A DIFFERENTIAL-GEOMETRIC APPROACH TO DEFORMATIONS OF PAIRS (X, E)

KWOKWAI CHAN AND YAT-HIN SUEN

ABSTRACT. This article gives an exposition of the deformation theory for pairs (X, E), where X is a compact complex manifold and E is a holomorphic vector bundle over X, adapting an analytic viewpoint à la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer–Cartan equation and differential graded Lie algebra (DGLA) governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of E, obtaining a chain level refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of E over X. As an application, we give examples where deformations of pairs are unobstructed.

CONTENTS

1. INTRODUCTION

The theory of deformations of pairs (X, E), where X is a compact complex manifold and E is a holomorphic vector bundle over X, has been studied using both algebraic [21, 16, 9, 15] and analytic [22, 6] approaches and is well-understood among experts. In this mostly expository paper, we revisit this problem from a viewpoint à la Kodaira-Spencer [12, 13, 11], emphasizing the use of differential-geometric notions such as connections and curvatures of E and the induced differential operators. What we obtain is a chain level refinement of the classical results.

To illustrate our strategy, recall that a family of deformations $\{X_t\}_{t\in\Delta}$ of a compact complex manifold X over a small ball Δ can be represented by elements $\{\varphi_t\}_{t\in\Delta} \subset \Omega^{0,1}(T_X)$, where T_X is the holomorphic tangent bundle of the complex manifold X. While the Dolbeault operator $\bar{\partial}_t : \Omega^0_{X_t} \to \Omega^{0,1}_{X_t}$ on X_t is not easy to write down explicitly, one may consider the more convenient operator

$$\bar{\partial} + \varphi_t \lrcorner \partial : \Omega^0_X \to \Omega^{0,1}_X.$$

Although $\bar{\partial} + \varphi_t \lrcorner \partial$ is *not* the same as $\bar{\partial}_t$, their kernels coincide (see Proposition 3.1), and hence $\bar{\partial} + \varphi_t \lrcorner \partial$ completely determines the local holomorphic functions with respective to the complex structure J_t on X_t. In fact, we have $\Omega^{0,1}_{X_t} = (id - \bar{\varphi}_t^*)\Omega^{0,1}_X$ and the commutative diagram

$$
\begin{array}{ccc}
\Omega^0_X & \xrightarrow{\bar{\partial}_t} & \Omega^{0,1}_{X_t} \\
& \searrow{\scriptstyle \bar{\partial}+\varphi_t\lrcorner\partial} & \downarrow{\scriptstyle \pi_t} \\
& & \Omega^{0,1}_X
\end{array}
$$

where π_t is the inverse of the canonical projection $P_t : \Omega^{0,1}_X \subset \Omega^1_X \to \Omega^{0,1}_{X_t}$ (see the proof of Proposition 3.13); in this way we can compute everything in terms of the holomorphic structure on X.

The same idea can be applied to deformations of pairs. First of all, given a family of deformations $\{(X_t, E_t)\}_{t\in\Delta}$ of (X, E), we have a family of elements $\{\varphi_t\}_{t\in\Delta} \subset \Omega^{0,1}(T_X)$ since $\{X_t\}$ is in particular a family of deformations of X. Using a smooth trivialization, we may further assume that $E_0 \cong E$ as smooth complex vector bundles. By choosing a hermitian metric on E and considering the associated Chern connection, we define a differential operator

$$\bar{D}_t : \Omega^{0,q}_X(E) \to \Omega^{0,q+1}_X(E),$$

which satisfies the Leibniz rule and $\bar{D}_t^2 = 0$ (see Section 3 for details). While $\bar{D}_t$ is certainly *not* the Dolbeault operator $\bar{\partial}_{E_t}$ on the holomorphic bundle E_t, its kernel gives precisely the space of holomorphic sections of E_t over X_t, and similar to the case when $E = \mathcal{O}_X$ (the trivial line bundle), we have a commutative diagram

