

LAGRANGIAN SECTIONS ON MIRRORS OF TORIC CALABI–YAU 3-FOLDS

KWOKWAI CHAN, DANIEL POMERLEANO, AND KAZUSHI UEDA

ABSTRACT. We construct Lagrangian sections of a Lagrangian torus fibration on a 3-dimensional conic bundle, which are SYZ dual to holomorphic line bundles over the mirror toric Calabi–Yau 3-fold. We then demonstrate a ring isomorphism between the wrapped Floer cohomology of the zero-section and the regular functions on the mirror toric Calabi–Yau 3-fold. Furthermore, we show that in the case when the Calabi–Yau 3-fold is affine space, the zero section generates the wrapped Fukaya category of the mirror conic bundle. This allows us to complete the proof of one direction of homological mirror symmetry for toric Calabi–Yau orbifold quotients of the form \mathbb{C}^3/\check{G} . We finish by describing some elementary applications of our computations to symplectic topology.

1. INTRODUCTION

The goal of this paper is to provide evidence for and connect homological mirror symmetry [Kon95] and the Strominger–Yau–Zaslow (SYZ) conjecture [SYZ96] in the context of *3-dimensional conic bundles* of the form

$$(1.0.1) \quad Y = \{(w_1, w_2, u, v) \in (\mathbb{C}^\times)^2 \times \mathbb{C}^2 \mid h(w_1, w_2) = uv\}.$$

Mirror symmetry for such varieties goes back at least to [HIV]. For simplicity, in this paper we will assume that the Newton polytope Δ of $h(w_1, w_2)$ is full-dimensional and that the Laurent polynomial $h(w_1, w_2)$ is general among those whose Newton polytope is Δ . The mirror partner \check{Y} is an open subvariety of a 3-dimensional toric variety whose fan is the cone over a triangulation of the Newton polytope of the Laurent polynomial $h(w_1, w_2)$. These examples have been a fertile testing ground for mathematical thinking on mirror symmetry, a pioneering example being [KS02, ST01].

These conic bundles have the distinguishing feature that they are among the few examples of Calabi–Yau threefolds which admit explicit Lagrangian torus fibrations. An intriguing feature of these examples is that the Lagrangian fibrations are only piecewise smooth and have codimension one discriminant locus, and thus exhibit important features of the general case. There have therefore been a number of recent papers focusing on these examples through the lens of the SYZ conjecture. The essential idea for constructing such fibrations appears in [Gol01, Gro01a, Gro01b], and the construction has been carried out in [CBM05, CBM09, AAK16]. The details of this construction with some modifications are presented in Section 2.

The manifolds Y are affine varieties and therefore have natural symplectic forms and Liouville structures. On the other hand, in algebro-geometric terms, Y is the *Kaliman modification* of $(\mathbb{C}^\times)^2 \times \mathbb{C}$ along the hypersurface Z of $(\mathbb{C}^\times)^2$ defined by $h(w_1, w_2) = 0$ (see e.g. [Zai]). When viewed from this perspective, there is an alternative symplectic form used in [AAK16], which arises by viewing Y as an open submanifold of the symplectic blow-up \bar{Y} of $(\mathbb{C}^\times)^2 \times \mathbb{C}$ along the codimension two subvariety $Z \times 0$. This form is convenient for the purposes of the SYZ conjecture, since one can construct a Lagrangian torus fibration whose behavior is easy to understand. We modify their construction by degenerating the hypersurface Z to a certain tropical localization introduced in [Mik04,

Abo06]. This modification plays an important role throughout the rest of the paper as we explain below.

The starting point for our work is the observation that the SYZ fibration comes equipped with certain natural *base-admissible Lagrangian sections*. A related construction of Lagrangian sections has been carried out independently by Gross and Matessi in [GM], though we place an emphasis on the asymptotics of these sections since these are important for Floer cohomology. In Section 3, we prove the following theorem:

Theorem 1.1. *For every holomorphic line bundle \mathcal{F} on the SYZ mirror \check{Y} , there is a base-admissible Lagrangian section $L_{\mathcal{F}}$ in Y whose SYZ transform is \mathcal{F} . Furthermore, such Lagrangian submanifolds are unique up to Hamiltonian isotopy.*

For the purposes of mirror symmetry for non-compact manifolds such as Y , it is essential to consider the *wrapped Floer cohomology* of non-compact Lagrangian submanifolds. The foundational theory of wrapped Fukaya categories on Liouville domains has been developed in [AS10]. However, wrapped Floer cohomology is very difficult to calculate directly and there are only a few computations in the literature. One of the main objectives of this paper is to prove the following theorem:

Theorem 1.2. *Let L_0 denote the section of the fibration which is SYZ dual to the structure sheaf $\mathcal{O}_{\check{Y}}$. Then the wrapped Floer cohomology $\mathrm{HW}^*(L_0)$ of L_0 is concentrated in degree 0, and there exists an isomorphism*

$$(1.2.1) \quad \mathrm{Mir}^{L_0} : \mathrm{HW}^0(L_0) \xrightarrow{\sim} H^0(\mathcal{O}_{\check{Y}})$$

of \mathbb{C} -algebras.

The wrapped Floer cohomology ring on the left hand side of (1.2.1) comes with a natural basis given by Hamiltonian chords. As suggested by Tyurin and emphasized by Gross, Hacking, Keel and Siebert [GHK15, GS], the images of these Hamiltonian chords on the right hand side of (1.2.1) give generalizations of theta functions on Abelian varieties. The proof of Theorem 1.2, which is a direct consequence of Theorem 1.3 and Theorem 1.4 below, is a higher dimensional analogue of [Pas14]. However, the approach taken in this paper is slightly different. While Pascaleff’s argument exploits a certain TQFT structure for Lefschetz fibrations due to Seidel, our argument is based upon a study of how Floer theory behaves under Kaliman modification. Though this birational geometry viewpoint governs our approach, we do not study this problem in maximal generality, but limit ourselves to this series of examples.

The first step in our argument is to construct a suitable family of Hamiltonians which are well behaved with respect to the conic bundle structure. The essential technical ingredient for wrapped Floer theory is the existence of suitable C^0 -estimates for solutions to Floer’s equation. However, base-admissible Lagrangian sections do not naturally fit into the setup of [AS10] and, for this reason, we adapt their theory of wrapped Floer cohomology to the above choice of symplectic forms and base-admissible Lagrangian sections.

In our setup, there are now two directions in which Floer curves can escape to infinity, namely, in the “base direction” and in the “fiber direction” of the conic fibration. We show that, outside of a compact set in the base, the Hamiltonian flow on the total space projects nicely to the Hamiltonian flow on the base. It follows that there is a maximum principle for solutions to Floer’s equation with boundary on admissible Lagrangians, which prevents curves from escaping to infinity in the “base direction”. On the other hand, it is more difficult to prevent curves from escaping to infinity in the “fiber direction” of the conic fibration. In fact, curves can indeed escape to infinity. However, we show

in Lemma 5.20 that they must break along *divisor chords*, which are Hamiltonian chords living completely inside the divisor at infinity. There is a natural auxiliary grading *relative* to the exceptional divisor E . The key observation is that, with respect to this grading, the grading of the divisor Hamiltonian chords becomes arbitrarily large as m gets larger. By restricting to generators for the Floer complex whose relative grading lies in a certain range, we are therefore able to exclude this breaking and obtain the compactness needed to define wrapped Floer cohomology.

We now turn to the computational aspect of our paper. The zero-section L_0 is a base-admissible Lagrangian and so it makes sense to consider its adapted wrapped Floer cohomology ring $\mathrm{HW}_{\mathrm{ad}}^*(L_0)$. In Section 6, we prove the following theorem:

Theorem 1.3. *The adapted wrapped Floer cohomology $\mathrm{HW}_{\mathrm{ad}}^*(L_0)$ is concentrated in degree 0, and there exists an isomorphism*

$$(1.3.1) \quad \mathrm{Mir}_{\mathrm{ad}}^{L_0} : \mathrm{HW}_{\mathrm{ad}}^0(L_0) \xrightarrow{\sim} H^0(\mathcal{O}_{\tilde{Y}})$$

of \mathbb{C} -algebras.

One case where wrapped Floer cohomology can sometimes be directly computed is in manifolds which are products of lower-dimensional manifolds and where the Lagrangians and Hamiltonians split according to the product structures. An important advantage of base-admissible Lagrangians is that they live away from the exceptional locus and hence can be regarded as Lagrangian submanifolds in $(\mathbb{C}^\times)^2 \times \mathbb{C}^\times$. When viewed in this way, these Lagrangians respect the product structure. Moreover, the admissible Hamiltonians we use interplay nicely with this product structure, and hence the Floer theory of base-admissible Lagrangians is amenable to direct calculation.

In the case of L_0 , we find that all Hamiltonian chords lie away from the exceptional locus and moreover that as a vector space, the Floer cohomologies agree when regarded as living in $(\mathbb{C}^\times)^2 \times \mathbb{C}^\times$ or Y . However, the product structure on Floer cohomology is deformed. This deformation can be formalized in terms of the *relative Fukaya category* of Seidel and Sheridan [Sei15, She15]. One slightly novel feature is that in typical situations, one works relative to a compactifying divisor at infinity, while here we work relative to the exceptional divisor E . In this case, one computes the deformed ring structure directly by exploiting the correspondence between holomorphic curves in $(\mathbb{C}^\times)^2 \times \mathbb{C}$ with incidence conditions relative to the submanifold $Z \times 0$ and holomorphic curves in Y (one direction of this correspondence is given by projection and the other direction is given by proper transform). The relevant enumerative calculation is then done in Lemma 6.12 based upon a simple degeneration-of-domain argument.

It is not difficult to see that the Lagrangian L_0 is also admissible in the sense of [AS10] when Y is equipped with its natural finite-type convex symplectic structure (for a definition, see [McL09, Definition 2.2]). In Section 7, we prove the following comparison theorem between the two types of Floer theories:

Theorem 1.4. *There exists an isomorphism*

$$(1.4.1) \quad \phi_{\mathrm{HW}} : \mathrm{HW}^*(L_0) \xrightarrow{\sim} \mathrm{HW}_{\mathrm{ad}}^*(L_0)$$

of graded \mathbb{C} -algebras.

The proof of this theorem is a modification of [McL09, Theorem 5.5], which applies to Lefschetz fibrations. While his arguments do not generalize to symplectic fibrations over higher dimensional bases, certain simplifying features of our situation allow us to adapt his arguments in a straightforward way.

We use this result to calculate the zero-th symplectic cohomology $\mathrm{SH}^0(Y)$ of Y . Abouzaid [Abo10] has introduced a map

$$(1.4.2) \quad \mathrm{CO}: \mathrm{SH}^0(Y) \rightarrow \mathrm{HW}^0(L_0).$$

In Section 8, we prove the following:

Theorem 1.5. *The map (1.4.2) is an isomorphism.*

The argument employed here is inspired by an argument of Pascaleff [Pas]. Namely, it is not difficult to attain that the map is injective which again follows essentially from the fact that the symplectic fibration is trivial away from the discriminant locus. This enables us to compute the “local” or “low energy” terms in the map CO by reduction to the case of $(\mathbb{C}^\times)^3$. Proving surjectivity is more delicate and rests upon making partial computations of the differential in symplectic cohomology. This relies on two recent ingredients: a local computation of the BV-operator in [Zha] and a trick in [Pas] which enables us to rule out higher energy terms in the differential of a certain “primitive” cochain.

Finally, we apply all of these calculations to give a proof of homological mirror symmetry in the simplest case when our polynomial $h(w_1, w_2)$ is $1 + w_1 + w_2$ (this corresponds to the case when the mirror \check{Y} is $\mathrm{Spec}(\mathbb{C}[x, y, z][[(xyz - 1)^{-1}]]$). In a dual picture to the one above, Abouzaid has introduced a map

$$(1.5.1) \quad \mathrm{OC}: \mathrm{HH}_3(\mathrm{HW}^*(L_0)) \rightarrow \mathrm{SH}^0(Y).$$

In our case, the machinery behind this map is vastly simplified because all of our chain complexes lie in degree zero. Abouzaid showed that if the unit $[1]$ is hit by the map OC, then L_0 split-generates the wrapped Fukaya category $\mathcal{W}(Y)$. It follows from what we have proven so far that the image of OC must be a principal ideal $(f) \in \mathrm{SH}^0(Y)$. Although it is difficult to compute the entire open string map directly, the surjectivity of the map therefore follows formally if we can show that there are non-zero elements f_1 , f_2 , and f_3 in the image of OC which have no common divisors other than units. This again uses the triviality of the symplectic fibration away from the discriminant locus to make enough partial computations of the map to produce these elements. Combining all of these results, we obtain the following:

Theorem 1.6. *L_0 generates the wrapped Fukaya category of Y . In particular, there is an equivalence*

$$(1.6.1) \quad \psi: D^b\mathcal{W}(Y) \xrightarrow{\sim} D^b \mathrm{coh} \check{Y}$$

of enhanced triangulated categories sending L_0 to $\mathcal{O}_{\check{Y}}$.

To push this result somewhat further, we note that the behavior of (wrapped) Fukaya categories under finite covers is well-understood. To be more precise, let N_0 be a finite index subgroup of the fundamental group N of Y . Note that N is a free abelian group of rank 2. Denote the quotient group by $G = N/N_0$ and let Y_{N_0} be the corresponding finite cover (which is itself a conic bundle). The set of connected components of the pull-back of L_0 by the covering map $\varpi: Y_{N_0} \rightarrow Y$ is a torsor over G , which we identify with G by choosing a base point; $\varpi^{-1}(L_0) = \bigoplus_{g \in G} L_g$. The group $\check{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ acts naturally on \check{Y} , and the quotient stack is denoted by $[\check{Y}/\check{G}]$. We write the one-dimensional representation of \check{G} associated with an element $g \in G$ as V_g . Theorem 1.6 admits the following generalization to finite covers:

Corollary 1.7. *There is an equivalence*

$$(1.7.1) \quad \psi_{N_0}: D^b \mathcal{W}(Y_{N_0}) \xrightarrow{\sim} D^b \text{coh} [\check{Y}/\check{G}]$$

of enhanced triangulated categories sending L_g to $\mathcal{O}_{\check{Y}} \otimes V_g$.

This computation represents to our knowledge the first complete computation appearing in the literature of the wrapped Fukaya category of a six-dimensional Liouville domain, which is neither a product of lower dimensional manifolds nor a cotangent bundle. This has applications to symplectic topology, an elementary example being Proposition 8.17. In a different direction, the abelian McKay correspondence implies homological mirror symmetry for divisor complements of toric Calabi–Yau varieties which are related by a variation of GIT quotients to orbifold quotients $[\mathbb{C}^3/\check{G}]$.

Now assume that G is sufficiently ‘big’ in the sense that the crepant resolution of \mathbb{C}^3/\check{G} has a compact divisor. Let $\mathcal{F}_0(Y_{N_0})$ be the full subcategory of $\mathcal{W}(Y_{N_0})$ consisting of Lagrangian spheres, and $\text{coh}_0[\check{Y}/\check{G}]$ be the full subcategory of $\text{coh}[\check{Y}/\check{G}]$ consisting of sheaves supported at the origin.

Corollary 1.8. *The equivalence (1.7.1) induces a fully faithful functor*

$$(1.8.1) \quad D^b \mathcal{F}_0(Y_{N_0}) \hookrightarrow D^b \text{coh}_0 \check{Y}/\check{G}.$$

It seems likely that the above embedding (1.8.1) is inverse to the embedding

$$(1.8.2) \quad D^b \text{coh}_0 \check{Y}/\check{G} \hookrightarrow D^b \mathcal{F}(Y_{N_0}),$$

given by combining [Sei10] and [AKO08]. To prove this, one would have to compare symplectic forms used in this paper with that in [Sei10]; we do not go into that problem here. Another interesting problem would be to study the meaning of Bridgeland stability conditions on the essential image of (1.8.1) in terms of symplectic or Kähler geometry of Y_{N_0} (see [Bri07] for the foundation of Bridgeland stability conditions, [Bri06, BM11] for the space of Bridgeland stability conditions on some toric Calabi–Yau manifolds, and [TY02, Joy15] for a conjectural relation with the Fukaya category). Nevertheless, the existence of this embedding already has a nice application:

Proposition 1.9. *Let S_1, \dots, S_r be a collection of Lagrangian spheres in Y_{N_0} which are pairwise disjoint. Then $r \leq |\check{G}|$.*

In fact, we give a slightly stronger result in Proposition 8.18. Finally, it is worth mentioning that although we have focused on the case of conic fibrations over $(\mathbb{C}^\times)^2$, our methods apply to the somewhat simpler case of (fiber products of) conic bundles over \mathbb{C}^\times as well. In particular, they enable us to prove that the collections of Lagrangians studied in [CU13] and [CPU16] generate the relevant wrapped Fukaya categories. The latter case is especially interesting, as this gives rise to examples, different from those given in [BS15, Smi15], of symplectic 6-manifolds whose space of Bridgeland stability conditions on Fukaya categories is reasonably well understood [Tod08, Tod09, IUU10].

Returning to the case of a general toric Calabi–Yau manifold, there are two natural ways in which one might try to extend our results to a homological mirror symmetry statement. The first is to establish a version of homological mirror symmetry between a suitable Fukaya categories generated by base-admissible Lagrangians and categories of coherent sheaves on the mirror \check{Y} . Base admissible Lagrangians L naturally fiber over Lagrangians \underline{L} in $(\mathbb{C}^\times)^2$. The calculation of A_∞ -operations can likely be reduced using the approach of Section 6 to the enumeration of Floer polygons with boundary on \underline{L} . It seems reasonable to hope that the latter polygons can be dealt with using techniques similar to those in [Abo09].

Let \mathbf{D} be the union of the toric divisors in \check{Y} . A different direction of research begins with the observation that we have a Bousfield localization sequence

$$(1.9.1) \quad 0 \rightarrow \mathrm{QC}_{\mathbf{D}}(\check{Y}) \rightarrow \mathrm{QC}(\check{Y}) \rightarrow \mathrm{QC}(\check{Y} \setminus \mathbf{D}) \rightarrow 0,$$

where $\mathrm{QC}(\check{Y})$ is the unbounded derived category of quasi-coherent sheaves on \check{Y} and $\mathrm{QC}_{\mathbf{D}}(\check{Y})$ is the full subcategory consisting of objects whose cohomologies are supported on \mathbf{D} . Since $D^b \mathrm{coh}_{\mathbf{D}} \check{Y}$ generates $\mathrm{QC}_{\mathbf{D}}(\check{Y})$ and $\check{Y} \setminus \mathbf{D}$ is affine, the bounded derived category $D^b \mathrm{coh} \check{Y}$ is split-generated by $\mathcal{O}_{\check{Y}}$ and $D^b \mathrm{coh}_{\mathbf{D}} \check{Y}$.

This paper describes the subcategory of the wrapped Fukaya category generated by L_0 . Furthermore, SYZ mirror symmetry predicts that there is a full-subcategory of this wrapped Fukaya category consisting of Liouville-admissible Lagrangians which is equivalent to $D^b \mathrm{coh}_{\mathbf{D}}(\check{Y})$. These Lagrangians are again constructed using ideas from [Abo09], and a similar construction has already appeared in [GM].

For example, when \check{Y} is the total space of the canonical line bundle over a toric Fano manifold, we expect to have a collection of objects consisting of L_0 and compact Lagrangian spheres which split-generate the wrapped Fukaya category. We expect that the results in [Abo09] together with an analysis similar to that in [Sei10] would give a direct approach to studying the Floer cohomology of these Lagrangian spheres. Combining this with the results of this paper then gives an approach to studying the wrapped Fukaya categories for general $h(w_1, w_2)$.

Acknowledgement: We thank Diego Matessi for sending a draft of [GM] and useful discussions. The work of K. C. described in this paper was substantially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CUHK400213). D. P. was supported by Kavli IPMU, EPSRC, and Imperial College. He would also like to thank Mohammed Abouzaid, Denis Auroux, and Mauricio Romo for helpful discussions concerning the construction of base-admissible Hamiltonians and Jonny Evans for a discussion about symplectic blowups. K. U. was supported by JSPS KAKENHI Grant Numbers 24740043, 15KT0105, 16K13743 and 16H03930.

2. LAGRANGIAN TORUS FIBRATIONS

2.1. Tropical hypersurface. Let $N = \mathbb{Z}^2$ be a free abelian group of rank 2, and $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual group. A *convex lattice polygon* in $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is the convex hull of a finite subset of M . We assume that all convex lattice polygons in this paper are full-dimensional. Let Δ be a convex lattice polygon in $M_{\mathbb{R}}$ and $A = \Delta \cap M$ be the set of lattice points in Δ . A function $\nu: A \rightarrow \mathbb{R}$ defines a piecewise-linear function $\bar{\nu}: \Delta \rightarrow \mathbb{R}$ by the condition that

$$(2.1.1) \quad \mathrm{Conv}\{(\boldsymbol{\alpha}, u) \in A \times \mathbb{R} \mid u \geq \nu(\boldsymbol{\alpha})\} = \{(\mathbf{m}, u) \in \Delta \times \mathbb{R} \mid u \geq \bar{\nu}(\boldsymbol{\alpha})\}.$$

The set \mathcal{P}_{ν} consisting of maximal domains of linearity of $\bar{\nu}$ and their faces is a polyhedral decomposition of Δ . A polyhedral decomposition \mathcal{P} of Δ is *coherent* (or *regular*) if there is a function $\nu: A \rightarrow \mathbb{Z}$ such that $\mathcal{P} = \mathcal{P}_{\nu}$. It is a *triangulation* if all the maximal-dimensional faces are triangles. A coherent triangulation is *unimodular* if every triangle can be mapped to the standard simplex (i.e., the convex hull of $\{(0, 0), (1, 0), (0, 1)\}$) by the action of $\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

Let \mathcal{P} be a unimodular coherent triangulation of Δ . A function $\nu: A \rightarrow \mathbb{R}$ is *adapted* to \mathcal{P} if $\mathcal{P} = \mathcal{P}_{\nu}$. Given a function $\nu: A \rightarrow \mathbb{R}$ and an element $(c_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in A} \in \mathbb{C}^A$, a *patchworking*

polynomial is defined by

$$(2.1.2) \quad h_t(\mathbf{w}) = \sum_{\alpha \in A} c_\alpha t^{-\nu(\alpha)} \mathbf{w}^\alpha \in \mathbb{C}[t^{\pm 1}][M].$$

For a positive real number $t \in \mathbb{R}^{>0}$, a hypersurface Z_t of $N_{\mathbb{C}^\times} := N_{\mathbb{R}}/N \cong \text{Spec } \mathbb{C}[M]$ is defined by

$$(2.1.3) \quad Z_t = \{\mathbf{w} \in N_{\mathbb{C}^\times} \mid h_t(\mathbf{w}) = 0\}.$$

We assume that Z_t is connected. The image

$$(2.1.4) \quad \Pi_t := \text{Log}(Z_t)$$

of Z_t by the logarithmic map

$$(2.1.5) \quad \text{Log}: N_{\mathbb{C}^\times} \rightarrow N_{\mathbb{R}}, \quad \mathbf{w} = (w_1, w_2) \mapsto \frac{1}{\log |t|} (\log |w_1|, \log |w_2|)$$

is called the *amoeba* of Z_t . The *tropical polynomial* associated with $h_t(\mathbf{w})$ is the piecewise-linear map defined by

$$(2.1.6) \quad L_\nu: N_{\mathbb{R}} \rightarrow \mathbb{R}, \quad \mathbf{n} \mapsto \max \{\alpha(\mathbf{n}) - \nu(\alpha) \mid \alpha \in A\}.$$

The *tropical hypersurface* (or the *tropical curve*) associated with L_ν is defined as the locus $\Pi_\infty \subset N_{\mathbb{R}}$ where L_ν is not differentiable. The polyhedral decomposition of $N_{\mathbb{R}}$ defined by the tropical hypersurface Π_∞ is dual to the triangulation \mathcal{P} of Δ . In particular, the connected components of the complement of Π_∞ can naturally be labeled by $\mathcal{P}^{(0)} = A$;

$$(2.1.7) \quad N_{\mathbb{R}} \setminus \Pi_\infty = \coprod_{\alpha \in A} C_{\alpha, \infty}.$$

The amoeba Π_t converges to Π_∞ in the Hausdorff topology as t goes to infinity [Mik04, Rul].

We next introduce some terminology and notation which will be used throughout the rest of this paper. A *leg* is a one-dimensional polyhedral subcomplex of the tropical hypersurface Π_∞ which is non-compact. In what follows, we will use the variables r_i to denote $\log |w_i|$. Let $\{\Pi_i\}_{i=1}^\ell$ be the set of legs of the tropical curve Π_∞ . For each leg Π_i , we write the endpoints of the edge in \mathcal{P} dual to Π_i as $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}), \beta_i = (\beta_{i,1}, \beta_{i,2}) \in A$, so that Π_i is defined by

$$(2.1.8) \quad \begin{cases} \mathbf{r}_{\alpha_i - \beta_i} := (\alpha_{i,1} - \beta_{i,1})r_1 + (\alpha_{i,2} - \beta_{i,2})r_2 = \nu(\alpha_i) - \nu(\beta_i), \\ \mathbf{r}_{(\alpha_i - \beta_i)^\perp} := (\beta_{i,2} - \alpha_{i,2})r_1 + (\alpha_{i,1} - \beta_{i,1})r_2 \geq a_i, \end{cases}$$

for some $a_i \in \mathbb{R}$. The maximal polyhedral subcomplex of Π_∞ which is compact will be denoted by $\Pi_{\infty, c}$. Define $c_i, c'_i, c''_i \in \mathbb{Q}$ by

$$(2.1.9) \quad c_i = \frac{1}{|\alpha_i - \beta_i|}, \quad c'_i = \frac{\alpha_i \cdot (\alpha_i - \beta_i)}{|\alpha_i - \beta_i|^2}, \quad c''_i = \frac{\alpha_i \cdot (\alpha_i - \beta_i)^\perp}{|\alpha_i - \beta_i|^2},$$

so that

$$(2.1.10) \quad |\mathbf{r}|^2 := r_1^2 + r_2^2 = c_i^2 \left((\mathbf{r}_{\alpha_i - \beta_i})^2 + (\mathbf{r}_{(\alpha_i - \beta_i)^\perp})^2 \right),$$

$$(2.1.11) \quad \alpha_i = c'_i(\alpha_i - \beta_i) + c''_i(\alpha_i - \beta_i)^\perp.$$

Then we have

$$(2.1.12) \quad |\mathbf{w}^{\alpha_i}| = |\mathbf{w}^{\alpha_i - \beta_i}|^{c'_i} |\mathbf{w}^{(\alpha_i - \beta_i)^\perp}|^{c''_i}$$

$$(2.1.13) \quad d\theta_{\alpha_i} = c'_i d\theta_{\alpha_i - \beta_i} + c''_i d\theta_{(\alpha_i - \beta_i)^\perp}$$

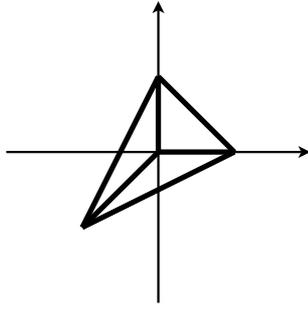


FIGURE 2.1. A triangulation

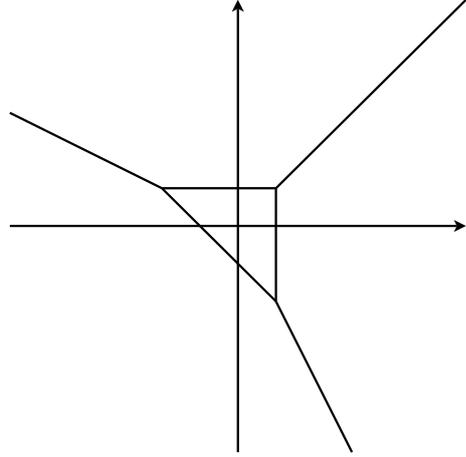


FIGURE 2.2. A tropical curve

Example 2.2. A prototypical example is

$$(2.2.1) \quad h_t(\mathbf{w}) = w_1 + w_2 + \frac{1}{w_1 w_2} + t^\varepsilon$$

for a small positive real number ε . The corresponding tropical polynomial is given by

$$(2.2.2) \quad L(\mathbf{n}) = \max\{n_1, n_2, -n_1 - n_2, \varepsilon\},$$

and the tropical curve Π_∞ is shown in Figure 2.2. The set

$$(2.2.3) \quad A = \{(0, 0), (1, 0), (0, 1), (-1, -1)\}$$

consists of four elements, corresponding to four connected components of $N_{\mathbb{R}} \setminus \Pi_\infty$. The tropical curve Π_∞ has three legs.

2.3. Tropical localization. Following [Abo06, Section 4], we set $C_{\alpha,t} := (\log t) \cdot C_\alpha$ for each $\alpha \in A$, and choose $\phi_\alpha: N_{\mathbb{R}} \rightarrow \mathbb{R}$ such that

- $d(\mathbf{n}, C_{\alpha,t}) := \min_{\mathbf{n}' \in C_{\alpha,t}} \|\mathbf{n} - \mathbf{n}'\| \leq (\varepsilon \log t)/2$ if and only if $\phi_\alpha(\mathbf{n}) = 0$,
- $d(\mathbf{n}, C_{\alpha,t}) \geq \varepsilon \log t$ if and only if $\phi_\alpha(\mathbf{n}) = 1$, and
- $\left| \frac{\partial \phi_\alpha}{\partial n_1} \right| + \left| \frac{\partial \phi_\alpha}{\partial n_2} \right| < \frac{4}{\varepsilon \log t}$

for a small positive real number ε . For an element $(c_\alpha)_{\alpha \in A} \in \mathbb{C}^A$, define a family $\{h_{t,s}\}$ of maps $N_{\mathbb{C}^\times} \rightarrow \mathbb{C}$ by

$$(2.3.1) \quad h_{t,s}(\mathbf{w}) = \sum_{\alpha \in A} c_\alpha t^{-\nu(\alpha)} (1 - s \phi_\alpha(\mathbf{w})) \mathbf{w}^\alpha,$$

where we write $\phi_\alpha(\mathbf{w}) := \phi_\alpha(\text{Log}_t(\mathbf{w}))$ by abuse of notation. For a face $\tau \in \mathcal{P}$, define

$$(2.3.2) \quad O_\tau = \{\mathbf{n} \in N_{\mathbb{R}} \mid (\phi_\alpha(\mathbf{n}) \neq 1 \text{ for } \forall \alpha \in \tau) \text{ and } (\phi_\alpha(\mathbf{n}) = 1 \text{ for } \forall \alpha \notin \tau)\}.$$

Then O_τ is contained in an ε -neighborhood of the face of Π_∞ dual to τ , and one has

$$(2.3.3) \quad N_{\mathbb{R}} = \coprod_{\tau \in \mathcal{P}} O_\tau.$$

The set

$$(2.3.4) \quad Z_{t,s} = \{\mathbf{w} \in N_{\mathbb{C}^\times} \mid h_{t,s}(\mathbf{w}) = 0\}$$

is a symplectic hypersurface in $N_{\mathbb{C}^\times}$ for a sufficiently large t [Abo06, Proposition 4.2], and the pairs $(N_{\mathbb{C}^\times}, Z_{t,s})$ for all $s \in [0, 1]$ are symplectomorphic to each other [Abo06,

Proposition 4.9] which for t sufficiently large can be taken C^k -small and to be supported inside of a tubular neighborhood of $Z_{t,0}$ of small norm in the Kahler metric [Abo09, Appendix A].

The *tropical localization* of Z_t is defined by

$$(2.3.5) \quad Z := Z_{t,1}.$$

We set $(c_\alpha)_{\alpha \in A} = \mathbf{1} := (1, \dots, 1) \in \mathbb{C}^A$ and

$$(2.3.6) \quad h(\mathbf{w}) := h_{t,s}(\mathbf{w}) \Big|_{s=1, (c_\alpha)_{\alpha \in A} = \mathbf{1}} = \sum_{\alpha_i \in A} t^{-\nu(\alpha_i)} (1 - \phi_{\alpha_i}(\mathbf{w})) \mathbf{w}^{\alpha_i}.$$

The hypersurface Z is localized in the following sense:

- Over O_τ where τ is dual to an edge of Π_∞ (i.e., when $\tau \in \mathcal{P}^{(1)}$), all but two terms of h vanish, and hence Z is defined by

$$(2.3.7) \quad h(\mathbf{w}) = t^{-\nu(\alpha)} (1 - \phi_\alpha(\mathbf{w})) \mathbf{w}^\alpha + t^{-\nu(\beta)} (1 - \phi_\beta(\mathbf{w})) \mathbf{w}^\beta = 0,$$

where $\alpha, \beta \in A$ are the endpoints of τ .

- Over O_σ where σ is dual to a vertex of Π_∞ (i.e., when $\sigma \in \mathcal{P}^{(2)}$), all but three terms of h vanish, whence Z is defined by

$$(2.3.8) \quad h(\mathbf{w}) = t^{-\nu(\alpha_0)} (1 - \phi_{\alpha_0}(\mathbf{w})) \mathbf{w}^{\alpha_0} + t^{-\nu(\alpha_1)} (1 - \phi_{\alpha_1}(\mathbf{w})) \mathbf{w}^{\alpha_1} \\ + t^{-\nu(\alpha_2)} (1 - \phi_{\alpha_2}(\mathbf{w})) \mathbf{w}^{\alpha_2}$$

$$(2.3.9) \quad = 0,$$

where $\alpha_0, \alpha_1, \alpha_2 \in A$ are the vertices of σ .

It follows that the amoeba $\Pi = \text{Log}(Z)$ of the tropical localization Z agrees with the tropical hypersurface Π_∞ away from the ε -neighborhood of the 0-skeleton $\Pi_\infty^{(0)}$. An important property for us is that each connected component of the complement of a compact set in Z is defined by a single *algebraic* equation

$$(2.3.10) \quad t^{-\nu(\alpha)} \mathbf{w}^\alpha + t^{-\nu(\beta)} \mathbf{w}^\beta = 0,$$

and fibers over a subset of a leg of the tropical hypersurface Π_∞ . In a slight abuse of terminology, we will refer to this portion of the curve Z as a *leg* of Z .

