomology at the Mittag-Leffler Institute*, Edited by P. Aluffi, Scuola Normale Superiore, Pisa, 1997.

[5] M. Audin, *Torus actions on symplectic manifolds*, Second revised edition. Progress in Mathematics, 93. Birkhäuser Verlag, Basel, 2004.

[6] D. Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gokova Geom. Topol. 1 (2007), 51–91 (arXiv:0706.3207).

[7] D. Auroux, L. Katzarkov and D. Orlov, *Mirror symmetry for weighted projective planes and their noncommutative deformations*, Ann. of Math. (2) 167 (2008), no. 3, 867–943 (math.AG/0404281).

[8] ______, *Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves*, Invent. Math. 166 (2006), no. 3, 537–582 (math.AG/0506166).
[9] V. Batyrev, *On the classification of smooth projective toric varieties*, Tohoku Math. J. (2) 43 (1991), no. 4, 569–585.
[10] ______, *Quantum cohomology rings of toric manifolds*, Journées de Géométrie Algébrique d'Orsay (Orsay, 1992). Asterisque No. 218 (1993), 9–34 (alg-geom/9310004).
[11] K.-W. Chan and N.-C. Leung, *On SYZ mirror transformations*, to appear in Advanced Studies in Pure Mathematics, "New developments in Algebraic Geometry, Integrable Systems and Mirror Symmetry" (arXiv:0808.1551).
[12] C.-H. Cho and Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. 10 (2006), no. 4, 773–814 (math.SG/0308225).
[13] R. E. Edwards, *Fourier series. A modern introduction. Vol. 1*, Graduate Texts in Mathematics, 64. Springer-Verlag, New York-Berlin, 1979.
[14] K. Fukaya, *Multivalued Morse theory, asymptotic analysis and mirror symmetry*, Graphs and patterns in mathematics and theoretical physics, 205–278, Proc. Sympos. Pure Math., 73, Amer. Math. Soc., Providence, RI, 2005.
[15] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Lagrangian Floer theory on compact toric manifolds I, to appear in Duke Math. J. (arXiv:0802.1703).
[16] ______, Lagrangian Floer theory on compact toric manifolds II: Bulk deformations, preprint, 2008 (arXiv:0810.5654).
[17] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993.
[18] A. Givental, *Homological geometry and mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 472–480, Birkhäuser, Basel, 1995.
[19] ______, *Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture*, Topics in singularity theory, 103–115, Amer. Math. Soc. Transl. Ser. 2, 180, Amer. Math. Soc., Providence, RI, 1997 (alg-geom/9612001).
[20] ______, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), 141–175, Progr. Math., 160, Birkhäuser Boston, Boston, MA, 1998 (alg-geom/9701016).
[21] ______, *A tutorial on quantum cohomology*, Symplectic geometry and topology (Park City, UT, 1997), 231–264, IAS/Park City Math. Ser., 7, Amer. Math. Soc., Providence, RI, 1999.
[22] M. Gross, Mirror symmetry for $\mathbb{P}^2$ and tropical geometry, preprint, 2009 (arXiv:0903.1378).
[23] M. Gross and B. Siebert, *From real affine geometry to complex geometry*, preprint, 2007 (math.AG/0703822).
[24] V. Guillemin, *Moment maps and combinatorial invariants of Hamiltonian $T^n$-spaces*, Progress in Mathematics, 122. Birkhäuser Boston, Inc., Boston, MA, 1994.
[25] V. Guillemin and S. Sternberg, *The Gel̆fand-Cetlin system and quantization of the complex flag manifolds*, J. Funct. Anal. 52 (1983), no. 1, 106–128.
[26] N. Hitchin, The moduli space of special Lagrangian submanifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), no. 3-4, 503–515 (dg-ga/9711002).
[27] K. Hori and C. Vafa, *Mirror symmetry*, preprint, 2000 (hep-th/0002222).
[28] M. Kontsevich, *Lectures at ENS*, Paris, Spring 1998, notes taken by J. Bellaiche, J.-F. Dat, I. Marin, G. Racinet and H. Randriambololona.
[29] M. Kontsevich and Y. Soibelman, *Affine structures and non-Archimedean analytic spaces*, The unity of mathematics, 321–385, Progr. Math., 244, Birkhäuser Boston, Boston, MA, 2006 (math.AG/0406564).
[30] N.-C. Leung, *Mirror symmetry without corrections*, Comm. Anal. Geom. 13 (2005), no. 2, 287–331 (math.DG/0009235).
[31] N.-C. Leung, S.-T. Yau and E. Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, Adv. Theor. Math. Phys. 4 (2000), no. 6, 1319–1341 (math.DG/0005118).
[32] G. Mikhalkin, *Enumerative tropical algebraic geometry in $\mathbb{R}^2$*, J. Amer. Math. Soc. 18 (2005), no. 2, 313–377 (math.AG/0312530).
[33] ______, *Tropical geometry and its applications*, International Congress of Mathematicians. Vol. II, 827–852, Eur. Math. Soc., Zürich, 2006 (math.AG/0601041).
[34] ______, *Moduli spaces of rational tropical curves*, Proceedings of Gokova Geometry-Topology Conference 2006, 39–51, Gokova Geometry/Topology Conference (GGT), Gokova, 2007 (arXiv:0704.0839).

[35] D. Morrison, *The geometry underlying mirror symmetry*, New trends in algebraic geometry (Warwick, 1996), 283–310, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999 (alg-geom/9608006).
[36] T. Nishinou, *Disc counting on toric varieties via tropical curves*, preprint, 2006 (math.AG/0610660).
[37] T. Nishinou, Y. Nohara and K. Ueda, *Toric degenerations of Gelfand-Cetlin systems and potential functions*, preprint, 2008 (arXiv:0810.3470).
[38] T. Nishinou and B. Siebert, *Toric degenerations of toric varieties and tropical curves*, Duke Math. J. 135 (2006), no. 1, 1–51 (math.AG/0409060).
[39] P. Seidel, *More about vanishing cycles and mutation*, Symplectic geometry and mirror symmetry (Seoul, 2000), 429–465, World Sci. Publ., River Edge, NJ, 2001 (math.SG/0010032).
[40] B. Siebert, *An update on (small) quantum cohomology*, Mirror symmetry, III (Montreal, PQ, 1995), 279–312, AMS/IP Stud. Adv. Math., 10, Amer. Math. Soc., Providence, RI, 1999.
[41] B. Siebert and G. Tian, *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, Asian J. Math. 1 (1997), no. 4, 679–695 (alg-geom/9403010).
[42] H. Spielberg, *Multiple quantum products in toric varieties*, Int. J. Math. Math. Sci. 31 (2002), no. 11, 675–686 (math.SG/0107030).
[43] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B 479 (1996), no. 1-2, 243–259 (hep-th/9606040).
[44] K. Ueda, *Homological mirror symmetry for toric del Pezzo surfaces*, Comm. Math. Phys. 264 (2006), no. 1, 71–85 (math.AG/0411654).

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