$$
\begin{array}{ccc}
\Omega^0_X(E) & \xrightarrow{\bar{\partial}_{E_t}} & \Omega^{0,1}_{X_t}(E) \\
& \searrow{\scriptstyle \bar{D}_t} & \downarrow{\scriptstyle \pi_t} \\
& & \Omega^{0,1}_X(E)
\end{array}
$$

relating the operators $\bar{D}_t$ and $\bar{\partial}_{E_t}$. Furthermore, $\bar{D}_t$ determines a family of elements

$$A_t := \bar{D}_t - \bar{\partial}_E - \varphi_t \lrcorner \nabla \in \Omega^{0,1}(\mathrm{End}(E));$$

conversely, given any family of pairs of elements $A_t \in \Omega^{0,1}(\mathrm{End}(E))$, $\varphi_t \in \Omega^{0,1}(T_X)$, we can set

$$\bar{D}_t := \bar{\partial}_E + \varphi_t \lrcorner \nabla + A_t.$$

The upshot is the following Newlander–Nirenberg-type theorem for deformations of pairs:

Theorem 1.1 (=Theorem 3.12). *Given $\varphi_t \in \Omega^{0,1}(T_X)$ and $A_t \in \Omega^{0,1}(End(E))$, if the induced differential operator $\bar{D}_t$ defined above satisfies $\bar{D}_t^2 = 0$, then it defines a holomorphic pair (X_t, E_t) (i.e. an integrable complex structure J_t on X together with a holomorphic bundle structure on E over (X, J_t)).*

Applying this, we derive the *Maurer–Cartan equation*:

Theorem 1.2 (=Theorem 3.17). *Given a holomorphic pair (X, E) and a smooth family of elements $\{(A_t, \varphi_t)\}_{t \in \Delta} \subset \Omega^{0,1}(A(E))$. Then (A_t, φ_t) defines a holomorphic pair (X_t, E_t) if and only if the Maurer–Cartan equation*

$$\bar{\partial}_{A(E)}(A_t, \varphi_t) + \frac{1}{2}[(A_t, \varphi_t), (A_t, \varphi_t)] = 0$$

is satisfied. Here, $A(E)$ is the Atiyah extension *of E which is equipped with the Dolbeault operator $\bar{\partial}_{A(E)}$, and the bracket $[-, -]$ is defined in terms of connections and curvatures on E in Proposition 3.14.*

Moreover, the triple $(\Omega^{0,\bullet}(A(E)), \bar{\partial}_{A(E)}, [-, -])$ forms a *differential graded Lie algebra* (DGLA), which (as expected) is naturally isomorphic to the one obtained by algebraic means [21, 16] (see Appendix A). At this point, we should mention that the relation between deformation theories and DGLAs was first recognized in [19] by Nijenhuis and Richardson; later, it was suggested by Goldman and Millson [3, 4] and many others that deformation problems should always be controlled by DGLAs and solutions to the associated Maurer–Cartan equations form moduli spaces of the deformation problems.

From the Maurer–Cartan equation, we deduce that the space of first order deformations of (X, E) is given by the first cohomology group $H^{0,1}_{\bar{\partial}_{A(E)}} \cong H^1(X, A(E))$ (see Section 4), and that the obstruction theory is captured by the *Kuranishi map*

$$Ob_{(X,E)} : U \subset H^1(X, A(E)) \to H^2(X, A(E)),$$

$$\sum_{i=1}^{m} t^i(A_i, \varphi_i) \mapsto H[(A_t, \varphi_t), (A_t, \varphi_t)],$$

whence obstructions lie inside the second cohomology group $H^{0,2}_{\bar{\partial}_{A(E)}} \cong H^2(X, A(E))$ (see Section 5). Here, U is a small open neighborhood of the origin $0 \in H^1(X, A(E))$. We also give a proof of the existence of a locally complete (or versal) family (see Theorem 6.2; cf. [22]) using an analytic method originally due to Kuranishi [14].