For a sufficiently large number R_0 and a relatively small number $\varepsilon_n \ll R_0$, we may assume that the union

$$(2.3.11) \quad U_\Pi := U_{\Pi_c} \cup \bigcup_{i=1}^{\ell} U_{\Pi_i}$$

of

$$(2.3.12) \quad U_{\Pi_c} := \{\mathbf{r} \in N_{\mathbb{R}} \mid |\mathbf{r}| < R_0\}$$

and

$$(2.3.13) \quad U_{\Pi_i} = \{\mathbf{r} \in N_{\mathbb{R}} \mid |\mathbf{r}_{\alpha_i - \beta_i}| < \varepsilon_n \text{ and } \mathbf{r}_{(\alpha_i - \beta_i)^\perp} \geq a_i + \varepsilon_n\}$$

is a neighborhood of Π . We assume that ε_n is large enough so that it contains the line $\mathbf{r}_{\alpha_i - \beta_i} - (\nu(\alpha_i) - \nu(\beta_i)) = 0$, and that if we have non-compact parallel legs Π_i , we will assume that all of the corresponding neighborhoods U_{Π_i} are the same. We next choose a neighborhood U_Z of Z in $N_{\mathbb{C}^\times}$ and set

$$(2.3.14) \quad U_{Z_i} := U_Z \cap \text{Log}^{-1}(U_{\Pi_i}), \quad U_{Z_c} := U_Z \cap \text{Log}^{-1}(U_{\Pi_c}).$$

We require:

- $\text{Log}(U_Z) \subset U_\Pi$,

- $\left| \frac{h(\mathbf{w})}{w^\alpha} \right| < \varepsilon_h$ for some small constant $\varepsilon_h > 0$ and any $\mathbf{w} \in U_{Z_i}$,
- $h = t^{-\nu(\alpha)} \mathbf{w}^\alpha + t^{-\nu(\beta)} \mathbf{w}^\beta$ for any $\mathbf{w} \in U_{Z_i}$,
- the set $\{\alpha \in A \mid \phi_\alpha(\mathbf{w}) \neq 1\}$ consists of either two or three elements for any $\mathbf{w} \in U_Z$, and
- U_Z does not intersect $N_{\mathbb{R}^{>0}} := \{(w_1, w_2) \in (\mathbb{R}^{>0})^2\} \subset N_{\mathbb{C}^\times}$.

We write the natural projections as

$$(2.3.15) \quad \text{pr}_1: N_{\mathbb{C}^\times} \times \mathbb{C} \rightarrow N_{\mathbb{C}^\times}, \quad \text{pr}_2: N_{\mathbb{C}^\times} \times \mathbb{C} \rightarrow \mathbb{C}.$$

We choose a tubular neighborhood $U_{Z \times 0}$ of $Z \times 0$ in $N_{\mathbb{C}^\times} \times \mathbb{C}$ such that

- $\text{pr}_1(U_{Z \times 0}) \subset U_Z$ and
- $\frac{|u|^2}{|\mathbf{w}^{(\alpha_i - \beta_i)}|^{2c_i'}} + |t^{-\nu(\alpha)} + t^{-\nu(\beta)} \mathbf{w}^{\beta - \alpha}|^2 < \varepsilon_h^2$ if $(\mathbf{w}, u) \in U_{Z \times 0}$ and $\text{Log}(\mathbf{w}) \in U_{\Pi_i}$.

We set

$$(2.3.16) \quad U_i := \{(\mathbf{w}, u) \in U_{Z \times 0} \mid \text{Log}(\mathbf{w}) \in U_{\Pi_i}\}.$$

We fix an almost complex structure $J_{N_{\mathbb{C}^\times}}$ on $N_{\mathbb{C}^\times}$, which is adapted to Z in the following sense:

Definition 2.4. An $\omega_{N_{\mathbb{C}^\times}}$ compatible almost complex structure $J_{N_{\mathbb{C}^\times}}$ on $N_{\mathbb{C}^\times}$ is said to be *adapted to Z* if

- $J_{N_{\mathbb{C}^\times}}$ agrees with the standard complex structure $J_{N_{\mathbb{C}^\times}, \text{std}}$ of $N_{\mathbb{C}^\times}$ outside the inverse image by Log of a small neighborhood of the origin in $N_{\mathbb{R}}$, and
- the function $h(\mathbf{w})$ is $J_{N_{\mathbb{C}^\times}}$ -holomorphic in U_Z .

2.5. Symplectic blow-up. We follow [AAK16] in this subsection; see also [CBM09] for a closely related construction. As a smooth manifold, the blow-up

$$(2.5.1) \quad p: \bar{Y} := \text{Bl}_{Z \times 0}(N_{\mathbb{C}^\times} \times \mathbb{C}) \rightarrow N_{\mathbb{C}^\times} \times \mathbb{C}$$

of $N_{\mathbb{C}^\times} \times \mathbb{C}$ along the symplectic submanifold $Z \times 0$ is given by

$$(2.5.2) \quad \bar{Y} := \{(\mathbf{w}, u, [v_0 : v_1]) \in N_{\mathbb{C}^\times} \times \mathbb{C} \times \mathbb{P}^1 \mid uv_1 = h(\mathbf{w})v_0\}.$$

The compositions of the structure morphism (2.5.1) and the projections (2.3.15) will be denoted by

$$(2.5.3) \quad \bar{\pi}_{N_{\mathbb{C}^\times}} := \text{pr}_1 \circ p: \bar{Y} \rightarrow N_{\mathbb{C}^\times}, \quad \bar{\pi}_{\mathbb{C}} := \text{pr}_2 \circ p: \bar{Y} \rightarrow \mathbb{C}.$$

The exceptional set is given by

$$(2.5.4) \quad E := p^{-1}(Z \times 0) = \{(\mathbf{w}, u, [v_0 : v_1]) \in N_{\mathbb{C}^\times} \times \mathbb{C} \times \mathbb{P}^1 \mid h(\mathbf{w}) = u = 0\},$$

which forms a \mathbb{P}^1 -bundle $p|_E: E \rightarrow Z \times 0$ over $Z \times 0$. The total transform of the divisor $Z \times \mathbb{C} \subset N_{\mathbb{C}^\times} \times \mathbb{C}$ is given by

$$(2.5.5) \quad \bar{E} := \bar{\pi}_{N_{\mathbb{C}^\times}}^{-1}(Z) = E \cup F \subset \bar{Y},$$

where F is the strict transform of $Z \times \mathbb{C}$. There is an \mathbb{S}^1 -action on \bar{Y} defined by

$$(2.5.6) \quad (\mathbf{w}, u, [v_0 : v_1]) \mapsto \left(\mathbf{w}, e^{\sqrt{-1}\theta} u, [v_0 : e^{-\sqrt{-1}\theta} v_1] \right).$$

Let

$$(2.5.7) \quad D := \{(\mathbf{w}, u, [v_0 : v_1]) \in \bar{Y} \mid v_0 = 0\} \cong N_{\mathbb{C}^\times}$$

be the strict transform of the divisor

$$(2.5.8) \quad \bar{D} := \{(\mathbf{w}, u) \in N_{\mathbb{C}^\times} \times \mathbb{C} \mid u = 0\},$$

and write its complement as

$$(2.5.9) \quad Y := \overline{Y} \setminus D \cong \{(\mathbf{w}, u, v) \in N_{\mathbb{C}^\times} \times \mathbb{C}^2 \mid h(\mathbf{w}) = uv\},$$

where $v = v_1/v_0$. The restrictions of (2.5.3) to Y will be denoted by

$$(2.5.10) \quad \pi_{N_{\mathbb{C}^\times}} := \overline{\pi}_{N_{\mathbb{C}^\times}}|_Y : Y \rightarrow N_{\mathbb{C}^\times}, \quad \pi_{\mathbb{C}} := \overline{\pi}_{\mathbb{C}}|_Y : Y \rightarrow \mathbb{C}.$$

We also write

$$(2.5.11) \quad \overline{\pi}_{N_{\mathbb{R}}} := \text{Log} \circ \overline{\pi}_{N_{\mathbb{C}^\times}} : \overline{Y} \rightarrow N_{\mathbb{R}}, \quad \pi_{N_{\mathbb{R}}} := \text{Log} \circ \pi_{N_{\mathbb{C}^\times}} : Y \rightarrow N_{\mathbb{R}}.$$

A tubular neighborhood of D in \overline{Y} is given by

$$(2.5.12) \quad U_D = \{(\mathbf{w}, u, [v_0 : 1]) \in N_{\mathbb{C}^\times} \times \mathbb{C} \times \mathbb{P}^1 \mid u = h(\mathbf{w})v_0, |v_0| < \delta\}$$

for a small positive number δ . We identify U_D with $N_{\mathbb{C}^\times} \times \mathbb{D}_\delta$ by the map

$$(2.5.13) \quad \begin{array}{ccc} U_D & \rightarrow & N_{\mathbb{C}^\times} \times \mathbb{D}_\delta \\ \Downarrow & & \Downarrow \\ (\mathbf{w}, u, [v_0 : 1]) & \mapsto & (\mathbf{w}, v_0) \end{array}$$

where $\mathbb{D}_\delta := \{v_0 \in \mathbb{C} \mid |v_0| < \delta\}$ is an open disk of radius δ . The projection will be denoted by

$$(2.5.14) \quad \pi_{\mathbb{D}_\delta} : U_D \cong N_{\mathbb{C}^\times} \times \mathbb{D}_\delta \rightarrow \mathbb{D}_\delta.$$

For a function f on an almost complex manifold, we set

$$(2.5.15) \quad d^c f := df \circ J,$$

so that $-dd^c f = 2\sqrt{-1}\partial\bar{\partial}f$ whenever J is integrable (or more generally if f is pulled back from an integrable complex manifold along a J -holomorphic map).

We consider the two-form

$$(2.5.16) \quad \omega_\epsilon := p^* \left(\omega_{N_{\mathbb{C}^\times} \times \mathbb{C}} - \frac{\epsilon}{4\pi} dd^c (\chi(\mathbf{w}, u) \log(|u|^2 + |h(\mathbf{w})|^2)) \right)$$

on $\overline{Y} \setminus p^*(Z \times 0)$ for a sufficiently small ϵ , where

$$(2.5.17) \quad \omega_{N_{\mathbb{C}^\times} \times \mathbb{C}} = \frac{\sqrt{-1}}{2} \left(du \wedge d\bar{u} + \frac{dw_1}{w_1} \wedge \frac{d\bar{w}_1}{\bar{w}_1} + \frac{dw_2}{w_2} \wedge \frac{d\bar{w}_2}{\bar{w}_2} \right)$$

is the standard symplectic form on $N_{\mathbb{C}^\times} \times \mathbb{C}$, and the function $\chi : N_{\mathbb{C}^\times} \times \mathbb{C} \rightarrow [0, 1]$ is a smooth \mathbb{S}^1 -invariant cut-off function supported on the tubular neighborhood $U_{Z \times 0}$ of $Z \times 0$ and satisfying $\chi \equiv 1$ in a smaller tubular neighborhood $U'_{Z \times 0}$ of $Z \times 0$. We require that

$$(2.5.18) \quad \chi|_{U_i} = \chi_i \circ G_i$$

for a function $\chi_i : \mathbb{R}^{\geq 0} \rightarrow [0, 1]$, where

$$(2.5.19) \quad G_i : U_i \rightarrow \mathbb{R}^{\geq 0}, \quad (\mathbf{w}, u) \mapsto \frac{|u|^2}{|\mathbf{w}^{(\alpha_i - \beta_i)^+}|^{2c'_i}} + |t^{-\nu(\alpha_i)} + t^{-\nu(\beta_i)} \mathbf{w}^{\beta_i - \alpha_i}|^2.$$

For clarity, we emphasize that in (2.5.16) the operator d^c is defined with respect to the almost complex structure $J_{N_{\mathbb{C}^\times}}$ adapted to Z in the sense of Definition 2.4. A crucial feature for us is that in the neighborhoods U_i , this form is actually invariant under the \mathbb{T}^2 -action which preserves the monomial $\mathbf{w}^{\beta_i - \alpha_i}$. This 2-form extends to a 2-form on \overline{Y} , which we write as ω_ϵ again by abuse of notation, since we may rewrite (2.5.16) as

$$(2.5.20) \quad \omega_\epsilon = p^* \omega_{N_{\mathbb{C}^\times} \times \mathbb{C}} + \frac{\sqrt{-1}\epsilon}{2\pi} \partial\bar{\partial} (\log(|v_0|^2 + |v_1|^2))$$

when $\chi(\mathbf{w}, u) = 1$.

Proposition 2.6. *The two-form ω_ϵ is a symplectic form for sufficiently small ϵ .*

Proof. When $\chi = 1$, the symplectic form is the restriction of the form

$$(2.6.1) \quad \omega_\epsilon = p^* \omega_{N_{\mathbb{C}^\times} \times \mathbb{C}} + \frac{\sqrt{-1}\epsilon}{2\pi} \partial \bar{\partial} (\log (|v_0|^2 + |v_1|^2))$$

on $N_{\mathbb{C}^\times} \times \mathbb{C} \times \mathbb{P}^1$ to \bar{Y} . It follows that, whenever $\chi = 1$, ω_ϵ is the restriction of a compatible symplectic form (2.6.1) to an almost complex submanifold, and hence symplectic in this region. When $\chi(\mathbf{w}, u) \neq 1$, the first term in (2.5.16) is a symplectic form. Along the legs, one explicitly checks that the forms $d \log (|u|^2 + |h(\mathbf{w})|^2)$, $d^c \log (|u|^2 + |h(\mathbf{w})|^2)$ and $dd^c \log (|u|^2 + |h(\mathbf{w})|^2)$ have bounded coefficients in the forms generated by du , $d\bar{u}$, $d \log w_1$, $d \log \bar{w}_1$, $d \log w_2$, and $d \log \bar{w}_2$ when $\chi(\mathbf{w}, u) \neq 1$. This implies that $dd^c (\chi(\mathbf{w}, u) \log (|u|^2 + |h(\mathbf{w})|^2))$ also has bounded coefficients and hence ω_ϵ is a symplectic form for sufficiently small ϵ . \square

Remark 2.7. We make one comment concerning the choice of symplectic form here and that in [AAK16]. Observe that the above construction could be repeated by choosing $U_{Z_{t,0}}$ a suitable tubular neighborhood of $Z_{t,0} \subset N_{\mathbb{C}^\times}$, $U_{Z_{t,0} \times 0}$ a tubular neighborhood of $Z_{t,0} \times 0$ such that $\text{pr}_1(U_{Z_{t,0} \times 0}) \subset U_{Z_{t,0}}$ and $\chi(\mathbf{w}, u)$ be a smooth function whose support is in $U_{Z_{t,0} \times 0}$ and such that $dd^c \chi(\mathbf{w}, u) \log (|u|^2 + |h_{t,0}|^2)$ has bounded coefficients whenever $\chi(\mathbf{w}, u) \neq 1$. Let $Y_{t,1}$ be the variety defined by $uv = h_{t,0}$ and equip it with a symplectic form as in (2.5.16) with $h(\mathbf{w})$ replaced by $h_{t,1}(\mathbf{w})$. It follows from [Abo06, Proposition 4.9] that for ϵ sufficiently small and t sufficiently large and suitable $\chi(\mathbf{w}, u)$, $U_{Z_{t,i}}$, $i \in \{0, 1\}$, there is a symplectomorphism $\phi_{0,1}: Y_{t,0} \cong Y_{t,1}$ which is the identity away from the preimage of a tubular neighborhood of $Z_{t,0}$ of small size.

We fix a convenient choice of a primitive θ_ϵ for the restriction of ω_ϵ to Y , which we write ω_ϵ by abuse of notation. The form

$$(2.7.1) \quad \theta_{\text{vc}} := -\frac{\epsilon}{4\pi} d^c (\chi(\mathbf{w}, u) \log (|u|^2 + |h(\mathbf{w})|^2) - \log(|u|^2))$$

is well-defined on the subset $Y \subset \bar{Y}$ and gives a primitive for the form

$$(2.7.2) \quad -\frac{\epsilon}{4\pi} dd^c (\chi(\mathbf{w}, u) \log (|u|^2 + |h(\mathbf{w})|^2)).$$

Now we define

$$(2.7.3) \quad \theta_\epsilon := p^* \theta_{N_{\mathbb{C}^\times} \times \mathbb{C}} + \theta_{\text{vc}},$$

where $\theta_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ is the standard primitive of $\omega_{N_{\mathbb{C}^\times} \times \mathbb{C}}$, so that

$$(2.7.4) \quad d\theta_\epsilon = \omega_\epsilon.$$

The \mathbb{S}^1 -action (2.5.6) is Hamiltonian with respect to the symplectic form (2.5.16) with the moment map

$$(2.7.5) \quad \mu = \pi |u|^2 + \frac{\epsilon}{2} |u| \frac{\partial}{\partial |u|} (\chi(\mathbf{w}, u) \log (|h(\mathbf{w})|^2 + |u|^2)).$$

Our conventions for the moment map follow those of [AAK16] (in particular, it differs from the more standard convention by a factor of 2π). This formula specializes to

$$(2.7.6) \quad \mu = \begin{cases} \pi |u|^2 + \epsilon \frac{|u|^2}{|h(\mathbf{w})|^2 + |u|^2} & \text{where } \chi \equiv 1 \text{ (near } E), \\ \pi |u|^2 & \text{where } \chi \equiv 0 \text{ (away from } E). \end{cases}$$

The level set $\mu^{-1}(\lambda)$ is smooth unless $\lambda = \epsilon$, where it is singular along the fixed locus

$$(2.7.7) \quad \tilde{Z} = \{(\mathbf{w}, u, [v_0 : v_1]) \in \bar{Y} \mid h(\mathbf{w}) = u = v_1 = 0\}.$$

The level set $\mu^{-1}(0)$ is the divisor D defined in (2.5.7). The fiber $\pi_{N_{\mathbb{C}^\times}}^{-1}(\mathbf{w})$ over any point $\mathbf{w} \notin Z$ is a smooth conic. Let $d\theta$ denote the natural \mathbb{S}^1 -invariant angular one-form on the smooth fiber $\pi_{N_{\mathbb{C}^\times}}^{-1}(\mathbf{w})$. Then the primitive θ_ϵ restricted to the smooth fiber is given by

$$(2.7.8) \quad \theta_\epsilon|_{\pi_{N_{\mathbb{C}^\times}}^{-1}(\mathbf{w})} = |u| \frac{\partial}{\partial |u|} \left(\frac{1}{4} |u|^2 + \frac{\epsilon}{4\pi} \chi(\mathbf{w}, u) \log(|h(\mathbf{w})|^2 + |u|^2) - \frac{\epsilon}{4\pi} \log(|u|^2) \right) d\theta.$$

In view of (2.7.5) and (2.7.1), we may rewrite this in the much simpler form:

$$(2.7.9) \quad \theta_\epsilon|_{\pi_{N_{\mathbb{C}^\times}}^{-1}(\mathbf{w})} = \frac{1}{2\pi} (\mu - \epsilon) d\theta.$$

The same formula holds when $\mathbf{w} \in Z$, away from the singular points of $\pi_{N_{\mathbb{C}^\times}}^{-1}(\mathbf{w})$. For $\lambda \in \mathbb{R}^{>0} \setminus \{\epsilon\}$, the map $\pi_{N_{\mathbb{C}^\times}} : Y \rightarrow N_{\mathbb{C}^\times}$ induces a natural identification

$$(2.7.10) \quad Y_{\text{red}, \lambda} := \mu^{-1}(\lambda) / \mathbb{S}^1 \cong N_{\mathbb{C}^\times}$$

of the reduced space and $N_{\mathbb{C}^\times}$. The resulting reduced symplectic form $\omega_{\text{red}, \lambda}$ on $N_{\mathbb{C}^\times}$ can be averaged by the action of the torus $N_{\mathbb{S}^1} := N_{\mathbb{R}} / N$ to obtain a torus-invariant symplectic form $\omega'_{N_{\mathbb{C}^\times}, \lambda}$. [AAK16, Lemma 4.1] states that there exists a family $(\phi_\lambda)_{\lambda \in \mathbb{R}^{>0}}$ of diffeomorphisms of $N_{\mathbb{C}^\times}$ such that

- $\phi_\lambda^* \omega'_{N_{\mathbb{C}^\times}, \lambda} = \omega_{\text{red}, \lambda}$,
- $\phi_\lambda = \text{id}$ at every point whose $N_{\mathbb{S}^1}$ -orbit is disjoint from the support of χ .
- ϕ_λ depends on λ in a piecewise smooth manner.

We set $\pi_\lambda := \text{Log} \circ \phi_\lambda : N_{\mathbb{C}^\times} \rightarrow N_{\mathbb{R}}$ and define a continuous, piecewise smooth map by

$$(2.7.11) \quad \pi_B : Y \rightarrow B := N_{\mathbb{R}} \times \mathbb{R}^{>0}, \quad x \in \mu^{-1}(\lambda) \mapsto (\pi_\lambda([x]), \lambda).$$

One can easily see as in [AAK16, Section 4.2] that fibers of π_B^{-1} are smooth Lagrangian tori outside of the *discriminant locus* $\text{Log} \circ \phi_\epsilon(Z) \times \{\epsilon\}$.

2.8. SYZ mirror construction. We continue to follow [AAK16] in this subsection; see also [Aur07, Aur09, CLL12] for closely related constructions. The critical locus of the SYZ fibration $\pi_B : Y \rightarrow B$ is given by $Z \times \{(0, 0)\} \subset N_{\mathbb{C}^\times} \times \mathbb{C}^2$, which is the fixed locus of the \mathbb{S}^1 -action. Hence the discriminant locus of π_B is given by $\Gamma = \Pi' \times \{\epsilon\} \subset B$, where $\Pi' := \pi_\epsilon(Z) \subset N_{\mathbb{R}}$ is essentially the amoeba of Z , except that the map π_ϵ differs from the logarithm map Log by ϕ_ϵ . The complement of the discriminant locus will be denoted by $B^{\text{sm}} := B \setminus \Gamma$. The SYZ fibration induces an integral affine structure on B^{sm} . The corresponding local integral affine coordinates $\{x_j\}_{j=1}^3$ give local systems $T_{\mathbb{Z}} B^{\text{sm}}$ and $T_{\mathbb{Z}}^* B^{\text{sm}}$, generated by $\{\partial/\partial x_j\}_{j=1}^3$ and $\{dx_j\}_{j=1}^3$ respectively.

A choice of a section of π_B induces a symplectomorphism

$$(2.8.1) \quad \pi^{-1}(B^{\text{sm}}) \cong T^* B^{\text{sm}} / T_{\mathbb{Z}}^* B^{\text{sm}}$$

given by the action-angle coordinates [Dui80]. The *semi-flat mirror* of Y is defined by

$$(2.8.2) \quad \check{Y}^{\text{sf}} := T B^{\text{sm}} / T_{\mathbb{Z}} B^{\text{sm}},$$

equipped with the natural complex structure $J_{\check{Y}^{\text{sf}}}$ such that the holomorphic coordinates are given by $\{z_j = \exp 2\pi(x_j + \sqrt{-1}y_j)\}_{j=1}^3$. Here $\{y_j\}_{j=1}^3$ are the coordinates on the

fiber corresponding to $\{x_j\}_{j=1}^3$. To obtain the SYZ mirror \check{Y} , one first corrects the semi-flat complex structure by contributions of the holomorphic disks bounded by Lagrangian torus fibers, and then adds fibers over $\Gamma = B \setminus B^{\text{sm}}$.

Instead of correcting complex structures of the semi-flat mirror, [AAK16] considers the subset $B^{\text{reg}} \subset B$ obtained by removing $\pi(p^{-1}(U_Z \times \mathbb{C}))$ from B . Here $U_Z \subset N_{\mathbb{C}^\times}$ is a sufficiently small neighborhood of Z containing the support of χ . The connected components of B^{reg} are in one-to-one correspondence with elements of A , and all fibers over B^{reg} are *tautologically unobstructed* (i.e., they do not bound any non-constant holomorphic disks). Let U_α denote the connected component of B^{reg} corresponding to $\alpha \in A$. The semi-flat mirror $TU_\alpha/T_{\mathbb{Z}}U_\alpha$ with coordinates $(z_{\alpha,1}, z_{\alpha,2}, z_{\alpha,3})$ can be completed to a torus $\tilde{U}_\alpha := \text{Spec } \mathbb{C}[z_{\alpha,1}^{\pm 1}, z_{\alpha,2}^{\pm 1}, z_{\alpha,3}^{\pm 1}]$. Motivated by the counting of Maslov index two disks in a partial compactification of Y , [AAK16] glues \tilde{U}_α and \tilde{U}_β for $\alpha, \beta \in A$ together by

$$(2.8.3) \quad \begin{aligned} z_{\alpha,1} &= (1 + z_{\beta,3})^{\beta_1 - \alpha_1} z_{\beta,1}, \\ z_{\alpha,2} &= (1 + z_{\beta,3})^{\beta_2 - \alpha_2} z_{\beta,2}, \\ z_{\alpha,3} &= z_{\beta,3}. \end{aligned}$$

These local coordinates are related to coordinates (w_1, w_2, w_3) of the dense torus by

$$(2.8.4) \quad \begin{aligned} z_{\alpha,1} &= w_1 w_3^{-\alpha_1}, \\ z_{\alpha,2} &= w_2 w_3^{-\alpha_2}, \\ z_{\alpha,3} &= w_3 - 1. \end{aligned}$$

Let Σ be the fan in $M_{\mathbb{R}} \oplus \mathbb{R}$ associated with the coherent unimodular triangulation \mathcal{P} , and X_Σ be the associated toric variety. Let further K be an anticanonical divisor in X_Σ defined by the function $p := \chi_{\mathbf{0},1} - 1$, where $\chi_{\mathbf{n},k} : X_\Sigma \rightarrow \mathbb{C}$ is the function associated with the character $(\mathbf{n}, k) \in N \oplus \mathbb{Z}$ of the dense torus of X_Σ . By adding torus-invariant curves to $\bigcup_{\alpha \in A} \tilde{U}_\alpha$, one obtains the complement $\check{Y} := X_\Sigma \setminus K$ of the anti-canonical divisor [AAK16, Theorem 1.7].

2.9. Coordinate ring of the mirror manifold. One has $H^0(\mathcal{O}_{X_\Sigma}) = \bigoplus_{(\mathbf{n},k) \in \mathcal{C}} \chi_{\mathbf{n},k}$ where

$$(2.9.1) \quad \mathcal{C} := \{(\mathbf{n}, \ell) \in N \oplus \mathbb{Z} \mid \mathbf{n}(\mathbf{m}) + \ell \geq 0 \text{ for any } \mathbf{m} \in A\}.$$

If we define a function $\ell_1 : N \rightarrow \mathbb{Z}$ by

$$(2.9.2) \quad \ell_1(\mathbf{n}) := \min \{\ell \in \mathbb{Z} \mid (-\mathbf{n}, \ell) \in \mathcal{C}\},$$

then the set $\{p^i \chi_{-\mathbf{n}, \ell_1(\mathbf{n})}\}_{(\mathbf{n}, i) \in N \times \mathbb{Z}}$ forms a basis of the algebra $H^0(\mathcal{O}_{\check{Y}}) = H^0(\mathcal{O}_{X_\Sigma})[p^{-1}]$. The product structure is given by

$$(2.9.3) \quad p^i \chi_{-\mathbf{n}, \ell_1(\mathbf{n})} \cdot p^{i'} \chi_{-\mathbf{n}', \ell_1(\mathbf{n}')} = p^{i+i'} \chi_{-\mathbf{n}-\mathbf{n}', \ell_1(\mathbf{n})+\ell_1(\mathbf{n}')}$$

$$(2.9.4) \quad = p^{i+i'} \chi_{\mathbf{0}, \ell_2(\mathbf{n}, \mathbf{n}')} \cdot \chi_{-\mathbf{n}-\mathbf{n}', \ell_1(\mathbf{n}+\mathbf{n}')}$$

$$(2.9.5) \quad = p^{i+i'} (1+p)^{\ell_2(\mathbf{n}, \mathbf{n}')} \cdot \chi_{-\mathbf{n}-\mathbf{n}', \ell_1(\mathbf{n}+\mathbf{n}')}$$

$$(2.9.6) \quad = \sum_{j=0}^{\ell_2(\mathbf{n}, \mathbf{n}')} \binom{\ell_2(\mathbf{n}, \mathbf{n}')} {j} p^{i+i'+j} \chi_{-\mathbf{n}-\mathbf{n}', \ell_1(\mathbf{n}+\mathbf{n}')} ,$$

where the function $\ell_2 : N \times N \rightarrow \mathbb{Z}$ is defined by

$$(2.9.7) \quad \ell_2(\mathbf{n}, \mathbf{n}') = \ell_1(\mathbf{n}) + \ell_1(\mathbf{n}') - \ell_1(\mathbf{n} + \mathbf{n}').$$

3. BASE-ADMISSIBLE LAGRANGIAN SECTIONS

3.1. Liouville domains. A pair (X^{in}, θ) of a compact manifold X^{in} with boundary and a one-form θ on X^{in} is called a *Liouville domain* if

- $\omega := d\theta$ is a symplectic form on X^{in} ,
- the Liouville vector field V_θ , determined uniquely by the condition $\iota_{V_\theta}\omega = \theta$, points strictly outward along ∂X^{in} .

The one-form θ is called the *Liouville one-form*. The manifold

$$(3.1.1) \quad \widehat{X^{\text{in}}} := X^{\text{in}} \cup_{\partial X^{\text{in}}} [1, \infty) \times \partial X^{\text{in}}$$

obtained by gluing the positive symplectization $([1, \infty) \times \partial X^{\text{in}}, d(r(\theta|_{\partial X^{\text{in}}}))$ of the contact manifold $(\partial X^{\text{in}}, \theta|_{\partial X^{\text{in}}})$ to X^{in} along ∂X^{in} is called the *Liouville completion* of (X^{in}, θ) . An exact symplectic manifold obtained as the Liouville completion of a Liouville domain will be called a *Liouville manifold*. The extension of the one-form θ to X will be denoted by θ again by abuse of notation. The coordinate r on the symplectization end $[1, \infty) \times \partial X^{\text{in}}$ corresponding to $[1, \infty)$ is called the *Liouville coordinate*.

If (X, J) is a Stein manifold with an exhaustive plurisubharmonic function $S: X \rightarrow \mathbb{R}$ whose critical values are less than $K \in \mathbb{R}$, then the manifold $X^{\text{in}} := S^{-1}((-\infty, K])$ is a Liouville domain with a Liouville one-form $\theta := -d^c S$. Under the additional assumption that the gradient flow of S is complete, the Liouville completion may be identified with X .

3.2. Base-admissible Lagrangian sections. Let S be an exhaustive plurisubharmonic function on $N_{\mathbb{C}^\times}$ defined by

$$(3.2.1) \quad S(\mathbf{w}) = \frac{1}{2}|\mathbf{r}|^2 = \frac{1}{2}(r_1^2 + r_2^2)$$

in the logarithmic coordinates $\mathbf{w} = (w_1, w_2) = (e^{r_1 + \sqrt{-1}\theta_1}, e^{r_2 + \sqrt{-1}\theta_2})$. One has

$$(3.2.2) \quad dS = r_1 dr_1 + r_2 dr_2$$

and

$$(3.2.3) \quad \theta_{N_{\mathbb{C}^\times}} := -d^c S = r_1 d\theta_1 + r_2 d\theta_2.$$

Let $L_0 := N_{\mathbb{R}^{>0}} \times \mathbb{R}^{>0}$ be the positive real locus of $N_{\mathbb{C}^\times} \times \mathbb{C}^\times = (N_{\mathbb{C}^\times} \times \mathbb{C}) \setminus \overline{D}$, which is a Lagrangian submanifold diffeomorphic to \mathbb{R}^3 . Since L_0 is disjoint from the tubular neighborhood $U_{Z \times 0} \subset N_{\mathbb{C}^\times} \times \mathbb{C}$ of the center $Z \times 0$ of the blow-up, it lifts to a Lagrangian submanifold of Y . By abuse of notation, we write the lifted Lagrangian again as L_0 . More generally, we consider the following type of Lagrangians:

Definition 3.3. An exact Lagrangian section L of the SYZ fibration (2.7.11) of Y is *base-admissible* if the following conditions are satisfied:

- L is fibered over a Lagrangian submanifold \underline{L} in $N_{\mathbb{C}^\times} \setminus U_Z$;

$$(3.3.1) \quad L = \underline{L} \times \mathbb{R}^{>0} \subset (N_{\mathbb{C}^\times} \setminus U_Z) \times \mathbb{C} \subset Y.$$

- \underline{L} is Legendrian at infinity, i.e., $\theta_{N_{\mathbb{C}^\times}}|_{\underline{L}} = 0$ outside of a compact set.

It is clear from Definition 3.3 that base-admissible Lagrangian sections of $\pi_B: Y \rightarrow B$ are in one-to-one correspondence with Lagrangian sections \underline{L} of $\text{Log}: N_{\mathbb{C}^\times} \rightarrow N_{\mathbb{R}}$ which are disjoint from U_Z and satisfying the Legendrian condition at infinity.

3.4. Framed Lagrangian sections. For each lattice point $\alpha \in A$, consider the polynomial

$$(3.4.1) \quad h^\alpha(\mathbf{w}) = -t^{-\nu(\alpha)}(1 - \phi_\alpha(\mathbf{w}))\mathbf{w}^\alpha + \sum_{\beta \in A \setminus \{\alpha\}} t^{-\nu(\beta)}(1 - \phi_\beta(\mathbf{w}))\mathbf{w}^\beta$$

obtained by flipping the sign of one term in (2.3.6). The corresponding hypersurface will be denoted by

$$(3.4.2) \quad Z^\alpha := \{\mathbf{w} \in N_{\mathbb{C}^\times} \mid h_{t,1}^\alpha(\mathbf{w}) = 0\}.$$

Lemma 3.5. *The amoeba of Z^α coincides with that of Z .*

Proof. If $\text{Log}(\mathbf{w}) \in O_\tau$ for $\tau \in \mathcal{P}^{(1)}$ such that $\partial\tau = \{\alpha, \beta\}$, then one has

$$(3.5.1) \quad \begin{aligned} h(\mathbf{w}) &= t^{-\nu(\alpha)}(1 - \phi_\alpha(\mathbf{w}))\mathbf{w}^\alpha + t^{-\nu(\beta)}(1 - \phi_\beta(\mathbf{w}))\mathbf{w}^\beta, \\ h^\alpha(\mathbf{w}) &= -t^{-\nu(\alpha)}(1 - \phi_\alpha(\mathbf{w}))\mathbf{w}^\alpha + t^{-\nu(\beta)}(1 - \phi_\beta(\mathbf{w}))\mathbf{w}^\beta. \end{aligned}$$

If $\text{Log}(\mathbf{w}) \in O_\sigma$ where $\sigma \in \mathcal{P}^{(2)}$ is the simplex whose vertices are $\alpha, \beta, \gamma \in A$, then one has

$$(3.5.2) \quad \begin{aligned} h(\mathbf{w}) &= t^{-\nu(\alpha)}(1 - \phi_\alpha(\mathbf{w}))\mathbf{w}^\alpha + t^{-\nu(\beta)}(1 - \phi_\beta(\mathbf{w}))\mathbf{w}^\beta + t^{-\nu(\gamma)}(1 - \phi_\gamma(\mathbf{w}))\mathbf{w}^\gamma, \\ h^\alpha(\mathbf{w}) &= -t^{-\nu(\alpha)}(1 - \phi_\alpha(\mathbf{w}))\mathbf{w}^\alpha + t^{-\nu(\beta)}(1 - \phi_\beta(\mathbf{w}))\mathbf{w}^\beta + t^{-\nu(\gamma)}(1 - \phi_\gamma(\mathbf{w}))\mathbf{w}^\gamma. \end{aligned}$$

By choosing a coordinate of M in such a way that $\beta - \alpha = (1, 0)$ and $\gamma - \alpha = (0, 1)$, one can easily show that the amoebas are identical in both cases. \square

Choose a sufficiently large t so that the connected components of the complement of $\Pi := \text{Log}(Z)$ are labeled by A as

$$(3.5.3) \quad N_{\mathbb{R}} \setminus \Pi = \coprod_{\alpha \in A} \mathcal{Q}_\alpha$$

just as in (2.1.7). For an interior lattice point $\alpha \in A \cap \text{Int } \Delta$, a *tropical Lagrangian section* is an exact Lagrangian section of the restriction of $\text{Log}: N_{\mathbb{C}^\times} \rightarrow N_{\mathbb{R}}$ to the inverse image of \mathcal{Q}_α with boundary in Z^α which agrees with the parallel transport of ∂L along a segment in \mathbb{C} in a small neighborhood of ∂L [Abo09, Definitions 3.7 and 3.16]. The prototypical example of a tropical Lagrangian section is the restriction of the positive real Lagrangian

$$(3.5.4) \quad \underline{L}_0 := N_{\mathbb{R}^{>0}} = \{\mathbf{w} = (w_1, w_2) \in N_{\mathbb{C}^\times} \mid w_1, w_2 \in \mathbb{R}^{>0}\}$$

to the fibers over \mathcal{Q}_α .

Lemma 3.6. *A tropical Lagrangian section does not intersect a sufficiently small tubular neighborhood U_Z of Z .*

Proof. Since a tropical Lagrangian section is compact and Z is closed, it suffices to show that a tropical Lagrangian section does not intersect Z .

If $\text{Log}(\mathbf{w}) \in O_\tau$ for $\tau \in \mathcal{P}^{(1)}$ such that $\partial\tau = \{\alpha, \beta\}$, then it follows from (3.5.1) that $\mathbf{w}^{\beta-\alpha}$ is in $\mathbb{R}^{>0}$ for $\mathbf{w} \in Z$ and $\mathbb{R}^{<0}$ for $\mathbf{w} \in Z^\alpha$. It follows that a tropical Lagrangian section does not intersect Z in $\text{Log}^{-1}(O_\tau)$ for $\tau \in \mathcal{P}^{(1)}$.

If $\text{Log}(\mathbf{w}) \in O_\sigma$ for $\sigma \in \mathcal{P}^{(2)}$, then a tropical Lagrangian section agrees with the positive real Lagrangian \underline{L}_0 in the neighborhood $\partial\mathcal{Q} \cap O_\sigma$ of the vertex of Π_∞ dual to τ [Abo09, Lemma 3.18]. The positive real Lagrangian is clearly disjoint from Z , and Lemma 3.6 is proved. \square

One can use the complex structure of $N_{\mathbb{C}^\times}$ to view $N_{\mathbb{C}^\times}$ as the trivial $N_{\mathbb{R}}/N$ -bundle $TN_{\mathbb{R}}/T_{\mathbb{Z}}N_{\mathbb{R}}$ over $N_{\mathbb{R}}$, whose universal cover is the trivial $N_{\mathbb{R}}$ -bundle $TN_{\mathbb{R}}$. A section of $TN_{\mathbb{R}}$ can be identified with a function on $N_{\mathbb{R}}$ with values in $N_{\mathbb{R}}$. Since the tropical Lagrangian section agrees with the positive real Lagrangian near $\mathcal{Q} \cap O_\tau$ for $\tau \in \mathcal{P}^{(2)}$, a lift $\mathcal{Q} \rightarrow TN_{\mathbb{R}}$ of a tropical Lagrangian section $\mathcal{Q} \rightarrow N_{\mathbb{C}^\times}$ to the universal cover $TN_{\mathbb{R}} \rightarrow N_{\mathbb{C}^\times} \cong TN_{\mathbb{R}}/T_{\mathbb{Z}}N_{\mathbb{R}}$ takes values in N near $\mathcal{Q} \cap O_\tau$. The Hamiltonian isotopy class of a tropical Lagrangian section is determined by the values $(n_\tau)_{\tau \in \mathcal{P}^{(2)}} \in N^{\mathcal{P}^{(2)}}$ of its lifts near the vertices of Π [Abo09, Proposition 3.20]. Two lifts come from the same section if and only if they are related by an overall shift by N .

For an edge $\tau \in \mathcal{P}^{(1)}$ in the interior of Δ , choose a coordinate of M in such a way that τ is the line segment between $\boldsymbol{\alpha} = (0, 0)$ and $\boldsymbol{\beta} = (1, 0)$. Then one has

$$(3.6.1) \quad h^\alpha(\mathbf{w}) = -t^{-\nu(\boldsymbol{\alpha})} + t^{-\nu(\boldsymbol{\beta})}w_1$$

for $\mathbf{w} \in O_\tau$, so that

$$(3.6.2) \quad \Pi \cap O_\tau = \{(r_1, r_2) \in N_{\mathbb{R}} \mid r_1 = -\nu(\boldsymbol{\alpha}) + \nu(\boldsymbol{\beta})\}.$$

It follows that a tropical Lagrangian section is constant in the w_1 -variable above O_τ . A Lagrangian section $N_{\mathbb{R}} \rightarrow N_{\mathbb{C}^\times}$ is said to be *framed* if its restriction to \mathcal{Q}_α is bounded by Z^α for any interior lattice point $\boldsymbol{\alpha} \in A \cap \text{Int } \Delta$.

If σ and σ' are elements of $\mathcal{P}^{(2)}$ adjacent to an edge $\tau \in \mathcal{P}^{(1)}$ in the interior of Δ , then the condition that the boundary of a Lagrangian section lies in Z implies that

$$(3.6.3) \quad \langle n_\sigma - n_{\sigma'}, \boldsymbol{\alpha} - \boldsymbol{\beta} \rangle = 0.$$

For a collection $(n_\sigma)_{\sigma \in \mathcal{P}^{(2)}} \in N^{\mathcal{P}^{(2)}}$ of elements of N , there exists a framed Lagrangian section whose lift takes the value n_σ on O_σ if and only if (3.6.3) is satisfied for any edge $\tau \in \mathcal{P}^{(1)}$.

3.7. Legendrian condition at infinity. Recall from Definition 3.3 that a Lagrangian submanifold $\underline{L} \subset N_{\mathbb{C}^\times}$ is Legendrian at infinity if $d^c S|_{\underline{L}} = 0$ outside of a compact set. A direct calculation shows that the graph

$$(3.7.1) \quad \Gamma_{df} := \left\{ (r_1, \theta_1, r_2, \theta_2) \in N_{\mathbb{C}^\times} \mid \theta_1 = \frac{\partial f}{\partial r_1}, \theta_2 = \frac{\partial f}{\partial r_2} \right\}$$

of the differential of a function $f: N_{\mathbb{R}} \rightarrow \mathbb{R}$ satisfies the Legendrian condition $d^c S|_{\Gamma_{df}} = 0$ if and only if

$$(3.7.2) \quad \left(r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2} \right) \frac{\partial f}{\partial r_1} = \left(r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2} \right) \frac{\partial f}{\partial r_2} = 0.$$

This happens if f homogeneous of degree one:

$$(3.7.3) \quad \left(r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2} \right) f = f.$$

Proposition 3.8. *Any framed Lagrangian section can be made Legendrian at infinity by a Hamiltonian isotopy.*

Proof. We can choose a framed Lagrangian in such a way that it coincides with the positive real Lagrangian in the neighborhood of each leg of Π outside of a compact set. Then the potential f of the Lagrangian is linear in that neighborhood. Now one can choose arbitrary homogeneous function of degree one which coincides with f in the compact set and in the neighborhood of each leg, and the Lagrangian generated by this function has the desired property. \square

3.9. SYZ transformation. Let L be a base-admissible Lagrangian section of $\pi_B: Y \rightarrow B$ associated with a framed Lagrangian section \underline{L} of $\text{Log}: N_{\mathbb{C}^\times} \rightarrow N_{\mathbb{R}}$. Let further $\tau \in \mathcal{P}^{(1)}$ be an edge in the interior of Δ and $\ell \subset \check{Y}$ be the corresponding torus-invariant curve. We use the same coordinates as in Section 2.8.

For each interior lattice point $\alpha \in A \cap \text{Int } \Delta$, a framed Lagrangian section \underline{L} restricts to a tropical Lagrangian section \underline{L}_α over \mathcal{Q}_α . The fiberwise universal cover of the restriction of $\text{Log}: N_{\mathbb{C}^\times} \rightarrow N_{\mathbb{R}}$ to \mathcal{Q}_α can be identified with $T\mathcal{Q}_\alpha$, with the positive real Lagrangian as the zero-section. We write the lift of \underline{L}_α as the graph of a one-form

$$(3.9.1) \quad \omega = \xi_1 dy_1 + \xi_2 dy_2,$$

where ξ_1 and ξ_2 are functions on $N_{\mathbb{R}}$ satisfying

$$(3.9.2) \quad \frac{\partial \xi_1}{\partial y_2} - \frac{\partial \xi_2}{\partial y_1} = 0.$$

The *semi-flat SYZ transform* of \underline{L}_α is the trivial bundle on $T\mathcal{Q}_\alpha$, equipped with the connection

$$(3.9.3) \quad \nabla_\alpha := d + 2\pi\sqrt{-1}\omega = d + 2\pi\sqrt{-1}(\xi_1 dy_1 + \xi_2 dy_2).$$

In cases without quantum corrections, this gives a holomorphic line bundle mirror to the given Lagrangian section [LYZ00]. In general, however, quantum corrections have to be taken into account [Cha13, CU13, CPU16]. In our case, due to the nontrivial gluing formulas (2.8.3), the semi-flat SYZ transforms of \underline{L}_α and \underline{L}_β do not coincide over the intersection

$$(3.9.4) \quad \tilde{U}_\alpha \cap \tilde{U}_\beta = T\ell/T_{\mathbb{Z}}\ell,$$

where $\ell \subset \Pi_\infty$ is the edge of the intersection of the connected components $\mathcal{Q}_\alpha, \mathcal{Q}_\beta \subset N_{\mathbb{R}} \setminus \Pi$ which is dual to $\tau \in \mathcal{P}^{(1)}$, but are related by

$$(3.9.5) \quad \nabla_\alpha = \nabla_\beta + \sqrt{-1}\langle df, \beta - \alpha \rangle d\arg(1 + z_3),$$

where f is a primitive of \underline{L} , i.e. $\xi_i = \partial f / \partial x_i$ for $i = 1, 2$.

Since \underline{L}_α and \underline{L}_β share the same boundary in Z over ℓ and the defining equation for Z is given as in (2.3.10), we have $k_{\alpha\beta} := \langle df, \beta - \alpha \rangle \in \mathbb{Z}$, so we may modify ∇_β to the *gauge equivalent* connection

$$(3.9.6) \quad \nabla'_\beta := \nabla_\beta + \sqrt{-1}k_{\alpha\beta} d\arg(1 + z_3).$$

Now ∇_α and ∇'_β glue to give a connection $\nabla_{\alpha\beta}$ on the chart $\tilde{U}_\alpha \cup \tilde{U}_\beta \subset \check{Y}$. It is clear that the cocycle condition is satisfied, so the connections $\{\nabla_{\alpha\beta}\}$ define a global $U(1)$ -connection over \check{Y} whose curvature has trivial $(0, 2)$ -part since L is Lagrangian. This produces a holomorphic line bundle $\mathcal{F}(L)$ over \check{Y} , called the *SYZ transform* of L .

To determine the isomorphism class of $\mathcal{F}(L)$, let $\ell \subset \Pi_\infty$ be an edge on the boundary of a connected component $C_{\alpha, \infty} \subset N_{\mathbb{R}} \setminus \Pi_\infty$ of the complement of the tropical curve Π_∞ . We can choose a coordinate on M in such a way that the endpoints of the edge τ is given by $\alpha = (0, 0)$ and $\beta = (1, 0)$. A subset of the torus-invariant curve in \check{Y} associated with the edge $\tau \in \mathcal{P}^{(1)}$ dual to ℓ can naturally be identified with $T\ell/T_{\mathbb{Z}}\ell$. Let $\sigma, \sigma' \in \mathcal{P}^{(2)}$ be the faces adjacent to τ , then the degree of the restriction of $\mathcal{F}(L)$ to $T\ell/T_{\mathbb{Z}}\ell$ is given by

$$(3.9.7) \quad \frac{\sqrt{-1}}{2\pi} \int_{T\ell/T_{\mathbb{Z}}\ell} F_{\nabla'_\alpha}^{1,1} = - \int_{T\ell/T_{\mathbb{Z}}\ell} \frac{\partial \xi_2}{\partial y'_2} dx'_2 \wedge dy'_2 = - \int_\ell \frac{\partial \xi_2}{\partial y'_2} dy'_2 = \xi_2(s_\sigma) - \xi_2(s_{\sigma'}),$$

where $s_\sigma, s_{\sigma'} \in N_{\mathbb{R}}$ are the endpoints of ℓ dual to $\sigma, \sigma' \in \mathcal{P}^{(2)}$. More generally, it can be shown that the degree of the restriction of $\mathcal{F}(L)$ to $T\ell/T_{\mathbb{Z}}\ell$ is given by $\langle df, (\beta - \alpha)^\perp \rangle|_{s_{\sigma'}}^{s_\sigma}$.

This shows that the isomorphism class of $\mathcal{F}(L)$ depends only on the Hamiltonian isotopy class of L , hence proving Theorem 1.1.

4. STANDARD WRAPPED FLOER THEORY

4.1. Basic geometric objects. Let (X, θ) be a Liouville manifold. The induced contact structure on ∂X^{in} will be denoted by $\xi := \ker(\theta|_{\partial X^{\text{in}}})$, and the Liouville coordinate on the symplectization end $[1, \infty) \times \partial X^{\text{in}}$ will be denoted by r . We assume that the canonical bundle of X is trivial, and fix its trivialization.

4.2. The vector field \mathcal{Z} on X dual to the Liouville one-form θ with respect to the symplectic form $d\theta$ is called the *Liouville vector field*. It is given by $r\partial_r$ on the symplectization end.

4.3. The *Reeb vector field* \mathcal{R} on the contact manifold $(\partial X^{\text{in}}, \theta|_{\partial X^{\text{in}}})$ is defined by $\mathcal{R} \in \text{Ker } d\theta$ and $\theta(\mathcal{R}) = 1$.

4.4. A Lagrangian submanifold $L \subset X$ is *Liouville-admissible* if it is the completion $L^{\text{in}} \cup [1, \infty) \times \partial L^{\text{in}}$ of a Lagrangian submanifold $L^{\text{in}} \subset X^{\text{in}}$ such that $\theta|_{L^{\text{in}}} \in \Omega^1(L^{\text{in}})$ vanishes to infinite order along the boundary ∂L^{in} . We assume that all such Lagrangians are spin and are equipped with brane structures in the sense of [Sei08].

4.5. A *Liouville-admissible Hamiltonian* is a positive function $H: X \rightarrow \mathbb{R}^{>0}$ which is λr outside of a compact set for a positive real number λ called the *slope* of H . The set of Liouville-admissible Hamiltonians of slope λ is denoted by $\mathcal{H}_{\text{La}}(X)_\lambda$ and we set $\mathcal{H}_{\text{La}}(X) := \bigcup_{\lambda \in \mathbb{R}^{>0}} \mathcal{H}_{\text{La}}(X)_\lambda$.

4.6. A *Liouville-admissible almost complex structure* is a compatible almost complex which outside of a compact set in the symplectization is the direct sum of an almost complex structure on ξ and the standard complex structure on the rank 2 bundle spanned by the Liouville vector field and the Reeb vector field, i.e., $J\mathcal{Z} = \mathcal{R}$. The set of Liouville-admissible complex structures is denoted by $\mathcal{J}_{\text{La}}(X)$.

4.7. Let $H: X \rightarrow \mathbb{R}$ be a function on a symplectic manifold X and L be a Lagrangian submanifold. A *Hamiltonian chord* is a trajectory $x: [0, 1] \rightarrow X$ of the Hamiltonian flow such that $x(0) \in L$ and $x(1) \in L$. The set of Hamiltonian chords will be denoted by $\mathcal{X}(L, X; H)$. We sometimes write $\mathcal{X}(L; H) := \mathcal{X}(L, X; H)$, if X is clear from the context. We also write the set of Hamiltonian chords in a given relative homotopy class $\gamma \in \pi_1(X, L)$ as

$$(4.7.1) \quad \mathcal{X}(L; H)_\gamma := \{x \in \mathcal{X}(L; H) \mid [x] = \gamma\}.$$

A Hamiltonian chord x is *non-degenerate* if the image $\varphi_1(L)$ of L by the time-one Hamiltonian flow $\varphi_1: X \rightarrow X$ intersects L transversally at the intersection point corresponding to x .

4.8. Let Σ be a closed disc with $d + 1$ boundary punctures $\zeta = \{\zeta_0, \dots, \zeta_d\}$, which are called the *points at infinity*. We denote by $\bar{\Sigma} := \Sigma \cup \zeta$ the closed disk obtained by filling in the punctures. The connected component of the boundary of Σ between ζ_i and ζ_{i+1} , which is homeomorphic to an open interval, will be denoted by $\partial_i \Sigma$. We also write $\partial \Sigma := \bigcup_{i=0}^d \partial_i \Sigma$. The moduli space of such discs and its stable compactification will be denoted by \mathcal{R}^d and $\bar{\mathcal{R}}^d$ respectively.

Remark 4.9. As most of our calculations will actually take place at the cohomological level, we will be mostly interested in the cases when $d \leq 3$ in this paper.

4.10. A *strip-like end* around a point ζ_i is a holomorphic embedding

$$(4.10.1) \quad \begin{cases} \epsilon: \mathbb{R}^{\leq 0} \times [0, 1] \rightarrow \Sigma, \\ \epsilon^{-1}(\partial\Sigma) = \mathbb{R}^{\leq 0} \times \{0, 1\}, \\ \lim_{s \rightarrow -\infty} \epsilon(s, -) = \zeta_i \end{cases}$$

if $i = 0$, and

$$(4.10.2) \quad \begin{cases} \epsilon: \mathbb{R}^{\geq 0} \times [0, 1] \rightarrow \Sigma, \\ \epsilon^{-1}(\partial\Sigma) = \mathbb{R}^{\geq 0} \times \{0, 1\}, \\ \lim_{s \rightarrow \infty} \epsilon(s, -) = \zeta_i \end{cases}$$

otherwise. Strip-like ends can be chosen compatibly over the moduli spaces.

4.11. A *Liouville-admissible Floer data* is a pair

$$(4.11.1) \quad (H, J) \in C^\infty([0, 1], \mathcal{H}_{\text{La}}(X)) \times C^\infty([0, 1], \mathcal{J}_{\text{La}}(X))$$

of families of Liouville-admissible Hamiltonians and Liouville-admissible almost complex structures.

4.12. Let Σ be a closed disk with $d + 1$ boundary punctures. A *Liouville-admissible perturbation data* (K, J) consists of

- (1) a 1-form $K \in \Omega^1(\Sigma, \mathcal{H}_{\text{La}}(X))$ on Σ with values in Liouville-admissible Hamiltonians satisfying

$$(4.12.1) \quad K|_{\partial\Sigma} = 0, \text{ and}$$

$$(4.12.2) \quad \begin{aligned} X_K &= X_{H_e} \otimes \beta \text{ outside of a compact set, where } H_e \text{ is of slope one and} \\ \beta &\text{ is sub-closed (i.e., } d\beta \leq 0\text{), and} \end{aligned}$$

- (2) a family $J \in C^\infty(\Sigma, \mathcal{J}_{\text{La}}(X))$ of Liouville-admissible almost complex structures on X parametrized by Σ .

It is *compatible* with a sequence $(\mathbf{H}, \mathbf{J}) = (H_j, J_j)_{j=0}^d$ of Liouville-admissible Floer data if

$$(4.12.3) \quad \epsilon_j^* K = H_j(t) dt \quad \text{and} \quad J(\epsilon_j(s, t)) = J_j(t)$$

for any $j \in \{0, \dots, d\}$ and any $t \in [0, 1]$.

4.13. A sequence $\mathbf{x} := (x_k \in \mathcal{X}(L; H_k))_{k=0}^d$ of Hamiltonian chords and a perturbation data (K, J) allow us to define *Floer's equation*

$$(4.13.1) \quad \begin{cases} y: \Sigma \rightarrow \overline{Y}, \\ y(\partial\Sigma) \subset L, \\ \lim_{s \rightarrow \pm\infty} y(\epsilon_k(s, -)) = x_k, \quad k = 0, \dots, d, \\ (dy - X_K)^{0,1} = 0, \end{cases}$$

where X_K is the one-form with values in Hamiltonian vector fields on \overline{Y} associated with K . Outside of a compact set, our choices of perturbation data agrees with that studied in [AS10] and so the C^0 estimates of Section 7 of *loc. cit.* still hold.

4.14. Cohomological constructions. For a non-degenerate Liouville-admissible Hamiltonian H , the *Floer complex* is defined by

$$(4.14.1) \quad \mathrm{CF}^*(L; H) := \bigoplus_{x \in \mathcal{X}(L; H)} |\mathfrak{o}_x|,$$

where $|\mathfrak{o}_x|$ is the one-dimensional \mathbb{C} -normalized orientation space associated to x (see [Sei08, Section 12]). For a pair $\mathbf{x} = (x_0, x_1)$ of Hamiltonian chords, the matrix element of the *Floer differential* \mathfrak{m}_1 is defined by counting the number of solutions to Floer's equation (4.13.1) on the strip $\Sigma := \mathbb{R} \times [0, 1]$ with perturbation data $K = Hdt$ up to \mathbb{R} -translations. The cohomology of the Floer complex (4.14.1) is denoted by $\mathrm{HF}^*(L; H)$, and called the *Floer cohomology*.

4.15. A *Liouville-admissible sequence of Hamiltonians* is a sequence $(H_m)_{m=1}^\infty$ of Liouville-admissible Hamiltonians satisfying the following conditions:

- (1) For each $m \in \mathbb{Z}^{>0}$, the set $\mathcal{X}(L; H_m)$ consists only of non-degenerate chords.
- (2) The slopes λ_m of H_m satisfy $\lambda_m < \lambda_{m+1}$ and $\lambda_m + \lambda_{m'} \leq \lambda_{m+m'}$ for any $m, m' \in \mathbb{Z}^{>0}$.

Note that (2) implies $\lim_{m \rightarrow \infty} \lambda_m = \infty$.

4.16. We fix a Liouville-admissible sequence $(H_m)_{m=1}^\infty$ of Hamiltonians and a sequence $(J_m)_{m=1}^\infty$ of Liouville-admissible almost complex structures. In addition, for each $m \in \mathbb{Z}^{>0}$, we fix a Liouville-admissible perturbation data $K(m, m+1)$ on the strip $\mathbb{R} \times [0, 1]$, which is compatible with the pair $((H_m, J_m), (H_{m+1}, J_{m+1}))$ of Floer data. For any $n > m$, by gluing $(K(i, i+1))_{i=m}^{n-1}$, we obtain a perturbation data $K(m, n)$ on the strip $\mathbb{R} \times [0, 1]$, which is compatible with the pair $((H_m, J_m), (H_n, J_n))$ of Floer data. By counting numbers of solutions to Floer's equation (4.13.1) on the strip with respect to this perturbation data, we obtain the *continuation map*

$$(4.16.1) \quad \mathrm{CF}^*(L; H_m) \rightarrow \mathrm{CF}^*(L; H_n)$$

on the Floer cochain complex. A standard argument in Floer theory shows that the continuation map commutes with the Floer differential, and induces the continuation map

$$(4.16.2) \quad \kappa_{m,n} : \mathrm{HF}^*(L; H_m) \rightarrow \mathrm{HF}^*(L; H_n)$$

on the Floer cohomology. The colimit

$$(4.16.3) \quad \mathrm{HW}(L) := \varinjlim_m \mathrm{HF}(L; H_m)$$

with respect to the continuation map (4.16.2) is called the *wrapped Floer cohomology*.

4.17. The most important Floer theoretic operation on wrapped Floer cohomology in this paper will be the pair-of-pants product. To define it, for any $m, n \in \mathbb{Z}^{>0}$, we fix a Liouville-admissible perturbation data $K(m, n, m+n)$ on a disk with three punctures, which is compatible with the triple

$$(4.17.1) \quad ((H_m, J_m), (H_n, J_n), (H_{m+n}, J_{m+n}))$$

of Floer data. This allows us to define a linear map

$$(4.17.2) \quad \mathfrak{m}_2 : \mathrm{CF}^*(L; H_n) \otimes \mathrm{CF}^*(L; H_m) \rightarrow \mathrm{CF}^*(L; H_{m+n})$$

by counting numbers of solutions to Floer's equation. A standard arguments in Floer theory shows that \mathfrak{m}_2 satisfies the Leibniz rule with respect to \mathfrak{m}_1 , and hence induces a

map on the Floer cohomology. As is standard in Floer theory, the fact that this pair-of-pants product is well-behaved comes from the existence of certain auxiliary moduli spaces which are enhanced with suitable choices of Floer data. For example, in order to show that this product is actually well-defined on the direct limit, we make the following construction:

4.18. For any $m_1, m_2, m_3 \in \mathbb{Z}^{>0}$ with $m_1 < m_2$, we fix a one-parameter family $(K_\tau(m_1, m_2, m_3))_{\tau \in [0,1]}$ of Liouville-admissible perturbation data such that $K_0(m_1, m_2, m_3)$ is the gluing of $K(m_1, m_3, m_1 + m_3)$ with $K(m_1 + m_3, m_2 + m_3)$ and $K_1(m_1, m_2, m_3)$ is the gluing of $K(m_1, m_2)$ with $K(m_2, m_3, m_2 + m_3)$. We also fix the analogous data for the case when the continuation map occur along the other positive strip-like end. A standard cobordism argument using this family shows that the product is well-defined on the direct limit.

4.19. For any $m_1, m_2, m_3 \in \mathbb{Z}^{>0}$ and any $n \in \mathbb{Z}^{\geq 0}$, we set $m := \sum_{i=1}^3 m_i + n$. We then choose a family $K(m_1, m_2, m_3)$ of Liouville-admissible perturbation data on the universal family of disks with 4 punctures over the moduli space $\overline{\mathcal{R}}^3$, which is compatible with (H_m, J_m) along the negative end and (H_{m_i}, J_{m_i}) along the three positive ends. We assume that along one end of the boundary of $\overline{\mathcal{R}}^3$, the family $K(m_1, m_2, m_3)$ restricts to the fiber product of a perturbation datum in $K(m_1, m_2, m - m_3)$ (for some datum in the interior of the above homotopy) with $K(m - m_3, m_3, m)$, and at the other end of the boundary, it restricts to the fiber product of a perturbation datum in $K(m_2, m_3, m - m_1)$ (for some choice of data in the interior of the above homotopy) with $K(m_1, m - m_1, m)$. We also require that for sufficiently small gluing parameters, in the ‘thin’ regions of the holomorphic curves, the perturbation data restricts exactly to the gluing of these perturbation data following [Sei08, Section (9i)].

For generic choices of Floer data and perturbation data, all moduli spaces above may be regularized. A standard argument in Floer theory using the moduli space of solutions to Floer’s equation with respect to this perturbation data shows that the product on the Floer cohomology is associative.

Remark 4.20. The extra flexibility in the parameter n above may seem unusual, but is motivated by our intended applications where the theory is not as nicely behaved as in the standard Liouville case.

4.21. Chain level structures. We now explain how to enhance the above constructions to the chain level. The existence of this chain level construction is important when we discuss (the (split-)generation of) the derived category. Still, as most of our computations will take place at the cohomological level, we will be brief and refer the reader to [AS10]. To define the chain level structure, we assume that our Liouville-admissible families H_m satisfy $\lambda_m = m$ and that

$$(4.21.1) \quad H_m(x) = \lambda_m r$$

in the region of X defined by $r \geq 2$. We similarly assume that all our Liouville-admissible Lagrangians L are conical over this region, and that for any perturbation data, (4.12.2) holds over this region as well.

Definition 4.22. The *wrapped Floer complex* of a Liouville-admissible Lagrangian L is defined by

$$(4.22.1) \quad \text{CW}^*(L) := \bigoplus_m \text{CF}^*(L, H_m)[q],$$

where $\deg q = -1$ and $q^2 = 0$.

We fix perturbation data of the form $K(m, m+1)$ and define a differential on this complex via the formula

$$(4.22.2) \quad \mu^1(x + qy) = (-1)^{|x|} \mathbf{m}_1(x) + (-1)^{|y|} (q\mathbf{m}_1(y) + \kappa_{m, m+1}(y) - y).$$

The cohomology of this complex gives the wrapped Floer cohomology defined in (4.16.3);

$$(4.22.3) \quad H^*(\text{CW}^*(L)) \cong \text{HW}^*(L).$$

Fix $d \geq 1$ and labels $p_f \in \{1, \dots, d\}$ (possibly not distinct) indexed by a finite set F . Let $\mathbf{p}: F \rightarrow \{1, \dots, d\}$ be the map given by $f \mapsto p_f$.

Definition 4.23. A \mathbf{p} -flavored popsicle is a disk $\Sigma \in \mathcal{R}^d$ with $d+1$ boundary punctures together with holomorphic maps $\phi_f: \Sigma \rightarrow Z$ to the strip $Z := \mathbb{R} \times [0, 1]$ which extend to an isomorphism $\bar{\Sigma} \xrightarrow{\sim} \bar{Z}$ such that $\phi_f(z_0) = -\infty$ and $\phi_f(z_{p_f}) = \infty$. The moduli space of \mathbf{p} -flavored popsicles is denoted by $\mathcal{R}^{d, \mathbf{p}}$.

This moduli space admits a compactification $\bar{\mathcal{R}}^{d, \mathbf{p}}$ over stable discs. Moreover, we can choose universally consistent strip-like ends over $\bar{\mathcal{R}}^{d, \mathbf{p}}$ in the sense of [AS10, Section 2.4]. Observe that if \mathbf{p} is not injective, then there is a symmetry group of $\text{Sym}^{\mathbf{p}}$ of permutations of F preserving \mathbf{p} . This admits a natural action on $\mathcal{R}^{d, \mathbf{p}}$ which extends to the compactification $\bar{\mathcal{R}}^{d, \mathbf{p}}$. Fix flavors \mathbf{p} and weights $\mathbf{m} = (m_0, \dots, m_d) \in (\mathbb{Z}^{>0})^{d+1}$ satisfying

$$(4.23.1) \quad m_0 = \sum_{i=1}^d m_i + |F|.$$

We denote by $\bar{\mathcal{R}}^{d, \mathbf{p}, \mathbf{m}}$ the moduli space of popsicles with weights \mathbf{m} (although this is just a copy of $\bar{\mathcal{R}}^{d, \mathbf{p}}$, it is useful to separate these). We equip this with perturbation data compatible with the strip-like ends $H_i dt$ and admissible complex structures J_t .¹ This data is chosen universally consistently and equivariantly with respect to the $\text{Sym}^{\mathbf{p}}$ actions. Denote these choices by $(K^{\mathbf{p}, \mathbf{m}}, J^{\mathbf{p}, \mathbf{m}})$.

Given a collection of chords $x_i \in \mathcal{X}(L; H_i)$, we can form the moduli space $\bar{\mathcal{R}}^{d, \mathbf{p}, \mathbf{m}}(\mathbf{x})$ of solutions to Floer's equation. With generic choices of $(K^{\mathbf{p}, \mathbf{m}}, J^{\mathbf{p}, \mathbf{m}})$, these spaces of maps have the expected dimension. Whenever the expected dimension is zero, counting these solutions with appropriate signs gives rise to operations $\mu^{\mathbf{p}, \mathbf{m}}$. Out of these operations, Abouzaid and Seidel constructs an A_∞ -structure on the Floer complex $\text{CW}^*(L, L)$.

These constructions can easily be adapted to a collection of Liouville-admissible Lagrangians, giving rise to an A_∞ category $\mathcal{W}(X)$ whose objects are these Lagrangians and whose morphisms are the Floer complexes. Finally, we embed this, via the Yoneda embedding, into the larger category $D^\pi \mathcal{W}(X) := \text{perf } \mathcal{W}(X)$ of perfect A_∞ -modules over this category. The smallest full triangulated subcategory of $D^\pi \mathcal{W}(X)$ containing $\mathcal{W}(X)$ will be denoted by $D^b \mathcal{W}(X)$.

Remark 4.24. The reader will note that we have used slightly more general Hamiltonians than those of the form $H_m = mH$ for a fixed admissible Hamiltonian. This is for two reasons, the first being that as one of their genericity constraints on the Hamiltonians, Abouzaid and Seidel [AS10, (39)] impose the condition that no point on L is both an endpoint and a starting point of a Hamiltonian chord. This is to rule out certain solutions of zero geometric energy which can roughly speaking be thought of as constant curves

¹Some care must be taken in the choice of complex structure over the moduli space, see [AS10, Section 3.2].

landing in triple intersections of the Lagrangians after being perturbed by the Hamiltonian flow (these solutions are problematic for transversality). For the Hamiltonians we wish to choose, all the Hamiltonian chords on the cylindrical end are Hamiltonian orbits. However, in our setting, because we can choose more flexible families of Hamiltonians, we can choose our data so that such constant curves are excluded a priori. The second consideration is practical: by working with slightly more general Hamiltonians, we can obtain smaller models for the Floer cohomology.