Next we apply this analytic approach to look for situations where deformations of holomorphic pairs are unobstructed (Section 7). The main tool is the following proposition relating deformations of the pair (X, E) to that of X and E, which first appeared in [7, Appendix A] without proof:

Proposition 1.3 (=Proposition 7.1). *Denote the Kuranishi obstruction maps of the deformation theory of X, E and (X, E) by Ob_X, Ob_E and $Ob_{(X,E)}$ respectively. Then we have the following commutative diagram:*

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^1(X, End(E)) & \xrightarrow{\iota^*} & H^1(X, A(E)) & \xrightarrow{\pi^*} & H^1(X, T_X) & \xrightarrow{\delta} & \cdots \\
& & \downarrow{\scriptstyle Ob_E} & & \downarrow{\scriptstyle Ob_{(X,E)}} & & \downarrow{\scriptstyle Ob_X} & & \\
\cdots & \longrightarrow & H^2(X, End(E)) & \xrightarrow{\iota^*} & H^2(X, A(E)) & \xrightarrow{\pi^*} & H^2(X, T_X) & \xrightarrow{\delta} & \cdots
\end{array}
$$

Here, the connecting homomorphism δ is given by contracting with the Atiyah class*:*

$$\delta(\varphi) = \varphi \lrcorner [F_\nabla].$$

Remark 1.4. *The vertical maps (Kuranishi maps) in the above proposition are understood to be defined on small neighborhoods around the origins of the corresponding cohomology groups.*

Applying this proposition, we obtain results which generalize some of those in the recent work of X. Pan [20] (where only the case when E is a line bundle was considered). We also prove that when X is a K3 surface and E is a good bundle over X with $c_1(E) \neq 0$ (Proposition 7.7), deformations of pairs (X, E) are unobstructed.

Remark 1.5. *After we posted an earlier version of this article on the arXiv, Carl Tipler informed us that the paper [6] of L. Huang already contained most of our results, although we have more detailed expositions of first order deformations (Section 4) and the proof of existence of Kuranishi families (Section 6) than Huang's paper and we have a comparison with the algebraic approach (Appendix A) showing in particular that the isomorphism class of the DGLA is independent of the choice of hermitian metric on E. As a result, this article should be regarded as largely expository.*

2. Connections, curvature and the Atiyah class

In this section, we review some basic notions in the theory of holomorphic vector bundles over complex manifolds and fix our notations. Excellent references for these materials include the textbooks [5, 8].

Let E be a complex vector bundle over a smooth manifold X. For $k \geq 0$, we denote by Ω^k the sheaf of k-forms and by $\Omega^k(E)$ the sheaf of E-valued k-forms over X. Recall that a *connection* on E is a $\mathbb{C}$-linear sheaf homomorphism $\nabla : \Omega^0(E) \to \Omega^1(E)$ satisfying the Leibniz rule:

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$$

for $f \in \Omega^0$ and $s \in \Omega^0(E)$. We extend ∇ naturally to $\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$ by defining

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s$$

for $\alpha \in \Omega^k$ and any $s \in \Omega^0(E)$. The *curvature*

$$F_\nabla = \nabla \circ \nabla : \Omega^0(E) \to \Omega^2(E)$$

of ∇ can then be regarded as a global $\text{End}(E)$-valued 2-form. Also, ∇ induces a natural connection on $\text{End}(E)$ by

$$(\nabla A)(s) = \nabla(As) - A(\nabla s),$$

where $A \in \Omega^0(\text{End}(E))$ and $s \in \Omega^0(E)$, and we have the *Bianchi identity*

$$\nabla F_\nabla = 0.$$

Now suppose that X is a complex manifold. For $p, q \geq 0$, we denote by $\Omega^{p,q}$ the sheaf of (p,q)-forms and by $\Omega^{p,q}(E)$ the sheaf of E-valued (p,q)-forms over X. Recall that a holomorphic structure on a complex vector bundle E over X is uniquely determined by a $\mathbb{C}$-linear operator $\bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$ satisfying the Leibniz rule and the integrability condition $\bar{\partial}_E^2 = 0$. If we further equip E with a hermitian metric h, then there exists a unique connection ∇ on E which is hermitian (i.e. $dh(s_1, s_2) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$ for any $s_1, s_2 \in \Omega^0(E)$) and compatible with the holomorphic structure on E (i.e. $\nabla^{0,1} = \bar{\partial}_E$, where $\nabla^{0,1} = \Pi^{0,1} \circ \nabla$ and $\Pi^{p,q} : \Omega^{p+q}(E) \to \Omega^{p,q}(E)$ is the natural projection map). ∇ is usually called the *Chern connection* on (E, h). The curvature F_∇ of the Chern connection on (E, h) is real and of type $(1,1)$, so the Bianichi identity implies that $\bar{\partial}_{\text{End}(E)} F_\nabla = 0$, and thus this defines a class