5. ADAPTED WRAPPED FLOER THEORY

5.1. Let $S: N_{\mathbb{C}^\times} \rightarrow \mathbb{R}$ be the standard exhaustive plurisubharmonic function defined in (3.2.1). A function $H: N_{\mathbb{C}^\times} \rightarrow \mathbb{R}$ is *homogeneous of degree one* with respect to S if

$$(5.1.1) \quad -d^c S(X_H) = H, \text{ and}$$

$$(5.1.2) \quad dS(X_H) = 0.$$

5.2. A positive function $H_b: N_{\mathbb{C}^\times} \rightarrow \mathbb{R}^{>0}$ is an *admissible base Hamiltonian* if it is homogeneous of degree one outside of a compact set.

5.3. For a function $H: N_{\mathbb{R}} \rightarrow \mathbb{R}$, the composition $H \circ \text{Log}: N_{\mathbb{C}^\times} \rightarrow \mathbb{R}$ will also be denoted by H by abuse of notation. The Hamiltonian vector field is given by

$$(5.3.1) \quad X_H = \frac{\partial H}{\partial r_1} \frac{\partial}{\partial \theta_1} + \frac{\partial H}{\partial r_2} \frac{\partial}{\partial \theta_2}.$$

One has

$$(5.3.2) \quad -d^c S(X_H) = r_1 \frac{\partial H}{\partial r_1} + r_2 \frac{\partial H}{\partial r_2},$$

so that $-d^c S(X_H) = H$ if and only if H is homogeneous of degree one in the usual sense.

5.4. A positive function $H_{\text{ba}}: \bar{Y} \rightarrow \mathbb{R}^{>0}$ is a *base-admissible Hamiltonian* if there exists an admissible base Hamiltonian $H_b: N_{\mathbb{C}^\times} \rightarrow \mathbb{R}^{>0}$ and a compact set $K \subset N_{\mathbb{C}^\times}$ such that

- for any $y \in \bar{Y} \setminus \bar{\pi}_{N_{\mathbb{C}^\times}}^{-1}(K)$, one has

$$(5.4.1) \quad (\bar{\pi}_{N_{\mathbb{C}^\times}})_*(X_{H_{\text{ba}}}(y)) = X_{H_b}(\bar{\pi}_{N_{\mathbb{C}^\times}}(y)).$$

- Outside of $\bar{\pi}_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$, H_{ba} is a C^2 -small perturbation of $\bar{\pi}_{N_{\mathbb{C}^\times}}^{-1}(H_b)$

The set of base-admissible Hamiltonians on \bar{Y} is denoted by $\mathcal{H}_{\text{ba}}(\bar{Y})$.

Proposition 5.5. *There exists a base-admissible Hamiltonian.*

Proof. Recall from (2.5.19) that $\chi|_{U_i} = \chi_i \circ G_i$. The symplectic form on U_i is invariant under the \mathbb{S}^1 -action on $N_{\mathbb{C}^\times}$ which preserves $\mathbf{w}^{\alpha-\beta}$. The essential idea is to produce the base Hamiltonian by gluing together local moment maps for these actions. In more detail, set $F_i = |u|^2 + |h(\mathbf{w})|^2$. Then we have

$$(5.5.1) \quad \omega = \bar{\pi}_{N_{\mathbb{C}^\times}}^* \omega_{N_{\mathbb{C}^\times}} + \bar{\pi}_{\mathbb{C}^u}^* \omega_{\mathbb{C}^u} - \frac{\epsilon}{4\pi} dd^c(\chi(G_i) \log(F_i)).$$

Let X be the vector field on \bar{Y} , which is $\partial_{\theta_{(\alpha_i-\beta_i)^\perp}}$ with (u, \mathbf{w}) as coordinates; it is characterized by

$$(5.5.2) \quad \iota_X du = \iota_X d\mathbf{w}^{\alpha_i-\beta_i} = \iota_X dr_{(\alpha_i-\beta_i)^\perp} = 0, \quad \iota_X d\theta_{(\alpha_i-\beta_i)^\perp} = 1.$$

Then we have

$$(5.5.3) \quad \iota_X (\bar{\pi}_{N_{\mathbb{C}^\times}})^* \omega_{N_{\mathbb{C}^\times}} = -dr_{(\alpha_i-\beta_i)^\perp}.$$

By invariance of the functions F_i and G_i under the local circle action generated by X , we have

$$(5.5.4) \quad \iota_X dd^c (\chi(G_i) \log(F_i)) = (\mathcal{L}_X - d\iota_X) d^c (\chi(G_i) \log(F_i))$$

$$(5.5.5) \quad = -d (\iota_X d^c (\chi(G_i) \log(F_i))),$$

so that

$$(5.5.6) \quad -\iota_X \omega = d \left(r_{(\alpha_i - \beta_i)^\perp} - \frac{\epsilon}{4\pi} \iota_X d^c (\chi(G_i) \log(F_i)) \right).$$

If we define $\rho_i: U_i \rightarrow \mathbb{R}$ by

$$(5.5.7) \quad \rho_i := r_{(\alpha_i - \beta_i)^\perp} - \frac{\epsilon}{4\pi} \iota_X d^c (\chi(G_i) \log(F_i)),$$

then $\rho_i - r_{(\alpha_i - \beta_i)^\perp}$ is a bounded function whose support is contained in the support of χ . Let R be a positive number satisfying $R \gg R_0$. Let H_b be a positive function on $N_{\mathbb{R}}$ (which is also considered as a function on $N_{\mathbb{C}^\times}$ by composing with $\text{Log}: N_{\mathbb{C}^\times} \rightarrow N_{\mathbb{R}}$) such that

$$(5.5.8) \quad H_b \text{ is homogeneous of degree one outside of a compact set,}$$

$$(5.5.9) \quad H_b|_{U_{\Pi_i}} = c_i r_{(\alpha - \beta)^\perp} / R \text{ where } c_i \text{ is defined in (2.1.9), and}$$

$$(5.5.10) \quad H_b(\mathbf{r}) = |\mathbf{r}| / R \text{ outside of a neighborhood of } U_{\Pi}.$$

Since the function $\frac{c_i \rho_i}{R}$ agrees with $\bar{\pi}_{N_{\mathbb{C}^\times}}^* H_b$ outside the support of χ , we may glue $\frac{c_i \rho_i}{R}$ for $i = 1, \dots, \ell$ and $\bar{\pi}_{N_{\mathbb{C}^\times}}^* H_b$ together to obtain a positive function ρ defined on the complement of $\bar{\pi}_{N_{\mathbb{C}^\times}}^{-1}(K)$ for a compact subset K of $N_{\mathbb{C}^\times}$. We may extend this function to \bar{Y} arbitrarily to obtain a function which satisfies the necessary axioms. \square

We fix a function ρ appearing in the proof of Proposition 5.5 throughout the rest of this paper. We say that a base-admissible Hamiltonian H_{ba} has a *slope* $\lambda \in \mathbb{R}^{>0}$ if it is a C^2 -small perturbation of $\lambda\rho$ which coincides with $\lambda\rho$ outside of the inverse image by $\bar{\pi}_{N_{\mathbb{C}^\times}}$ of a compact set in $N_{\mathbb{C}^\times}$.

5.6. An ω -compatible almost complex structure $J_{\bar{Y}}$ on \bar{Y} is said to be *base-admissible* if the map $\bar{\pi}_{N_{\mathbb{C}^\times}}: \bar{Y} \rightarrow N_{\mathbb{C}^\times}$ is $(J_{\bar{Y}}, J_{N_{\mathbb{C}^\times}})$ -holomorphic outside of a compact set. The set of base-admissible almost complex structures on \bar{Y} will be denoted by $\mathcal{J}_{ba}(\bar{Y})$.

5.7. A *base-admissible Floer data* is a pair

$$(5.7.1) \quad (H, J) \in C^\infty([0, 1], \mathcal{H}_{ba}(\bar{Y})) \times C^\infty([0, 1], \mathcal{J}_{ba}(\bar{Y}))$$

of families of base-admissible Hamiltonians and base-admissible almost complex structures.

5.8. A *base-admissible perturbation data* (K, J) consists of

- (1) a 1-form $K \in \Omega^1(\Sigma, \mathcal{H}_{ba}(\bar{Y}))$ on Σ with values in base-admissible Hamiltonians satisfying
 - (a) $K|_{\partial\Sigma} = 0$, and
 - (b) outside of a compact set in the base, we have $\pi_{N_{\mathbb{C}^\times},*}(K) = X_{H_b} \otimes \gamma$ for γ sub-closed, and
- (2) a family $J \in C^\infty(\Sigma, \mathcal{J}_{ba}(\bar{Y}))$ of base-admissible almost complex structures on \bar{Y} parametrized by Σ .

It is *compatible* with a sequence $(\mathbf{H}, \mathbf{J}) = (H_j, J_j)_{j=0}^d$ of base-admissible Floer data if (4.12.3) holds for any $j \in \{0, \dots, d\}$ and any $t \in [0, 1]$.

Lemma 5.9. *Let $y: \Sigma \rightarrow \bar{Y}$ be a solution to Floer's equation (4.13.1) with respect to a base-admissible perturbation data. Set $p := S \circ \bar{\pi}_{N_{\mathbb{C}^\times}} \circ y: \Sigma \rightarrow \mathbb{R}$. If p is not a constant function, then p does not have a maximum on Σ whose maximum value is outside of the compact set appearing in the definitions of $\mathcal{H}_{\text{ba}}(\bar{Y})$ and $\mathcal{J}_{\text{ba}}(\bar{Y})$.*

Proof. It follows from the base-admissibility of K that the map $w := \pi_{N_{\mathbb{C}^\times}} \circ y: \Sigma \rightarrow N_{\mathbb{C}^\times}$ satisfies the Floer's equation

$$(5.9.1) \quad (dw - X_{H_b} \otimes \gamma)^{0,1} = 0$$

on X for the Hamiltonian H_b outside of a compact set in $N_{\mathbb{C}^\times}$. We write the almost complex structures on $N_{\mathbb{C}^\times}$ and Σ as J and j respectively, and set $\beta := -\partial^c S = -dS \circ J$ and $p := S \circ w$. Applying dS to both sides of Floer's equation

$$(5.9.2) \quad (dw - X_{H_b} \otimes \gamma) \circ j = J \circ (dw - X_{H_b} \otimes \gamma)$$

and using $dS(X_{H_b}) = 0$, one obtains

$$(5.9.3) \quad d^c p = -\beta \circ (dw - X_{H_b} \otimes \gamma).$$

By applying d to both sides and using $\omega = d\beta = -dd^c S$, one obtains

$$(5.9.4) \quad -dd^c p = w^* \omega - d(\beta(X_{H_b}) \cdot \gamma).$$

Since $\beta(X_{H_b}) = -H_b$ outside of a compact set in $N_{\mathbb{R}}$, one has

$$(5.9.5) \quad -dd^c p = w^* \omega - d(w^* H_b \cdot \gamma)$$

$$(5.9.6) \quad = w^* \omega - d(w^* H_b) \wedge \gamma - w^* H_b \cdot d\gamma$$

$$(5.9.7) \quad = \|dw - X_{H_b} \otimes \gamma\|^2 - w^* H_b \cdot d\gamma$$

$$(5.9.8) \quad \geq 0$$

since $H_b \geq 0$ and $d\gamma \leq 0$. Now $-dd^c$ is an operator of the form (9.3.1), so the function p satisfies the strong maximum principle. If the function p attains a maximum at $\Sigma = \bar{\Sigma} \setminus \zeta$, then Hopf's lemma implies that

$$(5.9.9) \quad dp(\nu) > 0$$

for an outward normal vector ν of $\partial\Sigma$ at some point $x \in \partial\Sigma$. Let $\tau \in T_x(\partial\Sigma)$ be the tangent vector such that $\nu = j\tau$. Then one has

$$(5.9.10) \quad dp(\nu) = dS \circ (dw \circ j)(\tau)$$

$$(5.9.11) \quad = dS \circ (X_{H_b} \otimes \gamma \circ j + J \circ (dw - X_{H_b} \otimes \gamma))(\tau)$$

$$(5.9.12) \quad = -\beta \circ (dw - X_{H_b} \otimes \gamma)(\tau),$$

where we used $dS(X_{H_b}) = 0$ and $\beta = -dS \circ J$. The first term vanishes by the Legendrian condition $\beta|_{\underline{L}} = 0$ at infinity, and the second term vanishes by $\gamma|_{\partial\Sigma} = 0$. This contradicts (5.9.9), and Lemma 5.9 is proved. \square

5.10. An almost complex structure J on \bar{Y} is *fibration-admissible* if

- (1) J is base-admissible,
- (2) the map $\bar{\pi}_{\mathbb{C}}: \bar{Y} \rightarrow \mathbb{C}$ is J -holomorphic on $\bar{\pi}_{\mathbb{C}}^{-1}(\{u \in \mathbb{C} \mid |u| > C_0\})$ for some C_0 ,
- (3) the divisor \bar{E} defined in (2.5.5) is J -holomorphic, and
- (4) the almost complex structure $J|_{U_D}$ is the product of $J_{N_{\mathbb{C}^\times}}$ and the standard complex structure on \mathbb{D}_δ under the identification (2.5.13).

The set of fibration-admissible almost complex structures on \bar{Y} will be denoted by $\mathcal{J}(\bar{Y})$.

The following stronger notion will be used later in Section 6:

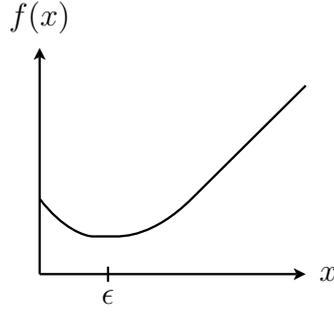


FIGURE 5.1. An admissible vertical Hamiltonian

5.11. A fibration-admissible almost complex structure J on \bar{Y} is *integrably fibration-admissible* if there exists an almost complex structure $J_{N_{\mathbb{C}^\times}}$ adapted to Z such that when one equips $N_{\mathbb{C}^\times} \times \mathbb{C}$ with the almost complex structure $(J_{N_{\mathbb{C}^\times}}, J_{\mathbb{C}})$, the structure map $p = (\bar{\pi}_{N_{\mathbb{C}^\times}}, \bar{\pi}_{\mathbb{C}}): \bar{Y} \rightarrow N_{\mathbb{C}^\times} \times \mathbb{C}$ of the blow-up is pseudo-holomorphic on the union of

- (1) $\bar{\pi}_{\mathbb{C}}^{-1}(\{u \in \mathbb{C} \mid |u| > C_0\})$ for some C_0 ,
- (2) $\bar{\pi}_{N_{\mathbb{C}^\times}}^{-1}(\{\mathbf{w} \in N_{\mathbb{C}^\times} \mid S(\mathbf{w}) > C_1\})$ for some $C_1 > 0$, and
- (3) $U_D \cup \pi^{-1}(U_Z)$.

The set of integrably fibration-admissible almost complex structures on \bar{Y} will be denoted by $\mathcal{J}_{\text{int}}(\bar{Y})$.

5.12. Fix $\mu_0, \mu_1 \in \mathbb{R}^{>0}$ such that $\mu_0 \ll \epsilon \ll \mu_1$. In the exact structure (2.7.9), if we set the boundary to be $\{\mu = \mu_0\}$ and $\{\mu = \mu_1\}$, then the Liouville coordinates become $c_-(\epsilon - \mu)$ and $c_+(\mu - \epsilon)$ for some constant c_- and c_+ . A function $H_v: \bar{Y} \rightarrow \mathbb{R}^{>0}$ is an *admissible vertical Hamiltonian* of slope $\lambda \in \mathbb{R}^{>0}$ if there exist a function $f: \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{>0}$ satisfying

- (1) $f''(x) \geq 0$ for any $x \in \mathbb{R}^{\geq 0}$,
- (2) $f(x) = \lambda c_-(\epsilon - x)$ when $x < \mu_0$,
- (3) $f'(x) = 0$ in a neighborhood of $x = \epsilon$,
- (4) $f(x) = \lambda c_+(x - \epsilon)$ when $x > \mu_1$, and
- (5) $H_v = f \circ \mu$ where $\mu: \bar{Y} \rightarrow \mathbb{R}^{\geq 0}$ is the moment map (2.7.5).

The set of admissible vertical Hamiltonians will be denoted by $\mathcal{H}_v(\bar{Y})$. The Hamiltonian vector field associated with H_v is given by

$$(5.12.1) \quad X_{H_v} = f'(\mu) \cdot X_\mu,$$

where X_μ is the fundamental vector field for the \mathbb{S}^1 -action (2.5.6). It follows that

$$(5.12.2) \quad (\pi_{N_{\mathbb{C}^\times}})_*(X_{H_v}) = 0.$$

5.13. A *fibration-admissible Hamiltonian* of slope $\lambda \in \mathbb{R}^{>0}$ is a function $H: \bar{Y} \rightarrow \mathbb{R}^{>0}$ satisfying the following conditions:

- (1) One has $(\bar{\pi}_{N_{\mathbb{C}^\times}})_*(X_H) = X_{\lambda\rho}$ outside of a compact set in $N_{\mathbb{C}^\times}$.
- (2) Whenever $|\mu| < \mu_0$ or $|\mu| > \mu_1$, one has $H = H_{\text{ba}} + H_v$, where H_{ba} is a base-admissible Hamiltonian of slope λ and H_v is an admissible vertical Hamiltonian of slope λ .
- (3) X_H is tangent to \bar{E} .

The set of fibration-admissible Hamiltonians of slope λ will be denoted by $\mathcal{H}_\lambda(\bar{Y})$.

Remark 5.14. To actually construct examples, we may assume that $H = H_{\text{ba}} + H_{\text{v}}$ holds everywhere. The extra flexibility of our definitions are included for Section 7.

5.15. A *fibration-admissible sequence of Hamiltonians* is a sequence $(H_m)_{m=1}^{\infty}$ of fibration-admissible Hamiltonians such that the slopes λ_m of H_m for $m \in \mathbb{Z}^{>0}$ satisfy $\lambda_m < \lambda_{m+1}$ and $\lambda_m + \lambda_{m'} \leq \lambda_{m+m'}$ for any $m, m' \in \mathbb{Z}^{>0}$.

5.16. A *fibration-admissible perturbation data* is a base-admissible perturbation data

$$(5.16.1) \quad (K, J) \in \Omega^1(\Sigma, H_{\text{ba}}(\bar{Y})) \times C^\infty(\Sigma, \mathcal{J}(\bar{Y}))$$

satisfying the following conditions:

- (1) X_K is tangent to \bar{E} and D .
- (2) For $\mu \gg \epsilon$, one has $(\bar{\pi}_{\mathbb{C}})_*(X_K) = (\bar{\pi}_{\mathbb{C}})_* X_{H_{\text{v}}} \otimes \gamma_+$ for a subclosed one form γ_+ and a vertical Hamiltonian H_{v} .
- (3) For points on $\bar{Y} \setminus \pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$ with $\mu \ll \epsilon$, one has

$$(5.16.2) \quad (\bar{\pi}_{\mathbb{C}})_*(X_K) = (\bar{\pi}_{\mathbb{C}})_* X_{H_{\text{v}}} \otimes \gamma_-$$

for a subclosed one form γ_- and a vertical Hamiltonian H_{v} .

5.17. Assume $d \leq 3$ and let $\mathcal{R}^d(Y, \mathbf{x})$ be the moduli space of solutions to Floer's equation for perturbation data in Sections 4.16, 4.17 and 4.19 (with Liouville admissible data replaced by the corresponding fibration-admissible data). Similarly, we let $\mathcal{R}_\tau^2(Y, \mathbf{x})$ be the moduli space of solutions to Floer's equation for the perturbation data $K_\tau(m_1, m_2, m_3)$ appearing in Section 4.18, whose union is denoted by

$$(5.17.1) \quad \mathcal{R}_\bullet^2(Y, \mathbf{x}) := \bigcup_{\tau \in [0,1]} \mathcal{R}_\tau^2(Y, \mathbf{x}).$$

Let \bar{L} be the closure in \bar{Y} of a base-admissible Lagrangian section L . By base-admissibility, solutions to Floer's equation are now constrained to lie in a compact subspace in \bar{Y} and so Gromov compactness applies as usual. Gromov compactness is typically stated for Lagrangians without boundary, but here it applies because we can extend \bar{L} slightly in the negative real direction as well.

Since \bar{L} is contractible, the relative homotopy group $\pi_2(\bar{Y}, \bar{L})$ is a torsor over $\pi_2(\bar{Y})$, and the possible relative homology classes of Floer curves in \bar{Y} is a torsor over the image of $\pi_2(\bar{Y})$ in $H_2(\bar{Y})$. From the general properties of blowing up, we have

$$(5.17.2) \quad H_2(\bar{Y}) \cong H_2(N_{\mathbb{C}^\times}) \oplus [E_{\mathbf{w}}],$$

where $[E_{\mathbf{w}}]$ is the class generated by any exceptional sphere over $\mathbf{w} \in Z$. It follows that the image of $\pi_2(\bar{Y})$ in $H_2(\bar{Y})$ is one-dimensional and generated by $[E_{\mathbf{w}}]$.

The moduli spaces $\mathcal{R}^d(Y, \mathbf{x})$ and $\mathcal{R}_\tau^2(Y, \mathbf{x})$ embeds naturally into the Gromov compactifications of maps into \bar{Y} with some relative homology class $A_{\mathbf{x}} \in H_2(\bar{Y}, \bar{L})$ which is uniquely determined by the intersection with either E or F . The closures of these spaces will be denoted by $\bar{\mathcal{R}}^d(\bar{Y}, \mathbf{x})$ and $\bar{\mathcal{R}}_\tau^2(\bar{Y}, \mathbf{x})$ respectively. Moreover, if every component of $u \in \bar{\mathcal{R}}^d(\bar{Y}, \mathbf{x})$ (or $\bar{\mathcal{R}}_\tau^2(\bar{Y}, \mathbf{x})$) avoids $D \cap \bar{L}$ and is asymptotic to chords in Y , then the image of u is contained in Y . We also set

$$(5.17.3) \quad \bar{\mathcal{R}}_\bullet^2(\bar{Y}, \mathbf{x}) := \bigcup_{\tau \in [0,1]} \bar{\mathcal{R}}_\tau^2(\bar{Y}, \mathbf{x}).$$

Definition 5.18. Hamiltonian chords for the Lagrangian \bar{L} which are completely contained in D are called *divisor chords*.

5.19. We assume that the Hamiltonian flow preserves $U_{Z \times 0}$, so that all Hamiltonian chords are disjoint from $\pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$.

Lemma 5.20. *Let (K, J) be a fibration-admissible perturbation data, and consider a sequence $(y_s)_{s=1}^\infty$ of maps $y_s: \Sigma \rightarrow Y$ in $\mathcal{R}^d(\mathbf{x})$ converging to $y_\infty \in \overline{\mathcal{R}}^d(\overline{Y}, \mathbf{x})$. Let $\{y_{k,\infty}\}_{k=1}^l$ be the irreducible components of the maps. If a component $y_{k,\infty}: \Sigma_k \rightarrow \overline{Y}$ intersects D at a point on the boundary $\partial\Sigma_k$, then the component $y_{k,\infty}$ lies entirely in D .*

Proof. Observe that there are four essentially distinct ways that a limiting component $y_{k,\infty}$ could intersect the divisor D , i.e.,

- in the interior of Σ ,
- on the boundary $\partial\Sigma$,
- $y_{k,\infty}$ lies completely in D , or
- $y_{k,\infty}$ limits to a divisor chord in D along some strip-like end ϵ .

It therefore remains to rule out intersections along a boundary, which we claim follows from the fact that $d^c(1/|u|)|_{\overline{L}} = 0$, where u is the base coordinate on \mathbb{C} discussed above. Consider a subsequence $y_s: \Sigma \rightarrow Y$ with boundary on L which meets the exceptional divisor E at points $z_{k,s}$. We can define the intersection number

$$(5.20.1) \quad y_s \cdot E = \sum_{z_{k,s}} d_{k,s},$$

where $d_{k,s} \geq 0$ are the local intersection numbers of y_s with E at $z_{k,s}$ on Σ . To define this number efficiently, observe that Gromov's trick [Gro85] (see e.g. [MS12, Section 8.1] for an exposition) allows us to view a solution y_s to Floer's equation as a pseudo-holomorphic section $\tilde{y}_s: \Sigma \rightarrow \Sigma \times \overline{Y}$ for a specific almost complex structure on $\Sigma \times \overline{Y}$. When the perturbation data (K, J) are admissible, both $\Sigma \times E$ and $\Sigma \times D$ are almost complex submanifolds of codimension 2. The local intersection number is then the intersection number of the section with $\Sigma \times E$. This number is constant in our sequence (y_s) . Assume for contradiction that a sequence (y_s) has a convergent subsequence which limits to u_∞ that has a component $y_{k,\infty}$ intersecting D along some \overline{L} at a point z_ℓ . Then the intersection points above limit to intersection points $z_{k,\infty}$ which are in the interior of y_∞ .

Fix a small ball $\mathbb{D}_{\epsilon_i}(z_{k,\infty})$ about these points which avoids z_ℓ . We choose s large enough so that all of the points $z_{k,s}$ lie in $\mathbb{D}_{\epsilon_i}(z_{k,\infty})$. Then for large s , there must be a local maximum of $1/|u|$ near z_ℓ . This is impossible by the same calculation as in Lemma 5.9 if we note that $d^c(1/|u|)|_{\overline{L}} = 0$. \square

5.21. There are two useful ways of grading Hamiltonian chords. The first is used to grade Hamiltonian chords in Y . Namely, one grades Y by the holomorphic volume form

$$(5.21.1) \quad \text{Res} \left[\frac{1}{h(\mathbf{w}) - uv} \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge du \wedge dv \right] = \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge \frac{du}{u}$$

on Y and graded Lagrangian submanifolds with respect to this grading of Y . Note that this grading of Y restricts to the standard grading on $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$. We denote this standard grading by $|x_i|$. If $d \geq 2$, then we have

$$(5.21.2) \quad \text{virt. dim } \mathcal{R}^d(Y, \mathbf{x}) = |x_0| - \sum_{i \neq 0} |x_i| + d - 2.$$

If $d = 1$ and the perturbation data is invariant under the \mathbb{R} -translation (this is the case when we define the Floer differential), then (5.21.2) continues to hold, after modding out

by automorphisms. If $d = 1$ and the perturbation data is not translation invariant (this is the case when we define the continuation map), then we have

$$(5.21.3) \quad \text{virt. dim } \mathcal{R}^1(Y, \mathbf{x}) = |x_0| - |x_1|.$$

In the case of perturbation data in Section 4.19, we have

$$(5.21.4) \quad \dim \mathcal{R}_\bullet^2(Y, \mathbf{x}) = |x_0| - \sum_{i \neq 0} |x_i| + 1.$$

For generic $J \in \mathcal{J}_{\text{int}}(\overline{Y})$, all moduli spaces can be cut out transversely so that the virtual dimension agrees with the actual dimension.

5.22. The second way of grading allows us to grade all chords, including divisor chords. This is by grading chords x_i with respect to the standard volume form on $N_{\mathbb{C}^\times} \times \mathbb{C}$. Equivalently, one may view this as choosing an algebraic volume form

$$(5.22.1) \quad \frac{dw_1}{w_1} \wedge \frac{dw_2}{w_2} \wedge du$$

with a simple zero along E , restricting this to $\overline{Y} \setminus E$, and then grading the Lagrangians. It follows from the index computation in [She15, Lemma 3.22] that for $d \geq 2$, one has

$$(5.22.2) \quad \dim(\overline{\mathcal{R}}^d(\overline{Y}, \mathbf{x})) = |x_0|_{\text{rel}} - \sum_{i \neq 0} |x_i|_{\text{rel}} - 2(A_{\mathbf{x}} \cdot E) + d - 2,$$

and similarly for $\overline{\mathcal{R}}^1(\overline{Y}, \mathbf{x})$ and $\overline{\mathcal{R}}_\bullet^2(\overline{Y}, \mathbf{x})$.

Remark 5.23. Floer theoretic operations will respect this second grading after one inserts a formal parameter t of cohomological degree -2 and curves y are weighted by $t^{y \cdot E}$, where $y \cdot E$ is their intersection with the divisor E (see [She15, Lemma 3.22 and Definition 5.1]). We will be primarily interested in the first grading, but use this second grading to rule out certain breaking configurations and in certain arguments in Section 6.

5.24. Now we restrict our attention to the Lagrangian L_0 , which is the example treated in this paper. Throughout the rest of this paper, we will equip L_0 with the trivial spin structure to view it as a Lagrangian brane. We may assume that all Hamiltonian chords are non-degenerate. For example, we may take a fixed H_{ba} and H_{v} of slope one and then take $H_{\text{ba},m}$ and $H_{\text{v},m}$ to be sufficiently small perturbations of mH_{ba} and mH_{v} supported away from Z .

Lemma 5.25. *The relative homotopy group $\pi_1(Y, L_0)$ is naturally isomorphic to N .*

Proof. The relative homotopy group $\pi_1(Y, L_0)$ is isomorphic to the fundamental group $\pi_1(Y)$ since L_0 is contractible. It is well-known that the blow-up $\overline{Y} \rightarrow N_{\mathbb{C}^\times} \times \mathbb{C}$ induces an isomorphism $\pi_1(\overline{Y}) \xrightarrow{\sim} \pi_1(N_{\mathbb{C}^\times} \times \mathbb{C}) \cong \pi_1(N_{\mathbb{C}^\times})$ of the fundamental group. The kernel of the map $\pi_1(Y) \rightarrow \pi_1(\overline{Y})$ is the normal subgroup generated by the class of a loop of the form $|w| = \text{pt}$, $|v_0| = \epsilon$ (cf. e.g. [Zai, Lemma 2.3(a)]). Such loops are contractible in Y due to the singular conic bundle structure. We therefore conclude that this map is an isomorphism. The fundamental group of $N_{\mathbb{C}^\times}$ is naturally isomorphic to N , and Lemma 5.25 is proved. \square

Recall that we have assumed that near D , our Hamiltonians have the form

$$(5.25.1) \quad H_m = H_{\text{ba},m} + H_{\text{v},m}$$

for $H_{\text{ba},m}$ and $H_{\text{v},m}$ sufficiently generic so that all chords are non-degenerate. It follows that for every Hamiltonian chord p in $\mathcal{X}(\underline{L}_0, N_{\mathbb{C}^\times}; H_{\text{b},m})$, there is a divisor chord in $\mathcal{X}(\overline{L}_0, \overline{Y}; H_m)$, which we write as p_{d} .

Lemma 5.26. *One has $|p_{\text{d}}|_{\text{rel}} = 2\lfloor c - \lambda_m \rfloor + 1 + |p|_{N_{\mathbb{C}^\times}}$.*

Proof. This is obtained by observing that we have a product splitting for the Lagrangians, Hamiltonian flow, and the trivializations under the product decomposition of $N_{\mathbb{C}^\times} \times \mathbb{C}$. So it suffices to compute the contribution from the \mathbb{C} factor. The short chord contributes 1 to the Maslov index and each rotation around the cylindrical end contributes $2\lfloor c - \lambda_m \rfloor$ (see [Sei08, Section (11e)]). \square

5.27. For

$$(5.27.1) \quad \mathbf{n} \in N \cong \pi_1(Y, L_0) \cong \pi_1(\overline{Y}, \overline{L}_0) \cong \pi_1(N_{\mathbb{C}^\times}, \underline{L}_0),$$

we set

$$(5.27.2) \quad \mathcal{X}(\underline{L}_0; H_{\text{b},m})_{\mathbf{n}} := \{p \in \mathcal{X}(\underline{L}_0; H_{\text{b},m}) \mid [p] = \mathbf{n}\}.$$

Let

$$(5.27.3) \quad \ell^\sharp: N \times \mathbb{Z}^{>0} \rightarrow \mathbb{Z}$$

be a function satisfying the following conditions for any $\mathbf{n} \in N$;

$$(5.27.4) \quad \ell^\sharp(\mathbf{n}, m) < \inf_{p \in \mathcal{X}(\underline{L}_0; H_{\text{b},m})_{\mathbf{n}}} |p_{\text{d}}|_{\text{rel}} - 1 \text{ for any } m \in \mathbb{Z}^{>0},$$

$$(5.27.5) \quad \ell^\sharp(\mathbf{n}, m) < \ell^\sharp(\mathbf{n}, m+1) \text{ for any } m \in \mathbb{Z}^{>0},$$

$$(5.27.6) \quad \lim_{m \rightarrow \infty} \ell^\sharp(\mathbf{n}, m) = \infty.$$

We set

$$(5.27.7) \quad \text{CF}^*(L_0; H_m) := \bigoplus_{x \in \mathcal{X}(L_0; H_m)} |\mathfrak{o}_x|,$$

where $|\mathfrak{o}_x|$ is a one-dimensional \mathbb{C} -normalized orientation space associated to x . We also set

$$(5.27.8) \quad \mathcal{X}(L_0; H_m)^{\leq \ell^\sharp} := \{x \in \mathcal{X}(L_0; H_m) \mid |x|_{\text{rel}} \leq \ell^\sharp([x], m)\}$$

and

$$(5.27.9) \quad \text{CF}^*(L_0; H_m)^{\leq \ell^\sharp} := \bigoplus_{x \in \mathcal{X}(L_0; H_m)^{\leq \ell^\sharp}} |\mathfrak{o}_x|.$$

The vector space $\text{CF}^*(L_0; H_m)^{> \ell^\sharp}$ is defined similarly.

Definition 5.28. A pair $((H_m)_{m=1}^\infty, \ell^\sharp)$ is said to satisfy *Assumption A* if the following conditions are satisfied:

1. For every pair $\mathbf{x} = (x_0, x_1)$ of chords $x_0 \in \mathcal{X}(L_0; H_{m'})$ and $x_1 \in \mathcal{X}(L_0; H_m)$ which lie in the same relative homotopy class, there is a topological strip $y_{\mathbf{x}}$ in \overline{Y} between x_1 and x_0 bounded by \overline{L}_0 such that $y_{\mathbf{x}}$ has intersection number zero with both components E and F of \overline{E} .
2. If a pair $\mathbf{x} = (x_0, x_1)$ of chords $x_0 \in \mathcal{X}(\overline{L}_0; H_m)^{> \ell^\sharp}$ and $x_1 \in \mathcal{X}(\overline{L}_0; H_m)^{\leq \ell^\sharp}$ satisfies $\text{virt. dim } \overline{\mathcal{R}}^1(\overline{Y}, \mathbf{x}) = 0$, then $\overline{\mathcal{R}}^1(\overline{Y}, \mathbf{x})$ is empty for generic $J \in C^\infty([0, 1], \mathcal{J}(\overline{Y}))$.

The following compactness lemma allows us to give $\text{CF}^*(L_0; H_m)^{\leq \ell^\sharp}$ the structure of a complex and to define continuation maps which preserve the relative grading.

Lemma 5.29. *If a pair $\mathbf{x} = (x_0, x_1)$ of chords $x_0 \in \mathcal{X}(L_0; H_{m'})^{\leq \ell^\sharp}$ and $x_1 \in \mathcal{X}(L_0; H_m)^{\leq \ell^\sharp}$ for $m \leq m'$ satisfies $|x_0|_{\text{rel}} < \ell^\sharp(\mathbf{n}, m)$, then for a generic almost complex structure $J \in C^\infty([0, 1], \mathcal{J}(\overline{Y}))$, the space $\overline{\mathcal{R}}^1(\overline{Y}, \mathbf{x})$ consists of curves which are disjoint from D and with no strip-breaking along chords in $\mathcal{X}(L_0)^{> \ell^\sharp}$.*

Proof. Since

- the intersection numbers of any Floer strip with E and F must be non-negative,
- the relative homology class of any topological strip can be obtained as the connected sum of $y_{\mathbf{x}}$ and an integer multiple of $[E_{\mathbf{w}}]$, and
- the intersection numbers of $[E_{\mathbf{w}}]$ with E and F are given by

$$(5.29.1) \quad [E_{\mathbf{w}}] \cdot E = -[E_{\mathbf{w}}] \cdot F = -1,$$

any Floer strip must have intersection number zero (and hence, in fact, be disjoint) with E (and F). It follows that any Floer strip must raise or at least preserve the relative grading, so that there can be no strip breaking along such chords with higher relative grading. \square

Definition 5.30. A pair $((H_m)_{m=1}^\infty, \ell^\sharp)$ is said to satisfy *Assumption B* if there exists a function $s: N^2 \rightarrow \mathbb{Z}^{\geq 0}$ satisfying the following conditions:

1. For any $(\mathbf{n}_1, \mathbf{n}_2) \in N^2$, any $(m_1, m_2) \in (\mathbb{Z}^{> 0})^2$ and any triple $\mathbf{x} = (x_0, x_1, x_2)$ of chords $x_0 \in \mathcal{X}(L_0; H_{m_1+m_2})_{\mathbf{n}_1+\mathbf{n}_2}^{\leq \ell^\sharp}$, $x_1 \in \mathcal{X}(L_0; H_{m_1})_{\mathbf{n}_1}^{\leq \ell^\sharp}$, and $x_2 \in \mathcal{X}(L_0; H_{m_2})_{\mathbf{n}_2}^{\leq \ell^\sharp}$ satisfying

$$(5.30.1) \quad \text{virt. dim}(\mathcal{R}(x_0, x_1, x_2)) \in \{0, 1\}$$

and

$$(5.30.2) \quad |x_0|_{\text{rel}} \geq |x_1|_{\text{rel}} + |x_2|_{\text{rel}} + s(\mathbf{n}_1, \mathbf{n}_2),$$

the space $\overline{\mathcal{R}}(\overline{Y}, \mathbf{x})$ is empty for a generic almost complex structures.

2. For any chords $x_1 \in \mathcal{X}(L_0; H_{m_1})_{\mathbf{n}_1}^{\leq \ell^\sharp}$ and $x_2 \in \mathcal{X}(L_0; H_{m_2})_{\mathbf{n}_2}^{\leq \ell^\sharp}$, one has

$$(5.30.3) \quad \ell^\sharp(\mathbf{n}_1 + \mathbf{n}_2, m_1 + m_2) \geq |x_1|_{\text{rel}} + |x_2|_{\text{rel}} + s(\mathbf{n}_1, \mathbf{n}_2).$$

3. Let x_1, x_2 be as above and $x_0 \in \mathcal{X}(L_0; H_{m_1+m_2})_{\mathbf{n}_1+\mathbf{n}_2}^{\leq \ell^\sharp}$. We assume that whenever $\text{virt. dim}(\mathcal{R}_\bullet^2(x_0, x_1, x_2)) \in \{0, 1\}$ and (5.30.2) holds, the moduli spaces $\mathcal{R}_\bullet^2(x_0, x_1, x_2)$ are empty for generic complex structures. We also require that the same holds when the continuation maps are along the other positive strip like end.

Lemma 5.31. *1. If a triple of chords $x_0 \in \mathcal{X}(L_0; H_{m+n})^{\leq \ell^\sharp}$, $x_1 \in \mathcal{X}(L_0; H_m)^{\leq \ell^\sharp}$, and $x_2 \in \mathcal{X}(L_0; H_n)^{\leq \ell^\sharp}$ satisfies*

$$(5.31.1) \quad |x_1|_{\text{rel}} + \inf_{p \in \mathcal{X}(L_0; H_{\text{ba}, n})_{\mathbf{n}_2}} |p_{\text{d}}|_{\text{rel}} > |x_0|_{\text{rel}},$$

$$(5.31.2) \quad |x_2|_{\text{rel}} + \inf_{p \in \mathcal{X}(L_0; H_{\text{ba}, m})_{\mathbf{n}_1}} |p_{\text{d}}|_{\text{rel}} > |x_0|_{\text{rel}}, \quad \text{and}$$

$$(5.31.3) \quad \text{virt. dim}(\mathcal{R}^2(x_0, x_1, x_2)) = \{0, 1\},$$

then for generic surface-dependent almost complex structures, the closure $\overline{\mathcal{R}}^2(x_0, x_1, x_2)$ consists of curves which are disjoint from D and with no breaking along chords in $\mathcal{X}(L_0, L_0)^{> \ell^\sharp}$.

2. If a triple of chords $x_0 \in \mathcal{X}(L_0; H_{m_2+m_3})^{\leq \ell^\sharp}$, $x_1 \in \mathcal{X}(L_0; H_{m_1})^{\leq \ell^\sharp}$ and $x_2 \in \mathcal{X}(L_0; H_{m_3})^{\leq \ell^\sharp}$ for $m_1 < m_2$ satisfies

$$(5.31.4) \quad |x_1|_{\text{rel}} + \inf_{p \in \mathcal{X}(\underline{L}_0; H_{\text{ba}, m_2})_{n_2}} |p_d|_{\text{rel}} > |x_0|_{\text{rel}},$$

$$(5.31.5) \quad |x_2|_{\text{rel}} + \inf_{p \in \mathcal{X}(\underline{L}_0; H_{\text{ba}, m_1})_{n_1}} |p_d|_{\text{rel}} > |x_0|_{\text{rel}},$$

$$(5.31.6) \quad |x_0|_{\text{rel}} < \ell^\sharp(\mathbf{n}, m_1 + m_3), \text{ and}$$

$$(5.31.7) \quad \text{virt. dim}(\mathcal{R}_t^2(x_0, x_1, x_2)) \in \{0, 1\},$$

then for generic surface-dependent almost complex structures, the closure $\overline{\mathcal{R}}_\bullet^2(x_0, x_1, x_2)$ consists of curves which are disjoint from D and which do not break along chords in $\mathcal{X}(L_0, L_0)^{> \ell^\sharp}$.

Proof. A broken curve in this moduli space will have one component which is a Floer triangle Σ and possibly many components which are spheres and strips. The most difficult case to rule out is breaking along a divisor chord. The curve cannot break along a divisor chord at a position in our tree after or as outputs of the curve Σ for the same reason as in the preceding compactness lemma. Now we consider the case of breaking along divisor chords before or as inputs of Σ . The two bulleted conditions rule that out since the Floer triangle would necessarily have negative virtual dimension and hence not exist for generic almost complex structures. \square

We define the product \mathbf{m}_2 of two (orientation lines corresponding to) chords $x_1 \in \mathcal{X}(L_0; H_m)_{\mathbf{n}_1}^{\leq \ell^\sharp}$ and $x_2 \in \mathcal{X}(L_0; H_n)_{\mathbf{n}_2}^{\leq \ell^\sharp}$ as follows. Let m_1 and m_2 be the smallest integers such that

$$(5.31.8) \quad \ell^\sharp(\mathbf{n}_1, m_1) > |x_1|_{\text{rel}} + s(\mathbf{n}_1, \mathbf{n}_2)$$

and

$$(5.31.9) \quad \ell^\sharp(\mathbf{n}_2, m_2) > |x_2|_{\text{rel}} + s(\mathbf{n}_1, \mathbf{n}_2).$$

If $m < m_1$ or $n < m_2$, we may define the Floer product by first applying the continuation maps into $\text{CF}^*(L_0; H_{m_1})^{\leq \ell^\sharp}$ and $\text{CF}^*(L_0; H_{m_2})^{\leq \ell^\sharp}$ and then applying the usual Floer product. This is compatible with continuation maps in both variables.

Definition 5.32. A pair $((H_m)_{m=1}^\infty, \ell^\sharp)$ is said to satisfy *Assumption C* if there is a function

$$(5.32.1) \quad s: (\pi_1(Y, L_0))^3 \rightarrow \mathbb{Z}^{\geq 0}$$

such that for any $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) \in (\pi_1(Y, L_0))^3$, any $(m_1, m_2, m_3) \in (\mathbb{Z}^{> 0})^3$, any $m_4 \in \mathbb{Z}^{\geq 0}$, any $\mathbf{x} = (x_0, x_1, x_2, x_3)$ consisting of $x_i \in \mathcal{X}(\underline{L}_0; H_{m_i})_{\mathbf{n}_i}^{\leq \ell^\sharp}$ for $i = 1, 2, 3$ and $x_0 \in \mathcal{X}(L_0; H_{m_1+m_2+m_3+m_4})^{\leq \ell^\sharp}$ satisfying

$$(5.32.2) \quad \text{virt. dim}(\overline{\mathcal{R}}^3(\overline{Y}, \mathbf{x})) \in \{0, 1\}$$

and

$$(5.32.3) \quad |x_0|_{\text{rel}} \geq s(\mathbf{n}_i) + \sum_{i=1}^3 |x_i|_{\text{rel}},$$

the space $\overline{\mathcal{R}}^3(\overline{Y}, (x_0, x_1, x_2, x_3))$ is empty.

A compactness result similar to Lemma 5.29 and Lemma 5.31 shows that under Assumption C, the restriction of the A_∞ -operation \mathfrak{m}_2 to $\text{CF}^*(L_0; H_m)^{\leq \ell^\sharp}$ gives rise to an associative product on the adapted wrapped Floer cohomology defined below.

Definition 5.33. A pair $((H_m)_{m=1}^\infty, \ell^\sharp)$ is *admissible* if it satisfies Assumptions A, B and C. Given an admissible pair, the *adapted wrapped Floer cohomology* is the colimit

$$(5.33.1) \quad \text{HW}_{\text{ad}}(L_0) := \varinjlim_m \text{HF}_{\text{ad}}(L_0; H_m)$$

of the cohomology $\text{HF}_{\text{ad}}(L_0; H_m)$ of the complex (5.27.9).

5.34. Let us assume that our data is chosen to satisfy (5.25.1) everywhere. For every p in $\mathcal{X}(\underline{L}_0; H_{\text{ba},m})$, in addition to p_{d} , there are chords p_i for every $i \in \mathbb{Z}$ satisfying $|i| < c_\pm \lambda_m$.

Lemma 5.35. *We have $|p_i|_{\text{rel}} = 2i + |p|_{N_{\mathbb{C}^\times}}$ and $|p_i| = |p_{\text{d}}| = |p|_{N_{\mathbb{C}^\times}}$.*

Remark 5.36. In fact, we can arrange that there is at most one generator $p_{\mathbf{n}}^m \in \mathcal{X}(\underline{L}_0, H_m)$ of grading 0 in each homotopy class $\mathbf{n} \in \pi_1(Y, L_0)$. See the explicit constructions of Section 6 for further details on this point.

Lemma 5.37. *There exist an admissible pair $((H_m)_{m=1}^\infty, \ell^\sharp)$.*

Proof. Let us assume that our data is chosen to satisfy (5.25.1) everywhere and to satisfy the condition stated in Remark 5.36. Then, there are no differentials since everything lies in degree zero, so we only have to consider continuation solutions to define the wrapped Floer groups, Floer triangles to define the multiplication, and one-dimensional moduli spaces of Floer quadangles to ensure that the multiplication is associative. A continuation solution y with input $p_{\mathbf{n}_1, i_1}^{m_1}$ does not intersect \bar{E} and in particular does not intersect E . Since the output must be of the form $p_{\mathbf{n}_1, i_2}^{m_2}$, we require that

$$(5.37.1) \quad \ell^\sharp(\mathbf{n}_1, m_1) < \ell^\sharp(\mathbf{n}_1, m_2).$$

Given a Floer triangle y with inputs $p_{\mathbf{n}_1, i_1}^{m_1}$ and $p_{\mathbf{n}_2, i_2}^{m_2}$, the intersection number $y \cdot \bar{E}$ is fixed, which is greater than or equal to $y \cdot E$, and the output is necessarily of the form $p_{\mathbf{n}_1 + \mathbf{n}_2, i_3}^{m_1 + m_2 + r}$. In view of (5.22.2), the degree of $p_{\mathbf{n}_1 + \mathbf{n}_2, i_3}^{m_1 + m_2 + r}$ is at most

$$(5.37.2) \quad \sum_{i=1}^2 |x_i|_{\text{rel}} + 2y \cdot \bar{E} + 1$$

and the condition (5.30.2) is satisfied. The condition (5.30.3) is also satisfied if

$$(5.37.3) \quad \ell^\sharp(\mathbf{n}_1, m_1) + \ell^\sharp(\mathbf{n}_2, m_2) + 2y \cdot \bar{E} + 1 \leq \ell^\sharp(\mathbf{n}_1 + \mathbf{n}_2, m_1 + m_2)$$

for every $\mathbf{n}_1, \mathbf{n}_2$. Conversely, these two conditions guarantee the admissibility of ℓ^\sharp . To see this, we consider the curves in \mathcal{R}^3 and the equations (5.32.3). (5.22.2) shows that for every $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$, the grading for x_0 is at most

$$(5.37.4) \quad \sum_{i=1}^3 |x_i|_{\text{rel}} + 2y \cdot \bar{E}$$

where $y \cdot \bar{E}$ is again determined by \mathbf{n}_i . Setting $s(\mathbf{n}_i) = 2y \cdot \bar{E} + 1$ shows that (5.32.3) is satisfied. \square

Remark 5.38. We will see that we can take the function $\ell^\sharp(\mathbf{n}) = 2m - 2\ell_1(\mathbf{n})$, where ℓ_1 is the function defined in Section 6 provided $c_- \lambda_m > m$.

Definition 5.39. Given two positive integers m and m' , two functions $k^\sharp, \ell^\sharp: \pi_1(Y, L_0) \times \mathbb{Z}^{>0} \rightarrow \mathbb{Z}$ is said to satisfy $k^\sharp(\bullet, m) < \ell^\sharp(\bullet, m')$ if $k^\sharp(\mathbf{n}, m) < \ell^\sharp(\mathbf{n}, m')$ for any $\mathbf{n} \in \pi_1(Y, L_0)$.

The following definition will be used in the proof of Theorem 7.17.

Definition 5.40. Two admissible pairs $\left((G_m)_{m=1}^\infty, \ell_G^\sharp\right)$ and $\left((H_m)_{m=1}^\infty, \ell_H^\sharp\right)$ are *equivalent* if

- (1) for every m there are admissible homotopies: $H_{s,m}$ between H_m and $G_{m'}$ for some $m' > m$ with $\ell_{m',G}^\sharp > \ell_{m,H}^\sharp$, and $G_{s,m}$ between G_m and $H_{m''}$ for $m'' > m$ with $\ell_{m'',H}^\sharp > \ell_{m,H}^\sharp$, and
- (2) for any m and m' , and $x \in \mathcal{X}(L_0; H_m)$ and $y \in \mathcal{X}(L_0; G_{m'})$ which lie in the same relative homotopy class, there is a topological strip Σ in \bar{Y} between x and y satisfies $\Sigma \cdot E = 0$ and $\Sigma \cdot F = 0$.

Lemma 5.41. *Given an equivalence between two fibered admissible families, the induced maps $\text{HW}_{\text{ad}}^*(L_0; H_m) \rightarrow \text{HW}_{\text{ad}}^*(L_0; G_m)$ is an equivalence.*

Proof. The continuation maps induced by $G_{s,m}$ and $H_{s,m}$ preserve the relative gradings by (5.22.2) and hence the equivalence follows by a standard Floer theoretic argument. \square

Remark 5.42. Fix finitely many base-admissible Lagrangian sections L_i . By restricting our attention to suitable Hamiltonians of the form $H = H_{\text{ba}} + H_v$, it is possible to formulate a generalization of the functions ℓ^\sharp for every pair L_i and L_j , to construct an A_∞ category with objects L_i . As we have emphasized, we will discuss only the Lagrangian L_0 in this paper, and hence do not pursue this generalization here.

6. WRAPPED FLOER COHOMOLOGY RING OF L_0

6.1. Approximations of a quadratic Hamiltonian. In this section, we compute the adapted wrapped Floer cohomology ring of the admissible Lagrangian submanifold L_0 . The first step in our computation is to construct a particular sequence $(H_m)_{m=1}^\infty$ of fibration-admissible Hamiltonians. Our calculation will be modeled on the following simplified situation: The positive real part $N_{\mathbb{R}^{>0}} \times \mathbb{R}^{>0}$ is an open Lagrangian submanifold of $N_{\mathbb{C}^\times} \times \mathbb{C}$, which can be separated from the center $Z \times 0$ of the blow-up by an open set. By choosing a function supported in this open set as the cut-off function χ appearing in the symplectic form (2.5.16) on the blow-up, this Lagrangian submanifold naturally lifts diffeomorphically to the Lagrangian submanifold L_0 of Y . Note that $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$ has a natural quadratic Hamiltonian

$$(6.1.1) \quad H_{\text{quad}}(\mathbf{w}, u) := \frac{1}{2}|\mathbf{r}|^2 + \frac{1}{2}(\log |u|)^2.$$

The image of $N_{\mathbb{R}^{>0}} \times \mathbb{R}^{>0}$ under the time 1 Hamiltonian flow with respect to this quadratic Hamiltonian is given by $\theta_1 = r_1$, $\theta_2 = r_2$, and $\arg u = \log |u|$, so that the set of Hamiltonian chords is naturally in bijection with $N \times \mathbb{Z}$. This bijection sends a chord to its class in the relative homotopy group $\pi_1(N_{\mathbb{C}^\times} \times \mathbb{C}^\times, N_{\mathbb{R}^{>0}} \times \mathbb{R}^{>0})$, which can naturally be identified with $N \times \mathbb{Z}$. It is well known in this case that one can identify the wrapped Floer cohomology ring of L_0 in $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$ with the group ring $\mathbb{C}[N \times \mathbb{Z}]$.

We now regard L_0 as a Lagrangian submanifold of Y and construct a sequence $(H_m)_{m=1}^\infty$ of Hamiltonians, such that H_m behaves like H_{quad} in a compact set and has a slope λ_m elsewhere. To be more precise, for each $m \in \mathbb{Z}^{>0}$, we consider a function of the form $H_m = H_{\text{ba},m} + H_{v,m}$. We require that $H_{v,m} = f_m \circ \mu$ and that there exist sequences $\{b_m\}_{m=1}^\infty, \{c_m\}_{m=1}^\infty$ of positive real numbers satisfying the following conditions:

- $\lim_{m \rightarrow \infty} b_m = \infty$ and $\lim_{m \rightarrow \infty} c_m = 0$.
- $f_m: \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{> 0}$ is a convex function.
- $f_m(x) = \lambda_m x$ for $x > b_m$.
- $f_m(x) = \lambda_m(\epsilon - x)$ for $x < c_m$.
- If $m < m'$, then any Hamiltonian chord of $H_{v,m}$ is also a Hamiltonian chord of $H_{v,m'}$.

To construct the function $H_{\text{ba},m}$, we may embed the neighborhoods U_{Π_c} and U_{Π_i} in neighborhoods W_{Π_i} and W_{Π_c} , where

$$(6.1.2) \quad W_{\Pi_c} := \{\mathbf{r} \in N_{\mathbb{R}} \mid |\mathbf{r}| < R_1\}$$

for $R_0 < R_1 \ll R$ and

$$(6.1.3) \quad W_{\Pi_i} = \{\mathbf{r} \in N_{\mathbb{R}} \mid |\mathbf{r}_{\alpha_i - \beta_i}| < 2\varepsilon_n \text{ and } \mathbf{r}_{(\alpha_i - \beta_i)^\perp} \geq a_i + \varepsilon_n\}$$

Let $g_{\text{deg}}: Y \rightarrow \mathbb{R}$ be a smooth convex function satisfying

$$(6.1.4) \quad g_{\text{deg}} = \begin{cases} 0 & \mathbf{r} \in U_c, \\ \frac{c_i^2}{2R^2}\rho^2 + \psi_i(r_{\alpha_i - \beta_i}) & \mathbf{r} \in W_i \setminus (W_i \cap W_c), \quad i = 1, \dots, \ell, \\ \frac{S(\mathbf{r})}{R^2} & \mathbf{r} \in N_{\mathbb{R}} \setminus \left(W_c \cup \bigcup_{i=1}^{\ell} W_i\right) \end{cases}$$

where $\psi_i: \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ is a smooth convex function satisfying

- $\psi_i^{-1}(0) = [0, \varepsilon_n]$ and
- $\psi_i(r) = \frac{c_i r^2}{2R^2}$ if $r \geq 2\varepsilon_n$

We let $\overline{H}_{\text{ba},m}$ be a function satisfying

$$(6.1.5) \quad \overline{H}_{\text{ba},m} = \begin{cases} g_{\text{deg}} & \rho < \lambda_m - \varepsilon_m, \\ \lambda_m \rho + \text{constant} & \rho > \lambda_m + \varepsilon_m \end{cases}$$

for some $0 < \varepsilon_m \ll \lambda_m$ and $\overline{H}_{\text{ba},m} = h_i(\rho)$ in each tubular neighborhood U_i for some function $h_i: \mathbb{R} \rightarrow \mathbb{R}$. Finally, we let $H_{\text{ba},m}$ be a small perturbation of this function, where the perturbation is supported outside of $\overline{\pi}^{-1}(U_Z)$ (in W_i , it suffices to take a small perturbation of the function ψ). We assume that the Hamiltonian flow preserves $U_{Z \times 0}$, so that the Hamiltonian chords for L_0 live entirely in the region where the fibration $\overline{\pi}_{N_{\mathbb{C}^\times}}: \overline{Y} \rightarrow N_{\mathbb{C}^\times}$ is trivial. By choosing suitable slopes λ_m and perturbations in the definition of $H_{\text{ba},m}$, we can assume

(6.1.6) all Hamiltonian chords are in the region where $H_{\text{ba},m}$ is quadratic,

(6.1.7) for any $m \in \mathbb{Z}^{> 0}$ and any $\mathbf{n} \in N$ satisfying $|\mathbf{n}| \leq \lambda_m$, there are exactly $2\lambda_m + 1$ Hamiltonian chords $\{p_{\mathbf{n},i}^m\}_{i=-m}^m$ of H_m in the homotopy class \mathbf{n} , and

(6.1.8) all $p_{\mathbf{n},i}^m$ are non-degenerate.

We assume $\lambda_m > m$ and define $\ell^\sharp(\mathbf{n}) = 2k - 2\ell_1(\mathbf{n})$. These functions are admissible and we may consider the adapted Floer groups $\text{HF}_{\text{ad}}(L_0; H_m)$. For each Hamiltonian chord $p_{\mathbf{n},i}$, there is an associated orientation space $\mathfrak{o}_{p_{\mathbf{n},i}}$. However, as noted by Pascaleff [Pas14, Section 4.6], in our situation, the kernel and cokernel of the local orientation operator $D_{p_{\mathbf{n},i}}$ are trivial. It follows that $|\mathfrak{o}_{p_{\mathbf{n},i}}|$ is canonically trivialized and has a canonical generator. We denote this generator by $p_{\mathbf{n},i}$. In constructing homotopies between H_m and $H_{m'}$, we can and will assume that our family of Hamiltonians satisfies $\partial_s H_{s,t} \leq 0$ everywhere.

Proposition 6.2. *If $m < m'$, then the continuation map*

$$(6.2.1) \quad \mathrm{HF}_{\mathrm{ad}}(L_0; H_m) \rightarrow \mathrm{HF}_{\mathrm{ad}}(L_0; H_{m'})$$

sends $p_{\mathbf{n},i}^m$ to $p_{\mathbf{n},i}^{m'}$.

Proof. Note that the continuation map preserves homotopy classes. Since there is a unique Hamiltonian chord on $N_{\mathbb{C}^\times}$ in each homotopy class, the limits of the curve are the same once projected to $N_{\mathbb{C}^\times}$. The projection of such a solution $\pi_{N_{\mathbb{C}^\times}}(y)$ is therefore a topological cylinder such that $\pi_{N_{\mathbb{C}^\times}}(y) \cdot Z = 0$. It follows that such solutions y have intersection number zero with \overline{E} and in particular E . This means that these solutions lie in $(N_{\mathbb{C}^\times} \setminus Z) \times \mathbb{C}^\times$.

However, in this space, the orbits of our Hamiltonian all lie in different homotopy classes (or alternatively we can reach the same conclusion by comparison of $|\cdot|_{\mathrm{rel}}$). Our family of Hamiltonians satisfies $\partial_s H_{s,t} \leq 0$ everywhere. Hence we have $E^{\mathrm{geom}}(y) \leq E^{\mathrm{top}}(y)$. The quantity on the right-hand side is zero since the solution has the same asymptotes at the positive and negative ends. We therefore conclude that the solution is constant. \square

Corollary 6.3. *One has*

$$(6.3.1) \quad \mathrm{HW}_{\mathrm{ad}}(L_0) := \varinjlim_m \mathrm{HF}_{\mathrm{ad}}(L_0; H_m) = \mathrm{span}\{p_{\mathbf{n},i}\}_{(\mathbf{n},i) \in N \times \mathbb{Z}},$$

where $p_{\mathbf{n},i} = \lim_{m \rightarrow \infty} p_{\mathbf{n},i}^m$.

6.4. We may also consider the family $(2H_m)_{m=1}^\infty$ of Hamiltonians. We equip these with the function $\ell^\#(\mathbf{n}, m) = 4m - 2\ell_1(\mathbf{n})$. For any positive integer m and any $\mathbf{n} \in N$ satisfying $|\mathbf{n}| \leq 2\lambda_m$, there are exactly $4\lambda_m + 1$ Hamiltonian chords of $2H_m$ in the homotopy class \mathbf{n} . All of them are non-degenerate, and will be denoted as $q_{\mathbf{n},i}^m$ for $-2\lambda_m \leq i \leq 2\lambda_m$.

6.5. Calculation of the product. Let ϕ_{H_m} denote the time-one flow of X_{H_m} . For this section, we adopt the notation that $L_1 = \phi_{H_m}(L_0)$ and $L_2 = \phi_{2H_m}(L_0)$. Time-one chords in $\mathcal{X}_{H_m}(L_0)$ (respectively $2H_m$) correspond to intersection points of $\phi_{H_m}(L_0)$ and $\phi_{2H_m}(L_0)$. The first step in our proof is to relate our problem of calculating the product in wrapped Floer cohomology to the ordinary product in Lagrangian Floer cohomology for these Lagrangians.

For the purposes of our calculation, we will define the Floer triangle product with respect to perturbation data of the form

$$(6.5.1) \quad X_K = X_{H_m} \otimes \gamma,$$

where γ is a *closed* form on the pair of pants such that $\epsilon_0^* \gamma = 2dt$, $\epsilon_1^* \gamma = dt$ and $\epsilon_2^* \gamma = dt$. This implies that a Floer triangle can then be recast as a pseudo-holomorphic curve for a *domain-dependent* almost complex structure on Y . Given $y: \Sigma \rightarrow (Y, J, H)$, we consider $\tilde{y} := \phi_{H_m}^\tau \circ y$, where $\tau: \Sigma \rightarrow [0, 2]$ is the primitive of the closed one form γ used to define the pair-of-pants product. In particular, the product structure then agrees with the usual product operation

$$(6.5.2) \quad \mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_0, L_1) \rightarrow \mathrm{Hom}(L_0, L_2)$$

in Lagrangian Floer theory for the domain dependent almost complex structure $J_\tau = (\phi^\tau)_*^{-1}(J)$. We remark that, after perhaps modifying our original J at infinity, we may assume that J_τ is fibration-admissible. As a consequence of this, for this section we will consider only pseudoholomorphic maps and all perturbation data X_K will be taken to be trivial.

Proposition 6.6. *Let J' be another fibration-admissible almost complex structure. Then the continuation maps*

$$(6.6.1) \quad \mathrm{HF}_{\mathrm{ad}}(L_i, L_0; J') \rightarrow \mathrm{HF}_{\mathrm{ad}}(L_i, L_0; J_\tau)$$

are the identity.

Proof. The proof follows that of Proposition 6.2 quite closely. Namely, observe that such continuation solutions have intersection number zero with E . By comparing the relative indices $|\cdot|_{\mathrm{rel}}$, we see that the continuation maps must be constant. \square

Lemma 6.7. *Any Floer triangle y with inputs $p_{\mathbf{n},i}$, $p_{\mathbf{n}',i'}$ has a fixed intersection number $y \cdot \overline{E}$ with \overline{E} .*

Proof. The projected input chords $\pi_{N_{\mathbb{C}^\times}}(p_{\mathbf{n},i})$ and $\pi_{N_{\mathbb{C}^\times}}(p_{\mathbf{n}',i'})$ determine (the projected output chords and) the relative homotopy class of the projected Floer triangle $\pi_{N_{\mathbb{C}^\times}}(y)$ uniquely. This fixes the intersection number $\pi_{N_{\mathbb{C}^\times}}(y) \cdot Z = p(y) \cdot (Z \times \mathbb{C})$, which is equal to $y \cdot \overline{E}$ since $\overline{E} = E \cup F$ is the total transform of $Z \times \mathbb{C}$. \square

We set $j := y \cdot E$, which is a non-negative integer satisfying $j \leq j_{\max} := y \cdot \overline{E}$. Note that j_{\max} depends only on \mathbf{n} and \mathbf{n}' . The relative gradings imply that one has

$$(6.7.1) \quad p_{\mathbf{n},i} \cdot p_{\mathbf{n}',i'} = \sum_{j=0}^{j_{\max}} N_j(p_{\mathbf{n},i}, p_{\mathbf{n}',i'}) q_{\mathbf{n}+\mathbf{n}',i+i'+j}$$

for some integers $N_j(p_{\mathbf{n},i}, p_{\mathbf{n}',i'})$.

The rest of this section is devoted to the computation of $N_j(p_{\mathbf{n},i}, p_{\mathbf{n}',i'})$. To simplify notation, we set $\mathbf{p}_1 := p_{\mathbf{n},i}$, $\mathbf{p}_2 := p_{\mathbf{n}',i'}$, $\mathbf{p}_0 := q_{\mathbf{n}+\mathbf{n}',i+i'+j}$, and $\mathbf{p} := (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2)$. Fix $J \in \mathcal{J}_{\mathrm{int}}(D \cup \overline{E})$ and $J_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ such that p is pseudoholomorphic. Recall that $\mathcal{R}^2(Y, \mathbf{p})$ is the moduli space of pseudoholomorphic maps $y: \Sigma \rightarrow Y$ from disks $(\Sigma, (\zeta_0, \zeta_1, \zeta_2))$ with three boundary punctures satisfying

$$(6.7.2) \quad y(\partial_i \Sigma) \subset L_i$$

and

$$(6.7.3) \quad \lim_{s \rightarrow \pm\infty} y(\epsilon_i(s, -)) = \mathbf{p}_i$$

for $i = 0, 1, 2$. We now introduce some relative moduli spaces which will be needed in our argument.

Definition 6.8. 1. Let $\mathcal{R}^{2,j}(Y, \mathbf{p})$ be the moduli space of triples $((\Sigma, \boldsymbol{\zeta}), \mathbf{z}, y)$ consisting of disks $(\Sigma, \boldsymbol{\zeta})$ with three boundary punctures $\boldsymbol{\zeta} = (\zeta_0, \zeta_1, \zeta_2)$, j marked points $\mathbf{z} = (z_1, \dots, z_j)$ in the interior of Σ , and pseudoholomorphic maps $y: \Sigma \rightarrow Y$ satisfying (6.7.2) and (6.7.3).

2. Let $\mathcal{R}^{2,j}(Y, E, \mathbf{p})$ be the subspace of $\mathcal{R}^{2,j}(Y, \mathbf{p})$ consisting of pseudoholomorphic maps $y: \Sigma \rightarrow Y$ satisfying

$$(6.8.1) \quad \{z_i\}_{i=1}^j = y^{-1}(E).$$

Signs can be associated to these moduli spaces exactly like their non-relative counterparts. Namely, given a pseudoholomorphic map y , there exists an isomorphism

$$(6.8.2) \quad |\mathfrak{o}_{\mathbf{p}_0}| \cong |\det(D_y)| \otimes |\mathfrak{o}_{\mathbf{p}_1}| \otimes |\mathfrak{o}_{\mathbf{p}_2}|,$$

where D_y is the extended linearized operator at y . Whenever all curves in $\mathcal{R}^{2,j}(Y, \mathbf{p})$ are regular and the natural evaluation map

$$(6.8.3) \quad \mathrm{ev}: \mathcal{R}^{2,j}(Y, \mathbf{p}) \rightarrow Y^j$$

is transverse to E^j , this enables us to assign a map between orientation spaces for each isolated map $y \in \mathcal{R}^{2,j}(Y, E, \mathbf{p})$. This is because in this situation, the determinant of the linearized operator can be expressed as $\det(D_y) \cong \otimes_i \det(TY_{y(z_i)}) \otimes \det(TE_{y(z_i)})^\vee$ which is canonically oriented. There is an orientation-preserving surjection from $\mathcal{R}^{2,j}(Y, E, \mathbf{p})$ to $\mathcal{R}^2(Y, \mathbf{p})$, which is $j!$ to 1 because of the choice of an ordering of the marked points.

Definition 6.9. 1. Let $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, \mathbf{p})$ be the moduli space of triples $((\Sigma, \zeta), \mathbf{z})$ consisting of disks (Σ, ζ) with three boundary punctures $\zeta = (\zeta_0, \zeta_1, \zeta_2)$, j marked points $\mathbf{z} = (z_1, \dots, z_j)$ in the interior of Σ , and pseudoholomorphic maps $y: \Sigma \rightarrow N_{\mathbb{C}^\times} \times \mathbb{C}$ satisfying (6.7.2) and (6.7.3).

2. Let $\mathcal{R}^{2,j}(Y, E, \mathbf{p})$ be the subspace of $\mathcal{R}^{2,j}(Y, \mathbf{p})$ consisting of pseudoholomorphic maps $y: \Sigma \rightarrow N_{\mathbb{C}^\times} \times \mathbb{C}$ satisfying

$$(6.9.1) \quad \{z_i\}_{i=1}^j = y^{-1}(Z \times 0).$$

For suitably generic $J_{N_{\mathbb{C}^\times} \times \mathbb{C}}$, the linearized operator is surjective at all curves in $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, \mathbf{p})$ and the evaluation map

$$(6.9.2) \quad \text{ev}: \mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, \mathbf{p}) \rightarrow (N_{\mathbb{C}^\times} \times \mathbb{C})^j$$

is transverse to $(Z \times 0)^j$. For such $J_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ and when $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, \mathbf{p})$ is isolated, the moduli space can be given a relative orientation in a similar fashion to $\mathcal{R}^{2,j}(Y, E, \mathbf{p})$. We now show that split almost complex structures of the form $J_{N_{\mathbb{C}^\times} \times \mathbb{C}} = (J_B, J_{\mathbb{C}})$ on $N_{\mathbb{C}^\times} \times \mathbb{C}$ are sufficiently generic to achieve these two conditions. The Lagrangians L_i are also split Lagrangians, i.e., can be written as $\underline{L}_i \times \pi_{\mathbb{C}}(L_i)$. Thus, for this class of almost complex structures, we view our holomorphic curve $p \circ y$ as pairs (y_1, y_2) where

$$(6.9.3) \quad \begin{cases} y_1: \Sigma \rightarrow N_{\mathbb{C}^\times}, \\ y_1(\partial_k \Sigma) \subset \underline{L}_k, \\ \lim_{s \rightarrow \pm\infty} y_1(\epsilon_k(s, -)) = \underline{q}_{\mathbf{n}+\mathbf{n}'}, \underline{p}_{\mathbf{n}}, \underline{p}_{\mathbf{n}'}, \end{cases}$$

and

$$(6.9.4) \quad \begin{cases} y_2: \Sigma \rightarrow \mathbb{C}, \\ y_2(\partial_k \Sigma) \subset \pi_{\mathbb{C}}(L_k), \\ \lim_{s \rightarrow \pm\infty} y_2(\epsilon_k(s, -)) = q_{i+i'+j}, p_i, p'_i. \end{cases}$$

We have

$$(6.9.5) \quad y_1 \cdot Z = \overline{E} \cdot y = j_{\max}.$$

For generic J , we can assume that $y_1^{-1}(Z)$ are distinct points, and we fix an ordering $(z_1, \dots, z_{j_{\max}})$ of these points. Given an ordered subset (i_1, \dots, i_j) of $\{1, \dots, j_{\max}\}$, the triple $((\Sigma, \zeta), \mathbf{z}, (y_1, y_2))$ with $\mathbf{z} = (z_{i_1}, \dots, z_{i_j})$ and (y_1, y_2) satisfying (6.9.3) and (6.9.4) lies in $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{p})$ if and only if

$$(6.9.6) \quad y_2^{-1}(0) = \{z_{i_1}, \dots, z_{i_j}\}.$$

Lemma 6.10. *If $J_{N_{\mathbb{C}^\times} \times \mathbb{C}} = (J_B, J_{\mathbb{C}})$ is the product of a generic almost complex structure J_B adapted to Z and the standard almost complex structure $J_{\mathbb{C}}$, then all pseudoholomorphic maps in $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, \mathbf{p})$ are Fredholm regular and the evaluation map (6.9.2) is transverse to $(Z \times 0)^j$.*

Proof. We have split Lagrangian boundary conditions and no curve is constant when projected to either of the product factors, so it is clear that we can achieve surjectivity of the operator D_y using split J . The transversality of the evaluation map (6.9.2) to $Z \times 0$ is equivalent to the transversality of y_1 with Z , which can be achieved by a small perturbation of J_B , and the transversality with 0^j of the evaluation map for the moduli space of curves of the form (6.9.6) to \mathbb{C}^j . It suffices to prove that the tangent space to y_2 of the moduli space of pseudoholomorphic maps of the form (6.9.6) is trivial. An element of this tangent space is precisely an element of $\text{Ker}(D_{y_1})$ which vanishes at z_{i_1}, \dots, z_{i_j} . The key for us is that $y_1^*(T\mathbb{C}_u)$ is a line bundle and our element is a section satisfying a $\bar{\partial}$ -equation with a prescribed number of zeroes. We may therefore apply [Sei08, Proposition 11.5], which controls the number of zeroes on such solutions on line bundles, to show that this element must be trivial. \square

Fix $J \in \mathcal{J}_{\text{int}}(D \cup \bar{E})$ and $J_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ be a split almost complex structure such that p is pseudoholomorphic. Given a pseudoholomorphic triangle y in $\mathcal{R}^2(Y, E, \mathbf{p})$ with boundary on L_i , we can compose it with the structure morphism $p: \bar{Y} \rightarrow N_{\mathbb{C}^\times} \times \mathbb{C}$ to obtain a curve $p \circ y$. Moreover given a curve $p \circ y$, we can reconstruct y as the proper transform of $p \circ y$. The condition that $y \in \mathcal{R}^{2,j}(Y, E, \mathbf{p})$ is equivalent to $p \circ y \in \mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{p})$. In particular, whenever curves in $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{p})$ are isolated, so are curves in $\mathcal{R}^{2,j}(Y, E, \mathbf{p})$. In fact, this persists at the level of linearized operators.

Lemma 6.11. *Let $J \in \mathcal{J}_{\text{int}}(D \cup \bar{E})$ and $J_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ be a split almost complex structure satisfying the conditions of Lemma 6.10 such that p is pseudoholomorphic. Let further y be the proper transform of a curve in $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{p})$. Then the linearized operator D_y is surjective.*

Proof. Since all of our moduli spaces are isolated, it suffices to check that any element $u \in \ker(D_y)$ such that the linearized evaluation map is tangent to E is zero. Given such a u , we may push this down to an element $p \circ u \in \ker(D_{p \circ y})$ such that the linearized evaluation maps at z_i are tangent to $Z \times 0$. By our hypothesis, such an element must be zero. For every point $z \neq z_i$ the map $y^*(TY)_z \rightarrow (p \circ y)^*(T(N_{\mathbb{C}^\times} \times \mathbb{C})_p(z))$ induced by the projection from y to $p \circ y$ is injective and hence it, it follows that $u = 0$. \square

The key enumerative calculation is the following:

Lemma 6.12. *The numbers $N_j(p_{\mathbf{n},i}, p_{\mathbf{n}',i'})$ appearing in (6.7.1) are $\binom{j_{\max}}{j}$.*

Proof. We first determine the signed count of elements in the moduli space $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{p})$, up to dividing by $j!$ to remove for the redundancy from the ordering of the marked points. After picking j points z_{i_1}, \dots, z_{i_j} , it suffices to prove that the signed count of pairs (y_1, y_2) is 1. Let μ be the standard moment map on the disc and consider a vertically admissible Hamiltonian $v(\mu)$ which is of slope $\lambda_v > 1$ near $\mu = 0$. Let α be the Hamiltonian orbit whose homotopy class in \mathbb{C}^\times is primitive. Let S be $\mathbb{P}^1 \setminus \{0\}$ with a distinguished marked point at $z = \infty$ and a negative cylindrical end near $z = 0$ given by

$$(6.12.1) \quad (s, t) \rightarrow e^{s + \sqrt{-1}t}.$$

Fix a subclosed one-form β which restricts to dt on the cylindrical end and which restricts to zero in a neighborhood of $z = \infty$. To be explicit, we consider a non-negative, monotone non-increasing cutoff function $\rho(s)$ such that

$$(6.12.2) \quad \rho(s) = \begin{cases} 0 & s \gg 0, \\ 1 & s \ll 0, \end{cases}$$

and set $\beta = \rho(s)dt$. The remainder of our proof is divided into two steps:

Step 1. Consider solutions y_I to the equation

$$(6.12.3) \quad \begin{cases} y_I : S \rightarrow \mathbb{C} \\ (dy_I - X_{v(\mu)} \otimes \beta)^{0,1} = 0 \\ y_I(0) = 0 \\ \lim_{s \rightarrow -\infty} u(\epsilon(s, -)) = \alpha \end{cases}$$

There is a well-known isomorphism

$$(6.12.4) \quad \mathrm{SH}^0(\mathbb{C}^\times) \cong H_0(C_{1-*}(\mathcal{L}\mathbb{S}^1)) \cong \mathbb{C}[\alpha, \alpha^{-1}].$$

We wish to prove that the signed count of curves y_I as above is $+1$. Let S_2 be the disc with a fixed boundary marked point ζ and with one interior puncture. Equip S_2 with a positive strip-like end near the puncture. Let $L = \mathbb{S}^1$ denote the zero section of $T^*\mathbb{S}^1 \cong \mathbb{C}^\times$ and $x \in L$. A consequence of the isomorphism (6.12.4) is that for suitable choices of Floer data, the signed count of curves with domain S_2 with Hamiltonian input α and with the marked point ζ passing through x is $+1$. Now view \mathbb{C} as a compactification of \mathbb{C}^\times and glue S to S_2 along α to obtain a disc with boundary on L and with a marked point passing through $z = 0$. If we then the perturbation data to zero, the result then follows from the obvious fact that there is a unique disc in \mathbb{C} with a marked point at $z = 0$ and a boundary marked point at any point on the Lagrangian L .

Step 2. With this established, let S_3 be a Riemann surface with j positive cylindrical ends, two positive strip like ends and one negative strip like end. There is also an isomorphism

$$(6.12.5) \quad \mathrm{HW}^0(T_q) \cong \mathbb{C}[p_1, p_1^{-1}]$$

After choosing suitable perturbation data it follows from (6.12.4), (6.12.5), together with the TQFT structure that the signed count of curves S_3 with chord inputs $p_i, p_{i'}$, Hamiltonian orbit inputs α and output $q_{i+i'+j}$ must be $+1$. At each Hamiltonian input, we may then glue on the above curves with domain S to obtain maps satisfying all of the requirements of (6.9.6) except that the marked points z'_{i_j} may be different and there will be non-trivial perturbation data X_K (these depend on the gluing parameters along the ends).

We may deform away the Floer data and deform the marked points to z_{i_j} as above. The signed count of these curves does not change because there is no possible breaking and it follows that the signed count of elements y_1 satisfying (6.9.6) is $+1$. The signed count of curves y_2 is also $+1$. Since we have split Lagrangian boundary conditions and almost complex structures, the count of configurations of curves of the form (y_1, y_2) contribute $+1$. One readily checks that the bijection between $\mathcal{R}^{2,j}(N_{\mathbb{C}^\times} \times \mathbb{C}, Z \times 0, \mathbf{p})$ and $\mathcal{R}^{2,j}(Y, E, \mathbf{p})$ preserves orientations, so the signed count of curves (y_1, y_2) contributes $+1$ to the Floer product as well. \square

Theorem 6.13. *Given two generators, $p_{\mathbf{n},i}$ and $p_{\mathbf{n}',i'}$, let $m = \overline{E} \cdot y$. Then we have*

$$(6.13.1) \quad p_{\mathbf{n},i} \cdot p_{\mathbf{n}',i'} = \sum_{j=0}^m \binom{m}{j} q_{\mathbf{n}+\mathbf{n}',i+i'+j}.$$

Proof. This is a consequence of Lemma 6.12. \square

We next observe the following fact concerning continuation maps. The above theorem allows us to calculate the continuation map:

$$(6.13.2) \quad \mathbf{c} : \mathrm{HF}_{\mathrm{ad}}(L_0; H) \rightarrow \mathrm{HF}_{\mathrm{ad}}(L_0; 2H)$$

Lemma 6.14. *For an arbitrary element $p_{\mathbf{n},i}$, one has $\mathbf{c}(p_{\mathbf{n},i}) = q_{\mathbf{n},i}$.*

Proof. The arguments concerning relative index in Proposition 6.2 and Proposition 6.6 together with the fact that all of our Floer products are in fact defined over \mathbb{Z} imply that the continuation map must send $p_{\mathbf{n},i} \rightarrow \pm q_{\mathbf{n},i}$. A standard Floer theoretic argument shows that the continuation map sends $\mathbf{c}(p_{\mathbf{n},i}) = p_{0,0} \cdot p_{\mathbf{n},i}$. The result follows immediately from the fact that the Floer triangle y with inputs $p_{0,0}$, $p_{\mathbf{n},i}$ and outputs $q_{\mathbf{n},i}$ must have intersection number zero since we already knew that $\mathbf{c}(p_{\mathbf{n},i}) = \pm q_{\mathbf{n},i}$. \square

Lemma 6.15. *The intersection number is given by*

$$(6.15.1) \quad \overline{E} \cdot y = \ell_2(\mathbf{n}, \mathbf{n}').$$

(2.9.6) and (6.13.1) show that the linear map

$$(6.15.2) \quad \mathrm{Mir}_{\mathrm{ad}}^{L_0} : \mathrm{HW}_{\mathrm{ad}}(L_0) \xrightarrow{\sim} H^0(\mathcal{O}_{\hat{Y}})$$

defined by

$$(6.15.3) \quad p_{\mathbf{n},i} \mapsto p^i \chi_{-\mathbf{n}, \ell_1(\mathbf{n})}$$

is an isomorphism of algebras, and Theorem 1.3 is proved.

7. COMPARISON OF FLOER THEORIES

7.1. The Liouville domain. For this section, we define

$$(7.1.1) \quad \pi_{\hat{B}} : Y \rightarrow \hat{B} := N_{\mathbb{R}} \times \mathbb{R}, \quad x \mapsto (\log |w_1|, \log |w_2|, \log \mu).$$

Let r_3 be the third coordinate on $N_{\mathbb{R}} \times \mathbb{R}$ and

$$(7.1.2) \quad \pi_{N_{\mathbb{R}}} : \hat{B} \rightarrow N_{\mathbb{R}}$$

be the natural projection. We will begin by considering a “toric” version T_R of our Liouville domain. The Liouville domain we will actually study, Y_R , will be constructed by modifying T_R along $\pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$. Define

$$(7.1.3) \quad \pi_{\hat{B}}^0 : N_{\mathbb{C}^\times} \times \mathbb{C}^\times \rightarrow \hat{B},$$

in which we replace the above moment map by the toric moment map $\mu_T = |u|^2/2$. In coordinates (r_i, θ_i) , the manifold $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$ is equipped with the symplectic form

$$(7.1.4) \quad \omega^T = dr_1 \wedge d\theta_1 + dr_2 \wedge d\theta_2 + e^{r_3} dr_3 \wedge d\theta_3$$

For some R , μ_0 and μ_1 with $R \gg 0$ and $0 < \epsilon - \mu_0 \ll \epsilon \ll \epsilon + \mu_1$, consider the region in \hat{B} defined by $\epsilon - \mu_0 \leq \mu_T \leq \epsilon + \mu_1$ and $H_b \leq R$. We choose μ_1 sufficiently large so that $\chi(u, \mathbf{w}) = 0$ for $\mu_T \geq \epsilon + \mu_1/2$. This forms a cornered region in \hat{B} , with corners given by

$$(7.1.5) \quad \{H_b = R, \mu_T = \epsilon - \mu_0\} \cup \{H_b = R, \mu_T = \epsilon + \mu_1\}.$$

We may “round corners”, i.e., make a deformation in a neighborhood of these corners to obtain a convex region $D_R \in \hat{B}$ such that the boundary ∂D_R is smooth and the region

$$(7.1.6) \quad T_R := (\pi_{\hat{B}}^0)^{-1}(D_R)$$

is a Liouville domain. To be more precise, we require the following conditions:

(7.1.7) For every leg Π_i and every point $x \in \partial D_R \cap \pi_{N_{\mathbb{R}}}^{-1}(U_{\Pi_i})$, the boundary ∂D_R is locally defined by $D_i(r_{\alpha-\beta}^\perp, \mu_T) = 0$ for some function $D_i: \mathbb{R}^2 \rightarrow \mathbb{R}$.

(7.1.8) Fix a number δ_R , which is relatively small compared to R . For the region bounded by $H_b \leq R - \delta_R$ under $\pi_{N_{\mathbb{R}}}$, ∂D_R agrees with $\{\mu_T = \epsilon - \mu_0\} \cup \{\mu_T = \epsilon + \mu_1\}$.

(7.1.9) In $\pi_{N_{\mathbb{R}}}^{-1}(U_{\Pi_i})$, when $\epsilon - \mu_0/2 \leq \mu_T \leq \epsilon + \mu_1/2$, ∂D_R agrees with $H_b = R$.

In particular, (7.1.7) implies that for any point $x \in \partial D_R \cap \pi_{N_{\mathbb{R}}}^{-1}(U_{\Pi_i})$, the projection to $N_{\mathbb{R}}$ of the normal vector to ∂D_R at x is parallel to Π_i . We wish to construct a version of this manifold which is adapted to Y . We first consider $Y_R^0 := \pi_{\widehat{B}}^{-1}(D_R)$, which is a Liouville domain for a suitable choice of D_R . However, for certain purposes, it is convenient to modify this domain further to have better control of the Reeb flow along the boundary near the discriminant locus. We remove a tubular neighborhood of each leg Π_i from the boundary to obtain $\partial Y_R^0 \setminus (\pi_{\widehat{B}} \circ \pi_{N_{\mathbb{R}}})^{-1}(\cup_i U_{\Pi_i})$. Over each region $(\pi_{\widehat{B}} \circ \pi_{N_{\mathbb{R}}})^{-1}(U_{\Pi_i})$, we may glue in the hypersurface given by $D_i(\rho, \mu) = 0$ to obtain a hypersurface ∂Y_R . Let Y_R be the Liouville domain bounded by this hypersurface.

Outside of U_Z , the manifolds Y_R and T_R are canonically identified. Let $\partial_i Y_R^0$ be the part of ∂Y_R^0 defined by $c_i r_{\alpha-\beta}^\perp = R$. Over $(\pi_{N_{\mathbb{R}}} \circ \pi_{\widehat{B}})^{-1}(\cup_i U_{\Pi_i})$, we glue in the piece of the hypersurface ∂Y_R where $D_i(\rho, \mu) = \rho$ to obtain a hypersurface with boundary which we denote by $\partial_i Y$. We view $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$ as $T^*\widehat{B}/\mathbb{Z}^3$ which we identify using the standard Euclidean inner product with $\widehat{B} \times \widehat{B}/\mathbb{Z}^3$. For any point $x \in \partial D_R$, let \vec{n}_x denote the normal vector to ∂D_R in the diagonal metric on \widehat{B} with coefficients

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{r_3} \end{pmatrix}.$$

Lemma 7.2. *The Reeb flow on ∂T_R fibers over ∂D_R and points in the direction of \vec{n}_x for every $x \in \partial D_R$.*

Proof. The tangent space to ∂T_R is given by the span of $(T_{D_R}, \partial_{\theta_i})$. Set $\vec{v} = \sum_i a_i \partial_{\theta_i}$. We have $\omega^T(\vec{v}, \vec{v}') = 0$ for any \vec{v}' which is a linear combination of the ∂_{θ_i} and for any $\vec{v}' \in T_{D_R}$ if and only if $a_1 \partial_{r_1} + a_2 \partial_{r_2} + a_3 \partial_{r_3}$ is normal to \vec{v}' in the above diagonal metric. \square

Definition 7.3. A non-zero element \mathbf{n} of N is *regular* if it does not point in the direction of any leg Π_i in $N_{\mathbb{R}}$.

We also consider the following conditions:

(7.3.1) Every Reeb orbit of ∂Y_R , whose homotopy class $\mathbf{n} \in \pi_1(Y) \cong N$ is regular, projects by $\pi_{N_{\mathbb{R}}} \circ \pi_{\widehat{B}}$ to the chamber C_{α_0} for the $\alpha_0 \in A$ such that $\alpha_0(\mathbf{n}) = \sup_{\alpha \in A} \alpha(\mathbf{n})$. Every Reeb orbit with non-regular homotopy class projects to the corresponding neighborhood U_i of the leg.

(7.3.2) The region where ∂Y_R agrees with $\{\mu = \epsilon - \mu_0\}$ or $\{\mu = \epsilon + \mu_1\}$ is precisely where $\rho \leq R - \delta_R$.

(7.3.3) For any $\mathbf{w} \in N_{\mathbb{C}^\times}$, the image of $\partial_i Y \cap \pi_{N_{\mathbb{C}^\times}}^{-1}(\mathbf{w})$ by μ is $[\epsilon - \mu_0/2, \epsilon + \mu_1/2]$.

We next deform the Liouville form θ_ϵ to a Liouville form θ'_ϵ on Y_R so that \overline{E} is preserved by both the Liouville flow and the Reeb flow along the boundary ∂Y_R . This will be necessary since for some arguments it will be necessary to choose the almost complex structure to be both adapted with respect to the Liouville domain and to preserve the divisor \overline{E} .

Lemma 7.4. *There is a Liouville one-form $\theta'_{N_{\mathbb{C}^\times}}$ on $N_{\mathbb{C}^\times}$ which agrees with $\theta_{N_{\mathbb{C}^\times}}$ outside of U_Z and such that the Liouville vector field preserves Z .*

Proof. Let $\iota_Z: Z \rightarrow N_{\mathbb{C}^\times}$ be the inclusion and $\pi_Z: Z \times \mathbb{D}_\delta \rightarrow Z$ be the first projection, where $\mathbb{D}_\delta := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \delta\}$ is a disk of radius δ . The symplectic tubular neighborhood theorem gives an embedding $\iota_{Z \times \mathbb{D}_\delta}: Z \times \mathbb{D}_\delta \rightarrow N_{\mathbb{C}^\times}$ satisfying

$$(7.4.1) \quad \iota_{Z \times \mathbb{D}_\delta}^* \omega_{N_{\mathbb{C}^\times}} = dx \wedge dy + \pi_Z^* \iota_Z^* \omega_{N_{\mathbb{C}^\times}}.$$

Note that $xdy + \pi_Z^* \iota_Z^* \theta_{N_{\mathbb{C}^\times}}$ is a primitive of $\iota_{Z \times \mathbb{D}_\delta}^* \omega_{N_{\mathbb{C}^\times}}$, and any other primitive differs from this by a closed form. Along each leg of Z , we can choose the standard tubular neighborhood of Z defined by $\{|\mathbf{w}^{\alpha_i - \beta_i} + t^{v(\beta) - v(\alpha)}| \leq \delta\}$, and the coordinate of the disc factor to be

$$(7.4.2) \quad x := \log |\mathbf{w}^{\alpha_i - \beta_i}| - (v(\beta) - v(\alpha)) \log t, \quad y := \text{Arg } \mathbf{w}^{\alpha_i - \beta_i} - \pi,$$

setting $x_{0,i} = -(v(\beta) - v(\alpha)) \log t$, we have

$$(7.4.3) \quad \iota_{Z \times \mathbb{D}_\delta}^* \theta_{N_{\mathbb{C}^\times}} = (x - x_{0,i})dy + \pi_Z^* \iota_Z^* \theta_{N_{\mathbb{C}^\times}}.$$

Therefore, we have

$$(7.4.4) \quad \iota_{Z \times \mathbb{D}_\delta}^* \theta_{N_{\mathbb{C}^\times}} - (xdy + \pi_Z^* \iota_Z^* \theta_{N_{\mathbb{C}^\times}}) = \pi_Z^* \eta + dh,$$

where η is a compactly supported closed one-form on Z and $h = x_{0,i}y$ along the legs of Z . Let $\rho := \sqrt{x^2 + y^2}$ be the radial coordinate on \mathbb{D}_δ and consider a cutoff function $\chi(\rho)$ such that $\chi(\rho) = 1$ for $\rho < \delta/4$ and $\chi(\rho) = 0$ for $\rho > \delta/2$. The the one-form

$$(7.4.5) \quad \theta'_{Z \times D_{\mathbb{D}_\delta}} := xdy + \pi_Z^* \iota_Z^* \theta_{N_{\mathbb{C}^\times}} + \pi_Z^* \eta + d(\chi h)$$

is a primitive of $\iota_{Z \times \mathbb{D}_\delta}^* \omega_{N_{\mathbb{C}^\times}}$ and the corresponding Liouville vector field is tangent to Z . Since $\theta'_{Z \times D_{\mathbb{D}_\delta}}$ agrees with $\iota_{Z \times \mathbb{D}_\delta}^* \theta_{N_{\mathbb{C}^\times}}$ when $\rho > \delta/2$, one can glue $(\iota_{Z \times \mathbb{D}_\delta}^{-1})^* \theta'_{Z \times D_{\mathbb{D}_\delta}}$ with $\theta_{N_{\mathbb{C}^\times}}$ to obtain a Liouville one-form $\theta'_{N_{\mathbb{C}^\times}}$ such that the corresponding Liouville vector field preserves Z . \square

Our proof of Lemma 7.4 shows that if we assume that the non-compact legs Π_i pass through the origin, then such a form can be chosen to agree with $\theta_{N_{\mathbb{C}^\times}}$ outside of small neighborhoods of the vertices of Z . Choose a primitive as in the proof of Lemma 7.4 and let $\theta'_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ be the induced primitive one form of $\omega_{N_{\mathbb{C}^\times} \times \mathbb{C}}$ on $N_{\mathbb{C}^\times} \times \mathbb{C}$. Finally, set

$$(7.4.6) \quad \theta'_\epsilon := \theta_{\text{vc}} + p^* \theta'_{N_{\mathbb{C}^\times} \times \mathbb{C}}.$$

The Liouville coordinate in the fiber is given by $C|\mu - \epsilon|$ for some $C > 0$. Let $V_{\theta'_\epsilon}$ be the Liouville vector field which is the ω_ϵ dual of θ'_ϵ .

Lemma 7.5. *There is a constant B such that when $\rho \geq R$ and $\mu \leq \epsilon - \mu_0$ or $\mu \leq \epsilon + \mu_1$, we have estimates*

$$(7.5.1) \quad d\rho(V_{\theta'_\epsilon}) \leq B\rho$$

and

$$(7.5.2) \quad d|\mu - \epsilon|(V_{\theta'_\epsilon}) \leq |\mu - \epsilon|.$$

Proof. The key to this estimate is again the local T^2 action in the tubular neighborhoods. Both (7.5.1) and (7.5.2) are immediate outside of the tubular neighborhood. Let us first consider (7.5.1), when, inside of the tubular neighborhood, we have

$$(7.5.3) \quad d\rho(V_{\theta'_\epsilon}) = \omega(V_{\theta'_\epsilon}, c_i \partial_{\theta_{\alpha-\beta}}^\perp)$$

$$(7.5.4) \quad = p^* \theta'_{N_{\mathbb{C}^\times} \times \mathbb{C}}(c_i \partial_{\theta_{\alpha-\beta}}^\perp) - d^c(\chi(G_i) \log(F_i))(c_i \partial_{\theta_{\alpha-\beta}}^\perp)$$

$$(7.5.5) \quad = \rho$$

The second estimate (7.5.2) is similar. Namely, we have $\theta'_\epsilon(\partial_\theta) = \mu - \epsilon$ and hence $d|\mu - \epsilon|(V_{\theta'_\epsilon}) = |\mu - \epsilon|$. \square

Let r_Y denote the Liouville coordinate on the completion of Y_R . It is not difficult to see that the Liouville flow is not complete as we approach the divisor D . However, given μ_0 , we can assume that we have an embedding of the region of the symplectic completion

$$(7.5.6) \quad Y_R \cup \partial Y_R \times [1, c_{\mu_0}] \rightarrow Y.$$

Lemma 7.5 implies the following direct analogue of [McL09, Lemma 5.7], which is crucial in McLean's proof of [McL09, Theorem 5.5].

Corollary 7.6. *When $\rho \geq R$ and $\mu \leq \epsilon - \mu_0$ or $\mu \leq \epsilon + \mu_1$, one has*

$$(7.6.1) \quad \rho \leq e^B r_Y$$

and

$$(7.6.2) \quad C|\mu - \epsilon| \leq r_Y.$$

Proof. This follows from Lemma 7.5 because r_Y is defined to be the integration of the Liouville vector field for time $\log(r_Y)$. \square

As a consequence of Corollary 7.6, we may take $c_{\mu_0} \rightarrow \infty$ as $\mu_0 \rightarrow 0$. For the moment take μ_0 sufficiently small so that $c_{\mu_0} \geq 2$. The key point of modifying the primitive can be seen in the following lemma:

Lemma 7.7. *For suitable choices of R and μ_1 , the Liouville flow on Y_R preserves \overline{E} for $r_Y \geq 1$.*

Proof. Consider equation (2.6.1). We may expand this out further to see that at points along points lying in E ,

$$(7.7.1) \quad \omega_\epsilon = \pi_{N_{\mathbb{C}^\times}}^* \omega_{N_{\mathbb{C}^\times}} + \frac{\sqrt{-1}}{4} |v_0|^2 dh \wedge d\bar{h} + \frac{\sqrt{-1}\epsilon}{2\pi} \partial \bar{\partial} (\log(1 + |v_0|^2))$$

Similarly, for the primitive θ_ϵ , we have

$$(7.7.2) \quad \theta'_\epsilon = \pi_{N_{\mathbb{C}^\times}}^* \theta'_{N_{\mathbb{C}^\times}} - \frac{\epsilon}{4\pi} d^c (\log(1 + |v_0|^2) - \log|v_0|^2)$$

along points lying in E . The second term of (7.7.1) vanishes on vector fields tangent to Z . Since the Liouville vector field $V_{\theta'}$ on $N_{\mathbb{C}^\times}$ is tangent to Z , we have that viewing this chart as a product manifold with coordinates (\mathbf{w}, v_0)

$$(7.7.3) \quad \pi_{N_{\mathbb{C}^\times}}^* \omega_{N_{\mathbb{C}^\times}} + \frac{\sqrt{-1}}{4} |v_0|^2 dh \wedge d\bar{h}((V_{\theta'}, 0), -) = \pi_{N_{\mathbb{C}^\times}}^* \theta'_{N_{\mathbb{C}^\times}}(-)$$

as required. On the other component of $\overline{E} \cap \partial Y_R$, after choosing μ_1 and R appropriately assume that the function $\chi(u, \mathbf{w}) = 0$ on $\overline{E} \cap \partial Y_R$ except when $h = t^{-v(\alpha)} w^\alpha + t^{-v(\beta)} w^\beta$. Following the notation and proof of Proposition 5.5, we have that the $\mathbf{w}^{\alpha-\beta}$ components of $d^c(G_i)$ and $d^c(F_i)$ vanish over this locus which implies the same for $d^c(\chi(G_i) \log(F_i))$.

For any point in this portion of $\overline{E} \cap \partial Y_R$, let S be the subspace of the tangent space spanned by $(\partial_{r_{\alpha-\beta}}, \partial_{\theta_{\alpha-\beta}})$. Restricting to S at the points in question, we have that the one form θ'_ϵ vanishes there. As can be seen from (5.5.1), we have that the symplectic orthogonal to the subspace S is given by the span of $(\partial_{r_{\alpha-\beta}^\perp}, \partial_{\theta_{\alpha-\beta}^\perp}, \partial_{|u|}, \theta_u)$. It follows that the projection of the Liouville field to those coordinates vanishes and the result is proven. \square

7.8. Let us discuss the Reeb flow. Away from $\pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$, the Reeb flow corresponds with that on ∂T_R as described in Lemma 7.2. Within $\pi_{N_{\mathbb{R}}}^{-1}(U_{\Pi_i})$, the Reeb flow acts on the T^2 orbits of the local torus action with slope $(\partial_\rho D_i(\rho, \mu), \partial_\mu D_i(\rho, \mu))$. For all other points in $\pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$, we have by (7.1.8) that the Reeb flow rotates the fibers of the conic bundle. In particular, the Reeb flow also preserves \overline{E} as well as the neighborhood $\pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$.

Let ∂L_0 denote the Legendrian $L_0 \cap \partial Y_R$. The Reeb flow on ∂T_R or ∂Y_R is degenerate. However, we have good control over the families of Reeb orbits and Legendrian Reeb chords of ∂L_0 . There are families $\mathcal{F}_{(\mathbf{n}, i)}$ of parameterized Reeb orbits such that the entire family avoids $\pi^{-1}(U_Z)$ which come in families of the following type:

- (Type I) For every $(\mathbf{n}, 0)$ with \mathbf{n} regular, there is a manifold with boundary of parameterized Reeb orbits diffeomorphic to $I \times T^3$. These Reeb orbits occur where the r_3 -component of the normal vector to ∂D_R vanishes. By the assumption (7.1.9) on the shape of D_R , this fibers over an interval in the \mathbb{R}^3 projection.
- (Type II) For every other class (\mathbf{n}, i) with \mathbf{n} regular, we may assume that we have a T^3 worth of orbits.

In both cases we denote the family by $\mathcal{F}_{(\mathbf{n}, i)}$ for \mathbf{n} regular. When viewed as Reeb orbits of ∂T_R , these identifications are T^3 equivariant with respect to the natural actions on both sides.

There are additional families of Reeb orbits in homotopy class with $\mathbf{n} \neq 0$ non-regular:

- (Type III) Those families of orbits beginning at any point in $\partial_i Y$. We will denote these by $\mathcal{F}_{(\mathbf{n}, 0)}$ for \mathbf{n} non-regular.
- (Type IV) Those families which contain Reeb orbits in $\pi^{-1}(U_{\Pi_j})$ with $\partial_\mu D_j(\rho, \mu) \neq 0$. We may assume that these families are diffeomorphic to $I \times T^3$. We will denote these by $\mathcal{F}_{(\mathbf{n}, i)}$ for \mathbf{n} non-regular and $i \neq 0$ corresponds to the winding number of the flow in the fibers of the conic fibration.

Finally there are contractible Reeb orbits:

- (Type V) When $\{\mu = \epsilon - \mu_0\}$ if $i > 0$ or $\{\mu = \epsilon - \mu_1\}$ if $i < 0$. These are denoted by $\mathcal{F}_{(0, i)}$.

For each of these families $\mathcal{F}_{(\mathbf{n}, i)}$, the Reeb flow gives $\mathcal{F}_{(\mathbf{n}, i)}$ the structure of an oriented \mathbb{S}^1 bundle over a manifold with corners. Every $\mathcal{F}_{(\mathbf{n}, i)}$ gives rise to a contractible family $\mathcal{F}_{(\mathbf{n}, i)}^0 := \partial L_0 \cap \mathcal{F}$ of Reeb chords and these are all Reeb chords of ∂L_0 . For families of Type I, $\mathcal{F}_{(\mathbf{n}, i)}^0 \cong I$, an interval and we may assume this for families of Type IV as well. For families of Type II, $\mathcal{F}_{(\mathbf{n}, i)}^0$ is a point (non-degenerate chord). Finally, we have an I^2 in the Type III case and D^2 in the Type V case.

Definition 7.9. Let \widehat{Y}_R denote the Liouville completion of Y_R and let $\widehat{\theta}'_\epsilon$ denote the standard extension of the Liouville form to the completion. We say that an almost complex structure on \widehat{Y}_R is Liouville-admissible in a neighborhood of a level set $r_Y = c$ if there exists a positive real number δ such that $\widehat{\theta}'_\epsilon \circ J = dr_Y$ holds in the region defined by $c - \delta < r_Y < c + \delta$.

It is these almost complex structures which are used in [AS10]. We will make use of almost complex structure which combine Definition 7.9 and 5.10.

Definition 7.10. We let $J(Y_R, \overline{E})$ denote the set of admissible almost complex structures on \overline{Y} which are Liouville-admissible in a neighborhood of the hypersurface where $r_Y = 2$.

Lemma 7.11. *In the above situation, the space $J(Y_R, \overline{E})$ is non-empty.*

Proof. By Lemma 7.7, the Liouville vector fields and Reeb fields are tangent to the two components of \overline{E} . It suffices to check that the almost complex structure on the contact distributions can be chosen so as to preserve both components of \overline{E} . This is a routine calculation left to the reader. \square

7.12. Floer theory for the Liouville domain. The Lagrangian L_0 satisfies $\theta'_\epsilon|L = 0$ everywhere on Y and is in particular Legendrian at infinity for the Liouville structure. In this subsection, we recall in more detail the Liouville-admissible families of Hamiltonians which are relevant to Abouzaid and Seidel's theory for Liouville domains. We then recall a variant of their construction which uses Hamiltonians which are constant outside of a compact region and review the equivalence of the two approaches. This is all completely standard material in symplectic cohomology, and it is easily adapted to the Lagrangian setting.

Choose a generic family of Liouville-admissible Hamiltonians H_m of slope λ_m . For the remainder of this article, we assume that all Liouville-admissible Hamiltonians satisfy

$$(7.12.1) \quad \begin{cases} H_m(x) = h_m(r_Y) \text{ for } r_Y \geq 1, \\ h_m''(r) \geq 0 \text{ for } r \geq 1, \text{ and,} \\ H_m(x) \text{ is } C^2\text{-small on } Y_R. \end{cases}$$

For any Liouville-admissible Hamiltonian H_m , notice that the flow of H_m preserves the neighborhood $\pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$ along the cylindrical end, and we may assume this is true everywhere. Furthermore, we can assume that it preserves $\pi_{N_{\mathbb{C}^\times}}^{-1}(Z)$. We first compute the wrapped Floer cohomology

$$(7.12.2) \quad \text{HW}^*(L_0) := \varinjlim_m \text{HF}^*(L_0; H_m)$$

as a vector space. Since the flow of H_m preserves the neighborhood $\pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$, all Hamiltonian chords of $\mathcal{X}(L_0, H_m)$ lie in $Y \setminus \pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$. We have seen in the discussion preceding Definition 7.9, that for each homotopy class (\mathbf{n}, i) , there is a contractible submanifold with corners of Hamiltonian chords for L_0 (diffeomorphic to either a point, an interval I , or I^2). For each of these submanifolds with corners, we may perform an arbitrarily small Hamiltonian perturbation supported in a neighborhood of these orbits, so that there is a unique chord of index 0. We may therefore label generators by $\theta_{\mathbf{n}, i}^m$, which corresponds to the unique chord of period less than or equal to λ_m (for the perturbed Hamiltonian) whose homotopy class in $N_{\mathbb{C}^\times} \times \mathbb{C}^\times$ is given by (\mathbf{n}, i) . As $\lambda_m \rightarrow \infty$, all homotopy classes are eventually realized.

Lemma 7.13. *If the Hamiltonian chord corresponding to $\theta_{\mathbf{n}, i}^m$ has period λ_0 , then for any $\lambda'_m \gg \lambda_m \gg \lambda_0$, the continuation map $\text{HF}^*(L_0; H_m) \rightarrow \text{HF}^*(L_0; H_{m'})$ sends $\theta_{\mathbf{n}, i}^m \rightarrow \theta_{\mathbf{n}, i}^{m'}$.*

Proof. Because $d = 0$ for grading reasons, we may use any Hamiltonians of slope λ_m and λ'_m to compute this continuation map. We again approximate by quadratic Hamiltonians, e.g. by a Hamiltonian which is

- r_Y^2 for $r_Y < a_{\lambda_m}$

- $\lambda' r_Y$ for $r_Y \geq a_{\lambda_m} + d_{\lambda_m}$.

Then the continuation maps are the identity on the Reeb chords which arise in the quadratic part. \square

Corollary 7.14. *HW $^*(L_0)$ is freely generated by the images $\theta_{\mathbf{n},i}$ of $\theta_{\mathbf{n},i}^m$ for sufficiently high m .*

Assume for the rest of this section that all Liouville-admissible Hamiltonians satisfy (4.21.1). We now turn to the Hamiltonians which are constant outside of a compact set following [McL09]. Choose a generic almost complex structure which is Liouville-admissible in a neighborhood of $r_Y = 2$. For any λ_m , define $\mu(\lambda_m) > 0$ to be some constant smaller than the distance between λ_m and the action spectrum. Define $A(\lambda_m) := 3\lambda_m/\mu(\lambda_m)$. By choosing $\mu(\lambda_m)$ small enough, we may assume that $A(\lambda_m) \geq 4$. We let H_m^c be a function satisfying the following conditions. For $r_Y \geq 1$, H_m^c depends only on r_Y and is a (smoothing of a) function which satisfies

- for $r_Y \leq A - 1$, $H_m^c = H_m$,
- for $r_Y > A$ we assume that $H_m^c = \lambda_m(A - 1)$
- for $r_Y \geq A - 1$, $(H_m^c)'' \leq 0$.

Recall that the Floer differential increases the action. We therefore have a subcomplex $\text{CF}^*(L_0; H_m^c)_{(0,\infty)}$ of generators whose action is greater than zero. Define

$$(7.14.1) \quad \text{CF}^*(L_0; H_m^c)_p := \text{CF}^*(L_0; H_m^c) / \text{CF}^*(L_0; H_m^c)_{(0,\infty)}.$$

We may choose H_m so that all time one chords x satisfy

$$(7.14.2) \quad A_{H_m}(x) \leq 0.$$

An elementary calculation shows that the action of any time one chord when $r_Y \geq 2$ all satisfy $A_{H_m^c}(x) \geq \lambda_m$. It therefore follows that:

Lemma 7.15. *We have an isomorphism of cochain-complexes*

$$(7.15.1) \quad \text{CF}^*(L_0; H_m^c)_p \cong \text{CF}^*(L_0; H_m).$$

In the limit, we obtain an isomorphism

$$(7.15.2) \quad \text{HW}^*(L_0) \cong \varinjlim_m \text{HF}^*(L_0; H_m^c)_p$$

of vector spaces.

Proof. The above discussion gives the identification on generators, so it only remains to give the identification of differentials. Since all generators of both complexes lie in the region where $r_Y \leq 2$, this follows from the “no-escape lemma” in [AS10, Lemma 7.2]. This lemma also applies to continuation solution to the result passes to the limit as claimed. \square

7.16. The main comparison theorem. This section compares the ring structure on the two flavours of wrapped Floer cohomology. For this section, we choose all of our almost complex structures to lie in $J(Y_R, \overline{E})$. As we will need to vary μ_0 in this section, for clarity we use the notation Y_{R,μ_0} to indicate the dependence on μ_0 . Notice that for $\mu'_0 < \mu''_0$ the manifolds Y_{R,μ'_0} are Liouville isomorphic in a way which preserves L_0 . As a result, there are continuation isomorphisms

$$(7.16.1) \quad \text{HW}^*_{Y_{R,\mu'_0}}(L_0) \rightarrow \text{HW}^*_{Y_{R,\mu''_0}}(L_0)$$

between the wrapped Floer cohomologies. In this case, the continuation maps are given by interpolating between μ'_0 and μ''_0 so that for every s in the continuation family, the

Hamiltonian preserves the divisor E . It follows that continuation map solutions have intersection number zero with the divisor E and thus the isomorphism respects the free homotopy class of the chord in $Y \setminus \pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$. We finally are in a position to state and prove our main comparison theorem.

Theorem 7.17. *There is an isomorphism*

$$(7.17.1) \quad \text{HW}^*(L_0) \cong \text{HW}_{\text{ad}}(L_0)$$

of \mathbb{C} -algebras.

Proof. This is a modification of the proof of [McL09, Theorem 5.5]. In this proof, we will have to discuss both fibered admissible and Liouville-admissible Hamiltonians. For this proof we will adopt the convention that fibered admissible Hamiltonians will be denoted by G and Liouville-admissible Hamiltonians are denoted by H . We may consider a family of fibered admissible Hamiltonians G_m which has slope λ_m in both directions, which vanish on Y_{R,μ_0} and such that for any time-one chord x , $A_{G_m}(x) \leq 0$.

Next, by shrinking μ_0 to $\mu_{0,m}$, we may ensure that $c_{\mu_{0,m}}$ above is arbitrarily large. Thus, for any slope λ'_m and for sufficiently small $\mu_{0,m}$, the Hamiltonian H_m^c makes sense as a Hamiltonian on Y . Given H_m^c , we will construct a fibered admissible Hamiltonian $G_{h,m}$ which for $r < A$ agrees with H_m^c and for which all chords in $r_Y \geq 2$ have very positive action. To do this, we consider a shifted version of a standard fibered admissible Hamiltonian G_A which vanishes when

$$(7.17.2) \quad (C|\mu - \epsilon| \leq A + 1) \cup (\rho \leq B \cdot A + 1)$$

We also require that whenever $\rho - B \cdot A \geq 1$ or $C|\mu - \epsilon| - A \geq 1$, G_A takes the form:

$$(7.17.3) \quad v(C|\mu - \epsilon| - A) + h(\rho - B \cdot A)$$

where h and v are suitable base-admissible and admissible vertical Hamiltonians. Finally, we require that $v(C|\mu - \epsilon| - A)$ and $h(\rho - B \cdot A)$ are linear of slope $\sqrt{\lambda'_m}$ whenever $\rho - B \cdot A \geq 2$ or $C|\mu - \epsilon| - A \geq 2$. It follows as in McLean Lemma 5.6 that the chords of the function G_A have action $A_{G_A}(x) \geq -\sqrt{\lambda'_m}(A + B \cdot A)$. In fact, the argument here is even easier because the chords live over the part of the manifold where the symplectic fibration is trivial. Now the function $G_{h,m} = G_A + H_m^c$ is fibered admissible. It follows that for any chord which lies in the region where $r_Y \geq 2$

$$(7.17.4) \quad A_{G_{h,m}}(x) \geq \lambda'_m(A - 1) - \sqrt{\lambda'_m}(A + B \cdot A) > 0.$$

Furthermore, it is a consequence of Corollary 7.6 that for any λ_m , we may choose λ'_m , $\mu_{0,m}$ and $G_{h,m}$ such that $G_{h,m} > G_m$ and the slopes in both direction of $G_{h,m}$ are bigger than G_m . Next, we can construct fibered admissible Hamiltonians G'_m which have slope λ''_m in both directions, which vanish on $Y_{R,\mu_{0,m}}$ and such the the action of all chords is ≤ 0 . We assume that $\lambda''_m \gg \lambda'_m$. We therefore have, after choosing suitable functions ℓ^\sharp , continuation maps

$$(7.17.5) \quad \text{HF}_{\text{ad}}^*(L_0; G_m) \rightarrow \text{HF}_{\text{ad}}^*(L_0; G_{h,m})_p \rightarrow \text{HF}_{\text{ad}}^*(L_0; G'_m).$$

In the limit, the composition of these two maps becomes an isomorphism and, as a result, the second map is an isomorphism as well. Since these maps are ring maps, this concludes the proof. \square

8. HOMOLOGICAL MIRROR SYMMETRY FOR MCKAY QUIVERS

8.1. Computation of $\mathrm{SH}^0(Y_R)$. Fix a Liouville domain X^{in} and a Liouville-admissible family of Hamiltonians $(H_m)_{m=1}^\infty$ on the Liouville completion X of slope λ_m with $\lambda_m \rightarrow \infty$ satisfying (7.12.1) and for simplicity also (4.21.1). A fixed point of a time-one flow of the Hamiltonian vector field X_H associated with a Hamiltonian function H is called a *time-one Hamiltonian orbit* of H . The set of time-one Hamiltonian orbits of H will be denoted by $\mathcal{X}(X; H)$. We may choose a small \mathbb{S}^1 -dependent perturbation $H_m^{\mathrm{pert}}: \mathbb{S}^1 \times X \rightarrow \mathbb{R}$ supported in a neighborhood of the union of all time-one Hamiltonian orbits of H_m , so that any time-one Hamiltonian orbit of the perturbed Hamiltonian $H_m + H_m^{\mathrm{pert}}$, is non-degenerate. The Floer cochain complex is defined by

$$(8.1.1) \quad \mathrm{CF}^*(X; H_m + H_m^{\mathrm{pert}}) := \bigoplus_{x \in \mathcal{X}(X; H_m + H_m^{\mathrm{pert}})} |\mathfrak{o}_x|,$$

where $|\mathfrak{o}_x|$ is the orientation line on the real vector space \mathfrak{o}_x of rank one associated with $x \in \mathcal{X}(X; H_m + H_m^{\mathrm{pert}})$. The degree of $x \in \mathcal{X}(X; H_m + H_m^{\mathrm{pert}})$ is the dimension of X minus the Conley–Zehnder index of x ;

$$(8.1.2) \quad \deg x = \dim X - \mathrm{CZ}(x).$$

The differential ∂ on the Floer complex (8.1.1) is given by counting solutions to Floer’s equation

$$(8.1.3) \quad (du - X_{H_m + H_m^{\mathrm{pert}}} \otimes dt)^{0,1} = 0$$

for maps $u: \mathbb{R} \times \mathbb{S}^1 \rightarrow X$ from the cylinder, with respect to an \mathbb{S}^1 -dependent almost complex structure J , modulo translation in the \mathbb{R} -direction. A monotone homotopy from $H_m + H_m^{\mathrm{pert}}$ to $H_{m+1} + H_{m+1}^{\mathrm{pert}}$ gives the continuation map, and the symplectic cohomology is defined as the colimit

$$(8.1.4) \quad \mathrm{SH}^*(X) := \varinjlim_m \mathrm{HF}^*(X, H_m + H_m^{\mathrm{pert}}).$$

For any conical spin Lagrangian L , there is the *closed-open map*

$$(8.1.5) \quad \mathrm{CO}: \mathrm{SH}^*(X^{\mathrm{in}}) \rightarrow \mathrm{HW}^*(L)$$

from the symplectic cohomology to the wrapped Floer cohomology (see e.g., [Abo10, Section 5]). This is defined by counting solutions to Floer’s equation, from a disk Σ with one marked point on the boundary (with a negative strip-like end) and one marked point in the interior (with a positive cylindrical end) to X , which asymptotes to a time-one Hamiltonian chord in the negative strip-like end and a time-one Hamiltonian orbit in the positive cylindrical end.

The goal of this section is prove the following theorem:

Theorem 8.2. *The closed-open map*

$$(8.2.1) \quad \mathrm{CO}: \mathrm{SH}^0(Y_R) \rightarrow \mathrm{HW}^0(L_0)$$

is an isomorphism.

In view of (7.12.1), time-one Hamiltonian orbits on the cylindrical end are in bijection with Reeb orbits of period less than λ_m . We must begin by dealing with the fact that our Reeb flow is degenerate. For each family $\mathcal{F}_{(n,i)}$ of time-one Hamiltonian orbits corresponding to the family $\overline{\mathcal{F}}_{(n,i)}$ of Reeb orbits, we choose an isolating neighborhood

$U_{(\mathbf{n},i)} \subset \widehat{Y}_R$, which contains only the orbit set $\mathcal{F}_{(\mathbf{n},i)}$ and no others. If the set $\mathcal{F}_{(\mathbf{n},i)}$ occurs at a level $r_Y = r_0$, then we may assume that the neighborhood is the product

$$(8.2.2) \quad U_{(\mathbf{n},i)} = (r_0 - \delta, r_0 + \delta) \times \overline{U}_{(\mathbf{n},i)},$$

of a small interval and a small open set $\overline{U}_{(\mathbf{n},i)} \subset \partial Y_R$ containing $\mathcal{F}_{(\mathbf{n},i)}$ and invariant under the natural local \mathbb{S}^1 -action which extends the Reeb flow action on $\overline{\mathcal{F}}_{(\mathbf{n},i)}$. Let

$$(8.2.3) \quad \mu_{(\mathbf{n},i)}: U_{(\mathbf{n},i)} \rightarrow \mathbb{R}$$

denote the local moment map for this \mathbb{S}^1 -action. For definiteness, let us fix choices of these open sets $\overline{U}_{(\mathbf{n},i)}$:

- Recall from Section 7.8 that the family $\overline{\mathcal{F}}_{(\mathbf{n},0)}$ for regular \mathbf{n} is a T^3 -fibration over the product (point) $\times I$ of a point in $N_{\mathbb{R}}$ and a closed interval in \mathbb{R} under the map $\pi_{\widehat{B}}$ given in (7.1.1). We choose the set $\overline{U}_{(\mathbf{n},i)}$ to be the preimage by $\pi_{\widehat{B}}|_{\partial T_R}$ of the product of a small ball of the point in $N_{\mathbb{R}}$ and an open interval in \mathbb{R} which is slightly larger than I .
- When \mathbf{n} is regular and $i \neq 0$, the family $\overline{\mathcal{F}}_{(\mathbf{n},i)}$ is the fiber of $\pi_{\widehat{B}}$ over a point in $N_{\mathbb{R}} \times \mathbb{R}$, and we can choose $\overline{U}_{(\mathbf{n},i)}$ to be the preimage of a ball around this point.
- When \mathbf{n} is not regular, we choose the tubular neighborhood to be where μ lies in $[\mu_{0,\mathbf{n},i} - \delta'_{0,\mathbf{n},i}, \mu_0 + \delta'_{1,\mathbf{n},i}]$ and $|r_{\alpha-\beta}| \leq r_{\alpha-\beta,0}$ for appropriate intervals.
- For $\overline{\mathcal{F}}_{(0,i)}$, we choose the neighborhood $\overline{U}_{(\mathbf{n},i)}$ to be defined by $\rho < R - \delta'_R$ for δ'_R slightly smaller than δ_R .

We let $W_{(\mathbf{n},i)}^T$ be an open set in $U_{(\mathbf{n},i)}$ containing $\mathcal{F}_{(\mathbf{n},i)}^0$ and contained in the region where \widehat{Y}_R agrees with \widehat{T}_R .

Lemma 8.3. *There is a function $h_{t,(\mathbf{n},i)}: \mathbb{S}^1 \times \widehat{Y}_R \rightarrow \mathbb{R}$ supported in $U_{(\mathbf{n},i)}$, such that:*

- All Hamiltonian orbits of $H_{t,m,\epsilon} = H_m + \epsilon h_{t,(\mathbf{n},i)}$ inside of $U_{(\mathbf{n},i)}$ for ϵ sufficiently small are in bijection with the critical points α_j of an outward pointing Morse function $\tilde{h}: \mathcal{F}_{(\mathbf{n},i)} \rightarrow \mathbb{R}$.
- For ϵ sufficiently small, the gradings of the critical points coincide with the Morse-index of α_j .
- There is a unique orbit of index zero $\alpha_{(\mathbf{n},i)}$ which lies in $W_{(\mathbf{n},i)}^T$.

Proof. To do this, we apply elementary Morse-Bott theory for Reeb flows which generate \mathbb{S}^1 -actions (for a nice review with references to the original papers, we recommend [KvK16]), taking care of the fact that our Reeb orbits come in families which are manifolds with boundary. A standard trick allows us to “unwrap” the Hamiltonian flow and thereby allows us to work in an equivalent Hamiltonian system where all of the orbits along $\mathcal{F}_{(\mathbf{n},i)}$ of the flow are constant (and these are still the only orbits in $U_{(\mathbf{n},i)}$). We temporarily work in this equivalent Hamiltonian system.

The sets $\overline{U}_{(\mathbf{n},i)}$ naturally project to certain “thickenings” $\mathcal{F}_{(\mathbf{n},i)}^t$ of $\mathcal{F}_{(\mathbf{n},i)}$ of the same dimension. When the set $\mathcal{F}_{(\mathbf{n},i)}$ are codimension zero in ∂Y_R , these are just $\overline{U}_{(\mathbf{n},i)}$ and in the case when $\mathcal{F}_{(\mathbf{n},i)} = T^3$, then $\mathcal{F}_{(\mathbf{n},i)}^t = \mathcal{F}_{(\mathbf{n},i)}$. In the case, when \mathbf{n} is regular, $\mathcal{F}_{(\mathbf{n},0)}^t$ is the locus where $\mu \in [\mu_{\mathbf{n},0} - \delta'_{0,\mathbf{n}}, \mu_{\mathbf{n},0} + \delta'_{1,\mathbf{n}}]$ and $r_i = r_{i,\mathbf{n}}$. Finally in the case when \mathbf{n} is not regular and i is regular, $\mathcal{F}_{(\mathbf{n},i)}^t$ is the locus where $\mu = \mu_0$ and $|r_{\alpha-\beta}| \leq r_{\alpha-\beta,0}$. Choose an outward pointing Morse-function $\tilde{h}: \mathcal{F}_{(\mathbf{n},i)}^t \rightarrow \mathbb{R}$ such that all critical points lie in the interior of $\mathcal{F}_{(\mathbf{n},i)}$. When $\mathcal{F}_{(\mathbf{n},i)}^t$ is a manifold with boundary we postulate that on $\mathcal{F}_{(\mathbf{n},i)}^t \setminus \mathcal{F}_{(\mathbf{n},i)}$ near a boundary component cut out by an equation $f = c$ (either

$\mu = c, r_{\alpha-\beta} = c, \text{ or } \rho = c$) that $\tilde{h} = \tilde{h}(f)$. In the case when \mathbf{n} is not regular and $\mathcal{F}_{(\mathbf{n},0)}^t$ is a manifold with corners we similarly require that near a corner $\tilde{h} = \tilde{h}(r_{\alpha-\beta}, \mu)$ along an open smooth piece of the boundary that $h = h(r_{\alpha-\beta})$ or $h = h(\mu)$.

Denote by $\pi_{\mathbf{n},i} : U_{(\mathbf{n},i)} \rightarrow \mathcal{F}_{(\mathbf{n},i)}^t$. Choose a cutoff function $\psi : \widehat{Y}_R \rightarrow \mathbb{R}$ supported in $U_{(\mathbf{n},i)}$ and such that $\psi = 1$ on some smaller open set $V_{(\mathbf{n},i)}$ containing $\mathcal{F}_{(\mathbf{n},i)}$. Finally consider the function $\psi\pi_{\mathbf{n},i}^*(\tilde{h})$. We take this function and translate it back to the original Hamiltonian system to obtain $h_{t,(\mathbf{n},i)}$ (Undoing the unwrapping process makes this function time dependent). The first bullet and second bullets now follow from standard Morse-Bott theory together with the fact that our Hamiltonian perturbations are chosen so that new orbits cannot appear near the boundary and hence all critical points occur when the flow is transversely non-degenerate. Finally the last point can be achieved by choosing \tilde{h} to have a unique critical point of index 0 which furthermore lies in $W_{(\mathbf{n},i)}^T$. \square

Remark 8.4. There is a variant construction of a perturbation $h_{t,(\mathbf{n},i)}$ satisfying the hypotheses of the lemma which proceeds in two steps but allows one to make partial computations of the BV operator in symplectic cohomology. Namely, one may without unwrapping consider a perturbation function \tilde{h}_{eq} on $\mathcal{F}_{\mathbf{n},i}^t$ which is the pull-back of a Morse-function on $\mathcal{F}_{\mathbf{n},i}^t/\mathbb{S}^1$ satisfying the same properties near the boundary strata as \tilde{h}_{eq} did above. Extend this function to the neighborhoods $U_{\mathbf{n},i}$ by the formula $\psi\pi_{\mathbf{n},i}^*(\tilde{h}_{eq})$. Consider the Hamiltonians

$$(8.4.1) \quad H_{m,eq} = H_m + \epsilon\psi\pi_{\mathbf{n},i}^*(\tilde{h}_{eq}).$$

For small ϵ , we obtain a transversely non-degenerate \mathbb{S}^1 family of Reeb orbits for every critical point on $\mathcal{F}_{\mathbf{n},i}^t/\mathbb{S}^1$. Finally, we use a small time dependent Hamiltonian perturbation in a neighborhood of the orbits to obtain non-degeneracy. This construction can be done so that the third bullet point of the lemma is satisfied.

We set

$$(8.4.2) \quad H_{m,\text{pert}}^Y := \sum_{(\mathbf{n},i), \mathcal{F}_{\mathbf{n},i} \in \mathcal{X}(X, H_m)} h_{t,(b\mathbf{n},i)}.$$

For certain purposes, it makes sense to view

$$(8.4.3) \quad H_{m,\text{pert}}^Y = H_{m,\text{pert}}^{\text{reg}} + H_{m,\text{pert}}^{\text{irr}},$$

where the first term is the sum over regular (\mathbf{n}, i) and the other term is the sum over irregular (\mathbf{n}, i) . We set

$$(8.4.4) \quad H_m^Y := H_m + \epsilon H_{m,\text{pert}}^Y.$$

Similarly we can take a small (time-independent) perturbation of

$$(8.4.5) \quad H_m^L := H_m + \epsilon H_{m,\text{pert}}^L$$

(the perturbation can be chosen to be supported in $W_{(\mathbf{n},i)}^T$), so that there is a Reeb chord of H_m^L for every suitable component of $\mathcal{F}_{\mathbf{n},i}^0$.

Lemma 8.5. *The map*

$$(8.5.1) \quad \text{CO}: \text{SH}^0(Y_R) \rightarrow \text{HW}^*(L_0)$$

is injective.

Proof. Observe that an element of the kernel y must lie in some finite $\text{HF}^*(\widehat{Y}_R, H_{t,m,\epsilon})$. Because the maps $\text{HF}^*(L_0, H_{m,\epsilon}) \rightarrow \text{HW}^*(L_0)$ are injective, the map must further lie in the kernel of a map

$$(8.5.2) \quad \text{CO}_m: \text{HF}^*(\widehat{Y}_R, H_{t,m,\epsilon}) \rightarrow \text{HF}^*(L_0, H_{m,\epsilon})$$

This map depends only on the slope λ_m at infinity and is hence independent of ϵ on homology. Let $\mathfrak{o}_{\alpha(\mathbf{n},i)}$ denote the orientation space in $\text{CF}^*(L_0, H_{m,\epsilon})$ corresponding to orbit $\alpha(\mathbf{n},i)$. Choosing arbitrarily a generator for this space and giving it the same name by abuse of notation, the statement will follow from the fact that for ϵ small, on the cochain level the coefficient of $\theta_{(\mathbf{n},i)}$ in $\text{CO}_m(\mathfrak{o}_{\alpha(\mathbf{n},i)})$ is ± 1 .

To see this, a Gromov compactness argument ([McL12, proof of Lemma 2.3]) shows that for sufficiently small ϵ any curve contributing to this coefficient will necessarily lie in $W_{(\mathbf{n},i)}^T$. Observe that we may view the Hamiltonians H_m as being Hamiltonians on \widehat{T}_R simply by replacing the Liouville coordinate r_Y with r_T . We denote the resulting Hamiltonians by $H_m^T = H_m(r_T)$ for clarity. Similarly, we may construct perturbations $H_{m,\epsilon}^T$ and $H_{t,m,\epsilon}^T$ which agree over $W_{(\mathbf{n},i)}^T$ with the previously constructed perturbations over $W_{(\mathbf{n},i)}^T$ and so that for sufficiently small ϵ the orbit corresponding under this equivalence to $\alpha_{(\mathbf{n},i)}$ is a unique Hamiltonian orbit of degree 0 in homotopy class (\mathbf{n}, i) . Again, for sufficiently small ϵ , all curves contributing to the

$$(8.5.3) \quad \text{CO}_m^T: \text{HF}^*(\widehat{T}_R, H_{t,m,\epsilon}^T) \rightarrow \text{HF}^*(L_0, H_{m,\epsilon}^T)$$

lie in $W_{(\mathbf{n},i)}^T$. This map must be ± 1 along the diagonal by the well-known calculations of $\text{SH}^*(T_R)$ and $\text{HW}_{T_R}^*(L_0)$. The result follows. \square

For any homotopy class \mathbf{n} , let $\text{SH}^0(Y_R)_{\mathbf{n}}$ and $\text{HW}^*(L_0)_{\mathbf{n}}$ denote the subvector space generated by orbits in class \mathbf{n} . We have a direct sum decomposition

$$(8.5.4) \quad \text{SH}^0(Y_R) := \bigoplus_{\mathbf{n}} \text{SH}^0(Y_R)_{\mathbf{n}}$$

and similarly for wrapped Floer theory. The map CO respects these direct sum decompositions.

Lemma 8.6. *The map*

$$(8.6.1) \quad \text{CO}_{\mathbf{n}}: \text{SH}^0(Y_R)_{\mathbf{n}} \rightarrow \text{HW}^*(L_0)_{\mathbf{n}}$$

is an isomorphism for $\mathbf{n} \neq 0$ regular. Furthermore for any \mathbf{n} regular, let $\alpha_{(\mathbf{n},i)}$ be the Hamiltonian orbit on the cylindrical end of $H_{t,m,\epsilon}$ (as usual ϵ is sufficiently small) of index 0. Choose a generator, $\alpha_{(\mathbf{n},i)}^0$, of the corresponding orientation line. For $H_{\text{pert},n}$ sufficiently close to zero, $\text{CO}(\alpha_{(\mathbf{n},i)}^0) = \pm \theta_{(\mathbf{n},i)}$.

Proof. We may assume that the Hamiltonians H_m are actually unperturbed near $\pi_{N_{\mathbb{C}^\times}}^{-1}(Z)$ (i.e. $H_{\text{pert},n}$ is sent to zero) so that the contribution to the differential is local. The result then follows from a similar argument to Lemma 8.5. \square

Remark 8.7. Using the fact that the open-closed map is a ring homomorphism, Lemma 8.6 implies that many other elements are hit by the map CO as well. To be more precise, let $\alpha \in \text{SH}^0$ be an element such that $\text{CO}(\alpha) = \theta_{\mathbf{n},i}$ and let $\beta \in \text{SH}^0$ be an element such that $\text{CO}(\beta) = \theta_{\mathbf{n}',i'}$ with both \mathbf{n} and \mathbf{n}' regular. Then

$$(8.7.1) \quad \text{CO}(\alpha \cdot \beta) = \sum_{j=0}^{\ell_2(\mathbf{n},\mathbf{n}')} \binom{\ell_2(\mathbf{n},\mathbf{n}')}{j} \theta_{\mathbf{n}+\mathbf{n}',i+i'+j} = (1 + \theta_{0,1})^{\ell_2(\mathbf{n},\mathbf{n}')} \theta_{\mathbf{n}+\mathbf{n}',i+i'}$$

is in the image. By choosing \mathbf{n} and \mathbf{n}' appropriately, this implies that for any \mathbf{n}'' , the element $(1 + \theta_{0,1})^{d(\mathbf{n}'')} \theta_{\mathbf{n}'',i}$ is the image of CO for some $d(\mathbf{n}'') \geq 1$ and all i .

We now prove the following lemma, which allows us to compute some differentials in symplectic cohomology and which uses a technique in [Pas]. For the remainder of this subsection, we use the perturbations of Remark 8.4 on the cylindrical end. We also assume that our data is chosen so that the restriction $H_{t,m,\epsilon}|_{Y_R}$ is Morse and time-independent with unique critical point of index 0.

Lemma 8.8. *Let $\mathcal{F}_{\mathbf{n},i}$ be the family of Hamiltonian orbits of H_m of highest action in the homotopy class \mathbf{n} . Let $\alpha_{(\mathbf{n},i)}$ be the Hamiltonian orbit on the cylindrical end of $H_{t,m,\epsilon}$ (as usual ϵ is sufficiently small) of index 0. Choose a generator, $\alpha_{(\mathbf{n},i)}^0$, of the corresponding orientation line. Then*

$$(8.8.1) \quad \partial(\alpha_{(\mathbf{n},i)}^0) = 0 \in \text{CF}^*(\widehat{Y}_R, H_{t,m,\epsilon}).$$

Proof. We focus on the more difficult case where $\mathbf{n} = 0$ and we want to prove that the corresponding $\alpha_{(0,i)}^0, i \neq 0$ is closed. Choose a generator of the orientation line corresponding to the Morse critical point in the interior which we will denote by $\alpha_{(0,0)}^0$.

For suitable such $H_{t,m,\epsilon}$ with ϵ sufficiently small and complex structures J_t , we use the Morse-Floer correspondence to identify trajectories between orbits contained in $U_{(0,i)}$ with differentials of the Morse-complex of a time independent \tilde{h} for the restriction of an almost-Kähler metric corresponding to a time-independent complex structure J_0 . The pair (h_t, J_t) and (\tilde{h}, J_0) are again related by unwrapping. Again, the only difference between this case and the classical case is that we must ensure that Floer trajectories do not escape to the boundary of $\mathcal{F}_{\mathbf{n},i}$, which is easily arranged. The upshot is that we have

$$(8.8.2) \quad \partial(\alpha_{(0,i)}^0) = \beta_{(0,0)}^1$$

where $\beta_{(0,0)}^1$ is a sum of generators corresponding to critical points in the interior of Y_R . Let Δ denote the BV operator on symplectic cohomology. Using the local analysis in [Zha], we can see that there is a cochain $\alpha_{(0,i)}^1$ in $\text{CF}^1(\widehat{Y}_R, H_{t,m,\epsilon})$ such that

$$(8.8.3) \quad \Delta(\alpha_{(0,i)}^1) = \alpha_{(0,i)}^0 + c \cdot \alpha_{(0,0)}^0$$

for some c and such that

$$(8.8.4) \quad \partial(\alpha_{(0,i)}^1) = \beta_{(0,0)}^2$$

where $\beta_{(0,0)}^2$ is a sum of generators corresponding to critical points in the interior of Y_R . Now Δ is a chain map, so $\partial \circ \Delta = \Delta \circ \partial$. The assumption that $H_{t,m,\epsilon}$ is time-independent when restricted to the interior implies that $\Delta(\beta_{(0,0)}^2) = 0$ on the chain level. Combining these statements implies the required result. \square

We finally complete the proof of Theorem 8.2.

Proof of Theorem 8.2. We can assume that the contractible orbit of Lemma 8.8 is $\alpha_{(0,\pm 1)}$. Assume it is $\alpha_{(0,1)}$ as the other case is similar. Then this implies that $\theta_{0,1}$ is in the image of CO. It proves that $\theta_{0,i}$ is in the image of CO for $i \geq 0$. Combining this with Remark 8.7 implies that the map $\text{CO}_{[0]}$ is surjective. Applying Lemma 8.8, this implies that for any \mathbf{n} , there is an i_0 for which $\theta_{\mathbf{n},i_0}$ is in the image of $\text{CO}_{\mathbf{n}}$. This combined with the surjectivity of $\text{CO}_{[0]}$ implies that $\theta_{\mathbf{n},i}$ is in the image of $\text{CO}_{\mathbf{n}}$ for any i . \square

8.9. Generation for the mirror of the complement of an anti-canonical divisor in \mathbb{C}^3 . Let Y be the conic bundle defined by the equation

$$(8.9.1) \quad 1 + w_1 + w_2 = uv.$$

The mirror in this case is the affine space

$$(8.9.2) \quad \check{Y} := \text{Spec} \left(\mathbb{C}[x, y, z] \left[(xyz - 1)^{-1} \right] \right).$$

The identification between this ring and $H^0(\mathcal{O}_{\check{Y}})$ sends y to $\chi_{(0,-1),0}$, x to $\chi_{(-1,0),0}$ and z to $\chi_{(1,1),0}$. The goal of this section is to prove the following theorem:

Theorem 8.10. *There is an equivalence of categories*

$$(8.10.1) \quad \psi: D^b \mathcal{W}(Y_R) \rightarrow D^b \text{coh } \check{Y}$$

sending L_0 to $\mathcal{O}_{\check{Y}}$.

It suffices to prove that the Lagrangian L_0 split-generates the wrapped Fukaya category of Y_R . Then it follows that L_0 generates $D^\pi \mathcal{W}(Y_R)$ without the need to pass to direct summands. The proof of this theorem works by checking a generation criterion due to Abouzaid [Abo10] (for an implementation with linear Hamiltonians, see [RS]). We will now describe it in the setting that we need. Let $\text{HH}_3(\text{HW}^*(L_0))$ denote the third Hochschild homology of the wrapped Floer algebra. This is of course isomorphic to $\text{HH}_3(H^0(\mathcal{O}_{\check{Y}}))$, which in this case admits an explicit description via the HKR theorem as

$$(8.10.2) \quad \text{HKR: } \text{HH}_3(H^0(\mathcal{O}_{\check{Y}})) \xrightarrow{\sim} H^0(\Omega_{\check{Y}}^3).$$

Since \check{Y} is equipped with a natural choice of holomorphic volume form

$$(8.10.3) \quad \Omega_{\check{Y}} := \frac{dx \wedge dy \wedge dz}{xyz - 1},$$

the module $H^0(\Omega_{\check{Y}}^3)$ is free of rank one over $H^0(\mathcal{O}_{\check{Y}})$ generated by $\Omega_{\check{Y}}$.

Assume that $\lambda_m = m$ for all m and that (4.21.1) holds. Then parallel to (4.22.2), the construction of the symplectic cohomology may be promoted to a cochain complex $\text{SC}^*(Y_R)$ whose cohomology is $\text{SH}^*(Y_R)$. Central to Abouzaid's criterion is the map of complexes

$$(8.10.4) \quad \mathcal{OC}: \text{CC}_{3-*}(\text{CW}^*(L_0, L_0)) \rightarrow \text{SC}^*(Y_R)$$

from the Hochschild chain complex of the A_∞ -algebra. The induced map on cohomology is denoted by

$$(8.10.5) \quad \text{OC}: \text{HH}_3(\text{HW}^*(L_0)) \rightarrow \text{SH}^0(Y_R).$$

Theorem 8.11 ([Abo10, Theorem 1.1]). *L_0 split-generates the wrapped Fukaya category if the unit [1] is in the image of (8.10.5).*

In fact, given our choice of data, the wrapped Floer complex is concentrated in degree zero. This allows for a considerable simplification in the description of the map OC essentially because $\mu^{d,p} = 0$ for $d = 1, |F| = 0$ and $d > 2$. The underlying domains used for constructing this map will be Riemann surfaces $\Sigma_{4,1}$ with four positive strip like ends and one negative cylindrical end. Let $\overline{\mathcal{R}}^{4,1}$ denote the moduli space of these structures.

As before, we consider perturbing functions of the form $H_{\text{pert}}^Y: \mathbb{S}^1 \times \widehat{Y}_R \rightarrow \mathbb{R}$ and $H_{\text{pert}}^L: \widehat{Y}_R \rightarrow \mathbb{R}$. Along the positive ends, we consider Floer data $H_i = H_{m_i} + \epsilon H_{\text{pert}}^L$ and setting $\mathbf{m} = \sum_i m_i$. Along the negative cylindrical end, we consider Floer data $H_0 = H_{\mathbf{m}} + \epsilon H_{\text{pert}}^Y$. As usual, we choose perturbation K and admissible surface-dependent complex structure compatible with the Floer data along the ends over the moduli space

and consistent with boundary strata. Given chords x_i and a Hamiltonian orbit x_0 , we let $\overline{\mathcal{R}}^{4,1}(\mathbf{x})$ denote the moduli space of solutions. For generic data, these moduli spaces have the expected dimension. In particular, for x_i such that $\text{virt. dim}(\overline{\mathcal{R}}^{4,1}(\mathbf{x})) = 0$, every rigid solution u gives rise to an isomorphism

$$(8.11.1) \quad \text{OC}_u: |\mathfrak{o}_{x_4}| \otimes \cdots \otimes |\mathfrak{o}_{x_1}| \xrightarrow{\sim} |\mathfrak{o}_{x_0}|.$$

As usual, the map OC is defined as the sum of OC_u .

Since $\text{HW}^*(L_0)$ is concentrated in degree zero and hence formal, we may avoid the telescope complex and realize the Hochschild chains as a direct limit

$$(8.11.2) \quad \text{CC}_3(\text{HW}^*(L_0)) \cong \varinjlim_{m_i} \text{HF}^*(L_0, H_{m_1} + \epsilon H_{\text{pert}}^L) \otimes \cdots \otimes \text{HF}^*(L_0, H_{m_d} + \epsilon H_{\text{pert}}^L)$$

In particular, after writing any cycle Ω as a finite sum of simple tensors, we obtain a cycle $\text{OC}(\Omega) \in \text{SH}^0(Y_R)$. Standard arguments show that this is compatible with direct limits and is independent of class $[\Omega] \in \text{HH}_3(\text{HW}^*(L_0))$. This map coincides with the map in (8.10.5).

Crucial for us is the property that the map OC is a module map over $\text{SH}^0(Y_R)$, where the module structure on the left hand side is via the map (8.1.5). Since $\text{HH}_3(\text{HW}^*(L_0))$ is a free $\text{SH}^0(Y_R)$ -module of rank one, the image of (8.10.5) is necessarily a principal ideal $(f) \subset \text{SH}^0(Y_R)$. Theorem 8.10 therefore follows immediately from Theorem 8.11 if we can show that there are a set of non-zero elements f_1, \dots, f_n in the image of OC which have no common divisors (other than units). More precisely, let $\alpha \in A$ be one of the three monomials in the defining equation for Y and \mathbf{n} be a weight corresponding to a monomial in the chamber C_α .

Definition 8.12. Let $\text{CC}_*(\text{HW}^*(L_0))_{\mathbf{n}}$ be the subcomplex of $\text{CC}_*(\text{HW}^*(L_0))$ of total weight \mathbf{n} ;

$$(8.12.1) \quad \text{CC}_d(\text{HW}^*(L_0))_{\mathbf{n}} := \bigoplus_{\sum_{i=0}^d \mathbf{n}_i = \mathbf{n}} \text{HW}^*(L_0)_{\mathbf{n}_0} \otimes \cdots \otimes \text{HW}^*(L_0)_{\mathbf{n}_d}.$$

The subspace of $\text{CC}_d(\text{HW}^*(L_0))_{\mathbf{n}}$ generated by $\text{HW}^*(L_0)_{\mathbf{n}_0} \otimes \cdots \otimes \text{HW}^*(L_0)_{\mathbf{n}_d}$ for $(\mathbf{n}_0, \dots, \mathbf{n}_d) \in N^{d+1}$ satisfying

- all \mathbf{n}_i are in the chamber corresponding to α , and
- for any two neighboring $\mathbf{n}_i, \mathbf{n}_{i+1}$ with respect to the cyclic ordering, $\mathbf{n}_i + \mathbf{n}_{i+1}$ lies in the chamber corresponding to α .

is denoted by $\text{CC}_*^\alpha(\text{HW}^*(L_0))_{\mathbf{n}}$.

We have maps

$$(8.12.2) \quad \text{OC}_{\mathbf{n}}: \text{HH}_3(\text{HW}^*(L_0))_{\mathbf{n}} \rightarrow \text{SH}^0(Y_R, G_m)_{\mathbf{n}}.$$

Our approach to making partial computations of the open closed map is again via the relationship with fibration-admissible Floer theory. We therefore temporarily assume that when constructing our perturbations of Liouville-admissible Hamiltonians H_m , we take all of the terms $H_{m,\text{pert}}^{\text{irr}}$ to be zero. This is possible because we will only be making computations in regular homotopy classes from now on. Notice that in particular the map (8.12.2) can be defined with such Floer data. Let G be an admissible Hamiltonian in the sense of Section 5, and \mathbf{n} be a regular homotopy class satisfying the following condition:

- There is no Hamiltonian orbit of G in the homotopy class \mathbf{n} which intersects $\pi_{N_{\mathbb{C}^\times}}^{-1}(U_Z)$.

Then after making small time-dependent perturbations $G_t = G + G_{\text{pert}}$ in the neighborhood of the orbits in homotopy class \mathbf{n} , we may define $\text{HF}_{\text{ad}}^*(Y, G_t)_{\mathbf{n}}$ analogously to Section 5 (In fact, for the G we need and since we never need to consider multiplicative structures on HF, we can avoid altogether the use of the functions ℓ^\sharp). Let G_m now be an admissible family of such Hamiltonians and \mathbf{n} a homotopy class such that 8.9 is satisfied for all m . Then we may define the $\text{SH}_{\text{ad}}^*(Y, G_m)_{\mathbf{n}}$ to be the direct limit of these Floer groups.

Continue to fix a regular \mathbf{n} . The argument of Theorem 7.17 shows that we can find an admissible sequence of Hamiltonians G_m together with an isomorphism

$$(8.12.3) \quad \phi_{\text{SH}}: \text{SH}^0(Y_R)_{\mathbf{n}} \xrightarrow{\sim} \text{SH}_{\text{ad}}^0(Y, G_m)_{\mathbf{n}}$$

for all such \mathbf{n} . For any $\Omega_{\text{ad}} \in \text{HH}_3^\alpha(\text{HW}_{\text{ad}}(L_0))_{\mathbf{n}}$, we may define an element $\text{OC}_{\text{ad}, \mathbf{n}}(\Omega_{\text{ad}})$ just as in the case of the ordinary (i.e., not adapted) theory. Moreover, given a cycle $\Omega \in \text{CC}^\alpha(\text{HW}(L_0))_{\mathbf{n}}$, applying the isomorphisms of Theorem 7.17 to each factor of the tensors defining the Hochschild complex gives rise to a cycle $\phi_{\text{CC}}(\Omega) \in \text{CC}^\alpha(\text{HW}(L_0))_{\mathbf{n}}$ (the use of CC^α is again for simplicity so that we only multiply elements in the same chamber and hence can avoid using the functions ℓ^\sharp from Section 5).

Lemma 8.13. *For any such Ω , we have*

$$(8.13.1) \quad \text{OC}_{\text{ad}, \mathbf{n}}(\Omega_{\text{ad}}) = \phi_{\text{SH}} \circ \text{OC}(\Omega).$$

Proof. We may assume that $\text{CF}^{-1}(Y, G_m) = 0$ for all m . The result then follows by a standard degeneration of domain argument. \square

We are now in a position to make the final partial computation of the map OC.

Lemma 8.14. *For some $i > 0$, the elements $\alpha_{((-5, -5), -4i)}^0$, $\alpha_{((5, 0), -4i)}^0$, and $\alpha_{((0, 5), -4i)}^0$ are in the image of OC_{ad} , where here we are identifying SH_{ad}^0 and SH^0 .*

Proof. The conclusion is invariant under suitable deformations of the functions G_m . We may therefore assume this sequence satisfies $G_m = H_{\text{ba}, m} + H_{\text{v}, m}$. We may assume that $H_{\text{v}, m} = 1/2\mu^2$ for $c_{m, 0} - \delta \geq \mu \geq c_{m, 1}$ with $\delta > 0$ and $c_{m, 1} \gg c_{m, 0} \gg \epsilon$ so that $\mu = 1/2|u|^2$. Assume that all of our complex structures respect the standard product structure for all μ in an open neighborhood of $c_{m, 0} - \delta$. We can assume that for some $i, m > 0$, the chords $p_{(-1, -1), -i}$, $p_{(-1, -2), -i}$, $p_{(-2, -1), -i}$ all lie in the region where $c_{m, 0} - \delta \geq \mu \geq c_{m, 1}$. Consider the Hochschild cycle given by the anti-symmetrization

$$(8.14.1) \quad \Omega_{\text{HKR}} = \epsilon_3(p_{(-1, -1), -i} \otimes p_{(-1, -1), -i} \otimes p_{(-1, -2), -i} \otimes p_{(-2, -1), -i}).$$

Under the HKR map, this corresponds to

$$(8.14.2) \quad \frac{xy}{(xyz - 1)^i} d \left(\frac{xy}{(xyz - 1)^i} \right) \wedge d \left(\frac{xy^2}{(xyz - 1)^i} \right) \wedge d \left(\frac{x^2y}{(xyz - 1)^i} \right)$$

$$(8.14.3) \quad = -i \frac{(xy)^5}{(xyz - 1)^{4i}} \frac{dx \wedge dy \wedge dz}{xyz - 1}$$

$$(8.14.4) \quad = -ip_{(-5, -5), -4i} \cdot \Omega_{\check{Y}}.$$

By homotopy class restrictions, the output of Ω_{HKR} must be a non-trivial multiple of $\alpha_{((-5, -5), -4i)}^0$. We assume that this orbit lies in the region where $c_{m, 0} - \delta \geq \mu \geq c_{m, 1}$ as well. A barrier argument (see e.g. [McL, Appendix B]) then shows that solutions for the map $\text{OC}_{\text{ad}}(\Omega_{\text{HKR}})$ lie in the region where $\mu > c_{m, 0} - \delta$ and hence must agree with the OC map defined on $(\mathbb{C}^\times)^3$. The fact that $\alpha_{((-5, -5), -4i)}^0$ is in the image of OC_{ad} follows directly from this. The other cases are symmetric and hence the lemma is proved. \square

Now we return to considering the map OC and we assume for the following proposition that $\epsilon H_{\text{pert},n}$ is chosen sufficiently small so that the conclusion of Lemma 8.6 holds.

Lemma 8.15. *Identify $\text{SH}^0(Y_R)$ with $H^0(\mathcal{O}_{\tilde{Y}})$ via the map CO. Then $\theta_{((-5,-5),0)}$, $\theta_{((5,0),0)}$, $\theta_{((0,5),0)}$ are all in the image of (8.10.5).*

Proof. The previous lemma implies that $\theta_{((-5,-5),-4i)}$, $\theta_{((5,0),-4i)}$, $\theta_{((0,5),-4i)}$ are in the image of OC. Because OC is a module homomorphism, the result follows. \square

Notice that under the isomorphism described at the beginning of this section, these elements correspond to the polynomials $(xy)^5$, $(yz)^5$, $(xz)^5$ and hence have no common divisors (other than units). This completes the proof of Theorem 8.10.

8.16. Finite covers and applications. Theorem 8.10 has interesting applications to the symplectic topology of Y . For example, we have the following:

Proposition 8.17. *Let L be a compact oriented Maslov zero exact Lagrangian submanifold in Y . Then L has the cohomology group of a torus;*

$$(8.17.1) \quad H^*(L; \mathbb{C}) \cong H^*(T^3; \mathbb{C}).$$

Proof. The mirror of L is an object $\psi(L)$ of $D^b \text{coh}(\tilde{Y})$ satisfying

$$(8.17.2) \quad \text{Ext}^0(\psi(L), \psi(L)) \cong \mathbb{C}$$

and

$$(8.17.3) \quad \text{Ext}^i(\psi(L), \psi(L)) = 0 \quad \text{for any } i < 0.$$

Let \mathfrak{m} denote the annihilator of $\text{Ext}^0(\psi(L), \psi(L))$ as a module over $\Gamma(\mathcal{O}_{\tilde{Y}})$. It is a maximal ideal of $\Gamma(\mathcal{O}_{\tilde{Y}})$ since $\text{Ext}^0(\psi(L), \psi(L)) \cong \mathbb{C}$ is a field. Let y be the corresponding closed point of \tilde{Y} . It follows from [IU05, Lemma 4.8] that all cohomology sheaves $\mathcal{H}^i(\psi(L))$ are \mathcal{O}_y -modules. As a consequence, we have an isomorphism of all cohomology sheaves $\mathcal{H}^i(\psi(L)) \cong (\mathcal{O}_y)^{\oplus n_i}$ as $\mathcal{O}_{\tilde{Y}}$ -modules.

It suffices to prove that $\psi(L)$ has cohomology in a single degree, since this implies that $\psi(L)$ is isomorphic to a sheaf, which by the previous paragraph must be \mathcal{O}_y . Assume for contradiction that

$$(8.17.4) \quad l := \max \{i \in \mathbb{Z} \mid \mathcal{H}^i(\psi(L)) \neq 0\}$$

is greater than

$$(8.17.5) \quad k := \min \{i \in \mathbb{Z} \mid \mathcal{H}^i(\psi(L)) \neq 0\},$$

and consider the cohomological spectral sequence with second page

$$(8.17.6) \quad E_2^{p,q} = \bigoplus_i \text{Ext}^p(\mathcal{H}^i(\psi(L)), \mathcal{H}^{i+q}(\psi(L)))$$

and converging to $\text{Ext}^{p+q}(\psi(L), \psi(L))$. Then one has

$$(8.17.7) \quad E_2^{0,k-l} \cong \text{Hom}_{\mathcal{O}_y}(\mathcal{O}_y^{n_l}, \mathcal{O}_y^{n_k}) \neq 0,$$

which would survive the spectral sequence and lead to a non-vanishing Ext^{k-l} . This contradicts (8.17.3), and the conclusion follows. \square

We conclude with the observation that the generation criterion is easily seen to be compatible with passage to finite covers (this is a very easy special case of [RS, Corollary 1.4]). To be more precise, let $N_0 \subset N$ be a finite index subgroup of the fundamental group of Y . Denote the quotient group by $G = N/N_0$. Let Y_{N_0} be the corresponding finite cover and Y_{R,N_0} the corresponding Liouville domain. There is a canonical lift of L_0

to this finite cover and hence we may parameterize all possible lifts as L_g with $g \in G$. Observe that there is a natural action of $\check{G} := \text{Hom}(G, U(1))$ on \check{Y} coming from the SYZ construction. In local coordinates, these correspond to diagonal linear symmetries on \check{Y} which preserve the holomorphic volume form. As a result, we obtain a Calabi–Yau orbifold $[\check{Y}/\check{G}]$. Now we prove Corollaries 1.7 and Corollary 1.8:

Proof of Corollary 1.7. The generation criterion shows that the Lagrangians L_g generated the wrapped Fukaya category of Y_{R, N_0} . The objects $\mathcal{O}_{\check{Y}} \otimes V_g$ generate the category $D^b \text{coh}(\check{Y}/\check{G})$. The endomorphism algebras of these objects are easily seen to agree with the crossed product algebra $\mathcal{O}_{\check{Y}} \rtimes \check{G}$. \square

Proof of Corollary 1.8. Given any Lagrangian sphere S , consider the object $\psi_{N_0}(S)$ and suppose it is not supported at the origin. The object $\psi_{N_0}(S)$ viewed as a module over $\Gamma(\mathcal{O}_{\check{Y}})^{\check{G}}$ must then be supported at a single, regular, point. A variant of the argument from Proposition 8.17 then shows that $\text{Ext}^*(\psi_{N_0}(S), \psi_{N_0}(S))$ must be $H^*(T^3)$, which is impossible. \square

Proposition 8.18 below is a slight strengthening of Proposition 1.9:

Proposition 8.18. *Let S_1, \dots, S_r be a collection of Lagrangian spheres in Y_{N_0} which are pairwise disjoint. Then r is less than or equal to the maximal dimension of an isotropic subspace of the Grothendieck group of $\text{coh}_0[\check{Y}/\check{G}]$ with respect to the Mukai pairing.*

Proof. We follow the ideas of [Sei, Section 15]. Any one-dimensional subtorus acting on \mathbb{C}^3 which acts with weight one on the holomorphic volume form induces a dilating \mathbb{G}_m -action on the category $D^b \text{coh}_0 \check{Y}/\check{G}$ in the sense of [Sei, Definition 15.4]. The result then follows via the same technique as [Sei, Proposition 15.8], replacing Hochschild homology with K -theory as in [Sei, Remark 15.16]. \square

9. APPENDIX

9.1. A priori energy bounds. The *geometric energy* and the *topological energy* of solutions to Floer’s equation (4.13.1) are respectively defined by

$$(9.1.1) \quad E^{\text{top}}(u) := \int_{\Sigma} u^* \omega - d(u^* K)$$

and

$$(9.1.2) \quad E^{\text{geom}}(u) := \frac{1}{2} \int_{\Sigma} \|du - X_K\|.$$

Using $\omega(X, X) = 0$ and $dH = \iota_X \omega$, a standard calculation in local holomorphic coordinates $z = s + \sqrt{-1}t$ on Σ yields

$$(9.1.3) \quad E^{\text{geom}}(u) = E^{\text{top}}(u) + \int_{\Sigma} \Omega_K$$

where the curvature Ω_K of a perturbation datum K is the exterior derivative of K in the Σ direction.

Definition 9.2. We say that a perturbation datum K is *monotone* if $\Omega_K \leq 0$.

In practice, the most typical monotone perturbation data are of the form $H \otimes \gamma$ for $H: Y \rightarrow \mathbb{R}^{\geq 0}$ and γ a sub-closed one form on Σ . Whenever the perturbation datum is chosen to be monotone, we have

$$(9.2.1) \quad E^{\text{geom}}(u) \leq E^{\text{top}}(u).$$

Note that the topological energy is determined by the action

$$(9.2.2) \quad A_{H_i}(x) = \int_0^1 (-x^*\theta + H_i(x(t))dt) + h(x(1)) - h(x(0))$$

as

$$(9.2.3) \quad E^{\text{top}}(u) = A_{H_0}(x_0) - \sum_{k=1}^d A_{H_k}(x_k),$$

where $h: L \rightarrow \mathbb{R}$ is defined by $\theta|_L = dh$. Throughout this paper, all Floer data K are chosen to be monotone outside of a compact set in $\Sigma \times Y$. This gives an a priori bound for the geometric energy of a solution to Floer's equation (4.13.1) of the form $E^{\text{geom}}(u) \leq E^{\text{top}}(u) + C$ for some constant C which is independent of the given solution u .

9.3. Maximum principle. Let $u: \Omega \rightarrow \mathbb{R}^{\geq 0}$ be a smooth function on a connected open set $\Omega \subset \mathbb{R}^n$ satisfying $Lu \geq 0$ for an elliptic second order differential operator

$$(9.3.1) \quad L = \sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^n b_i(x)\partial_i$$

such that

- the matrix-valued function $(a_{ij}(x))_{i,j=1}^n$ is locally uniformly positive-definite, and
- the functions $b_i(x)$ are locally bounded for $i = 1, \dots, n$.

The *strong maximum principle* states that if u attains a maximum in Ω , then it must be constant. *Hopf's lemma* states that if

- u extends smoothly to the boundary, and
- there exists $x_0 \in \partial\Omega$ that $u(x_0) > u(x)$ for any $x \in \Omega$,

then one has $du(\nu) > 0$, where ν is the unit outward normal vector to $\partial\Omega$ at x_0 . Proofs of the strong maximum principle and Hopf's lemma can be found, e.g., in [Eva10, Section 6.4.2].

REFERENCES

- [AAK16] Mohammed Abouzaid, Denis Auroux, and Ludmil Katzarkov, *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces*, Publ. Math. Inst. Hautes Études Sci. **123** (2016), 199–282. MR 3502098
- [Abo06] Mohammed Abouzaid, *Homogeneous coordinate rings and mirror symmetry for toric varieties*, Geom. Topol. **10** (2006), 1097–1157 (electronic). MR MR2240909 (2007h:14052)
- [Abo09] ———, *Morse homology, tropical geometry, and homological mirror symmetry for toric varieties*, Selecta Math. (N.S.) **15** (2009), no. 2, 189–270. MR MR2529936
- [Abo10] ———, *A geometric criterion for generating the Fukaya category*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 112, 191–240. MR 2737980
- [AKO08] Denis Auroux, Ludmil Katzarkov, and Dmitri Orlov, *Mirror symmetry for weighted projective planes and their noncommutative deformations*, Ann. of Math. (2) **167** (2008), no. 3, 867–943. MR MR2415388 (2009f:53142)
- [AS10] Mohammed Abouzaid and Paul Seidel, *An open string analogue of Viterbo functoriality*, Geom. Topol. **14** (2010), no. 2, 627–718. MR 2602848 (2011g:53190)
- [Aur07] Denis Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. GGT **1** (2007), 51–91. MR 2386535 (2009f:53141)
- [Aur09] ———, *Special Lagrangian fibrations, wall-crossing, and mirror symmetry*, Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry, Surv. Differ. Geom., vol. 13, Int. Press, Somerville, MA, 2009, pp. 1–47. MR MR2537081

- [BM11] Arend Bayer and Emanuele Macrì, *The space of stability conditions on the local projective plane*, Duke Math. J. **160** (2011), no. 2, 263–322. MR 2852118
- [Bri06] Tom Bridgeland, *Stability conditions on a non-compact Calabi-Yau threefold*, Comm. Math. Phys. **266** (2006), no. 3, 715–733. MR MR2238896
- [Bri07] ———, *Stability conditions on triangulated categories*, Ann. of Math. (2) **166** (2007), no. 2, 317–345. MR 2373143 (2009c:14026)
- [BS15] Tom Bridgeland and Ivan Smith, *Quadratic differentials as stability conditions*, Publ. Math. Inst. Hautes Études Sci. **121** (2015), 155–278. MR 3349833
- [CBM05] R. Castaño-Bernard and D. Matessi, *Some piece-wise smooth Lagrangian fibrations*, Rend. Semin. Mat. Univ. Politec. Torino **63** (2005), no. 3, 223–253. MR 2201567 (2006j:53115)
- [CBM09] Ricardo Castaño Bernard and Diego Matessi, *Lagrangian 3-torus fibrations*, J. Differential Geom. **81** (2009), no. 3, 483–573. MR 2487600 (2010e:53146)
- [Cha13] Kwokwai Chan, *Homological mirror symmetry for A_n -resolutions as a T -duality*, J. Lond. Math. Soc. (2) **87** (2013), no. 1, 204–222. MR 3022713
- [CLL12] Kwokwai Chan, Siu-Cheong Lau, and Naichung Conan Leung, *SYZ mirror symmetry for toric Calabi-Yau manifolds*, J. Differential Geom. **90** (2012), no. 2, 177–250. MR 2899874
- [CPU16] Kwokwai Chan, Daniel Pomerleano, and Kazushi Ueda, *Lagrangian torus fibrations and homological mirror symmetry for the conifold*, Comm. Math. Phys. **341** (2016), no. 1, 135–178. MR 3439224
- [CU13] Kwokwai Chan and Kazushi Ueda, *Dual torus fibrations and homological mirror symmetry for A_n -singularities*, Commun. Number Theory Phys. **7** (2013), no. 2, 361–396. MR 3164868
- [Dui80] J. J. Duistermaat, *On global action-angle coordinates*, Comm. Pure Appl. Math. **33** (1980), no. 6, 687–706. MR 596430 (82d:58029)
- [Eva10] Lawrence C. Evans, *Partial differential equations*, second ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010. MR 2597943 (2011c:35002)
- [GHK15] Mark Gross, Paul Hacking, and Sean Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 65–168. MR 3415066
- [GM] Mark Gross and Diego Matessi, *On homological mirror symmetry of toric Calabi-Yau threefolds*, arXiv:1503.03816.
- [Gol01] Edward Goldstein, *Calibrated fibrations on noncompact manifolds via group actions*, Duke Math. J. **110** (2001), no. 2, 309–343. MR 1865243 (2002j:53065)
- [Gro85] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347. MR 809718 (87j:53053)
- [Gro01a] Mark Gross, *Examples of special Lagrangian fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 81–109. MR 1882328 (2003f:53085)
- [Gro01b] ———, *Topological mirror symmetry*, Invent. Math. **144** (2001), no. 1, 75–137. MR 1821145 (2002c:14062)
- [GS] Mark Gross and Bernd Siebert, *Theta functions and mirror symmetry*, arXiv:1204.1991.
- [HIV] Kentaro Hori, Amer Iqbal, and Cumrun Vafa, *D-branes and mirror symmetry*, hep-th/0005247.
- [IU05] Akira Ishii and Hokuto Uehara, *Autoequivalences of derived categories on the minimal resolutions of A_n -singularities on surfaces*, J. Differential Geom. **71** (2005), no. 3, 385–435. MR MR2198807
- [IUU10] Akira Ishii, Kazushi Ueda, and Hokuto Uehara, *Stability conditions on A_n -singularities*, J. Differential Geom. **84** (2010), no. 1, 87–126. MR 2629510
- [Joy15] Dominic Joyce, *Conjectures on Bridgeland stability for Fukaya categories of Calabi-Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow*, EMS Surv. Math. Sci. **2** (2015), no. 1, 1–62. MR 3354954
- [Kon95] Maxim Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120–139. MR MR1403918 (97f:32040)
- [KS02] Mikhail Khovanov and Paul Seidel, *Quivers, Floer cohomology, and braid group actions*, J. Amer. Math. Soc. **15** (2002), no. 1, 203–271 (electronic). MR MR1862802 (2003d:53155)
- [KvK16] Myeonggi Kwon and Otto van Koert, *Brieskorn manifolds in contact topology*, Bull. Lond. Math. Soc. **48** (2016), no. 2, 173–241. MR 3483060

- [LYZ00] Naichung Conan Leung, Shing-Tung Yau, and Eric Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, Adv. Theor. Math. Phys. **4** (2000), no. 6, 1319–1341. MR 1894858 (2003b:53053)
- [McL] Mark McLean, *A spectral sequence for symplectic homology*, arXiv:1011.2478.
- [McL09] ———, *Lefschetz fibrations and symplectic homology*, Geom. Topol. **13** (2009), no. 4, 1877–1944. MR 2497314 (2011d:53224)
- [McL12] ———, *Local Floer homology and infinitely many simple Reeb orbits*, Algebr. Geom. Topol. **12** (2012), no. 4, 1901–1923. MR 2994824
- [Mik04] Grigory Mikhalkin, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*, Topology **43** (2004), no. 5, 1035–1065. MR MR2079993 (2005i:14055)
- [MS12] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, second ed., American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2012. MR 2954391
- [Pas] James Pascaleff, *On the symplectic cohomology of log Calabi-Yau surfaces*, arXiv:1304.5298.
- [Pas14] ———, *Floer cohomology in the mirror of the projective plane and a binodal cubic curve*, Duke Math. J. **163** (2014), no. 13, 2427–2516. MR 3265556
- [RS] Alexander F. Ritter and Ivan Smith, *The open-closed string map revisited*, arXiv:1201.5880v3.
- [Rul] Hans Rullgard, *Polynomial amoebas and convexity*, Research Reports in Mathematics, Number 8, Stockholm University, 2001.
- [Sei] Paul Seidel, *Categorical dynamics and symplectic topology*, lecture notes for the graduate course at MIT in Spring 2013, available at <http://www-math.mit.edu/~seidel/937/index.html>.
- [Sei08] ———, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR MR2441780
- [Sei10] ———, *Suspending Lefschetz fibrations, with an application to local mirror symmetry*, Comm. Math. Phys. **297** (2010), no. 2, 515–528. MR 2651908
- [Sei15] ———, *Homological mirror symmetry for the quartic surface*, Mem. Amer. Math. Soc. **236** (2015), no. 1116, vi+129. MR 3364859
- [She15] Nick Sheridan, *Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space*, Invent. Math. **199** (2015), no. 1, 1–186. MR 3294958
- [Smi15] Ivan Smith, *Quiver algebras as Fukaya categories*, Geom. Topol. **19** (2015), no. 5, 2557–2617. MR 3416110
- [ST01] Paul Seidel and Richard Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37–108. MR MR1831820 (2002e:14030)
- [SYZ96] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), no. 1-2, 243–259. MR MR1429831 (97j:32022)
- [Tod08] Yukinobu Toda, *Stability conditions and crepant small resolutions*, Trans. Amer. Math. Soc. **360** (2008), no. 11, 6149–6178. MR 2425708
- [Tod09] ———, *Stability conditions and Calabi-Yau fibrations*, J. Algebraic Geom. **18** (2009), no. 1, 101–133. MR 2448280 (2009k:14076)
- [TY02] R. P. Thomas and S.-T. Yau, *Special Lagrangians, stable bundles and mean curvature flow*, Comm. Anal. Geom. **10** (2002), no. 5, 1075–1113. MR 1957663
- [Zai] Mikhail Zaidenberg, *Lectures on exotic algebraic structures on affine spaces*, arXiv:math/9801075.
- [Zha] Jingyu Zhao, *Periodic symplectic cohomologies*, arXiv:1405.2084.

DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG

E-mail address: kwchan@math.cuhk.edu.hk

IMPERIAL COLLEGE LONDON, 180 QUEEN'S GATE, LONDON SW7 2AZ, UK

E-mail address: daniel.pomerleano@gmail.com

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU, TOKYO 153-8914, JAPAN

E-mail address: kazushi@ms.u-tokyo.ac.jp