$$[F_\nabla] \in H^{1,1}(X, \text{End}(E)),$$

called the *Atiyah class* of E [1]. We have the following lemma.

Lemma 2.1 ([8], Proposition 4.3.10)**.** *The Atiyah class is independent of the choice of the Hermitian metric.*

Using the Atiyah class, one can define an extension of $\text{End}(E)$ by T_X; indeed, in the language of algebraic geometry, we can interpret the Atiyah class as an element in the extension group $\text{Ext}^1(E \otimes T_X, E) = \text{Ext}^1(T_X, \text{End}(E))$. Consider the smooth vector bundle $A(E) := \text{End}(E) \oplus T_X$ and the differential operator $\bar{\partial}_{A(E)_B} : \Omega^0(A(E)) \to \Omega^{0,1}(A(E))$ on $A(E)$ defined by

$$\bar{\partial}_{A(E)_B} := \begin{pmatrix} \bar{\partial}_{\text{End}(E)} & B \\ O & \bar{\partial}_{T_X} \end{pmatrix},$$

where $B \in \Omega^{0,1}(\text{Hom}(T_X, \text{End}(E)))$ acts on $\Omega^0(T_X)$ by

$$B \wedge \varphi := -(-1)^{|\varphi|} \varphi \lrcorner F_\nabla.$$

To simplify notations, from this point on, we will denote the vector bundles $\text{End}(E)$ and $\text{Hom}(T_X, \text{End}(E))$ by Q and H respectively unless specified otherwise.

Proposition 2.2. $B \in \Omega^{0,1}(H)$ *is* $\bar{\partial}_H$*-closed.*

Proof. This follows from the Bianchi identity $\bar{\partial}_Q F_\nabla = 0$: For any $v \in T_X$,

$$(\bar{\partial}_H B)(v) = \bar{\partial}_Q(Bv) + B(\bar{\partial}_{T_X} v) = -\bar{\partial}_Q(v \lrcorner F_\nabla) + \bar{\partial}_{T_X} v \lrcorner F_\nabla = v \lrcorner \bar{\partial}_Q F_\nabla = 0.$$

$\square$

Proposition 2.3. $\left(A(E), \bar{\partial}_{A(E)_B}\right)$ *defines a holomorphic vector bundle over X whose holomorphic structure depends only on the class $[B]$.*

Proof. Clearly $\bar{\partial}_{A(E)_B}$ satisfies the Leibniz rule, so it suffices to prove that $\bar{\partial}^2_{A(E)_B} = 0$. But $\bar{\partial}^2_{A(E)_B} = 0$ if and only if $\bar{\partial}_H B = 0$ which holds by Proposition 2.2. This proves the first part of the proposition.

To see the second part, suppose that $B' - B = \bar{\partial}_H f$ for some $f \in \mathrm{Hom}(T_X, Q)$. Define the smooth bundle isomorphism $F : A(E) \to A(E)$ by

$$F : (A, v) \longmapsto (A - fv, v),$$

and extend to $A(E)$-valued p-forms. We compute

$$\bar{\partial}_{A(E)_{B'}} F(A, v) = (\bar{\partial}_Q(A - fv) + B'v, \bar{\partial}_{T_X} v) = (\bar{\partial}_Q A - \bar{\partial}_Q fv + Bv + \bar{\partial}_H fv, \bar{\partial}_{T_X} v)$$
$$= (\bar{\partial}_Q A + Bv - f\bar{\partial}_{T_X} v, \bar{\partial}_{T_X} v) = F\bar{\partial}_{A(E)_B}(A, v).$$

Hence F in fact defines a holomorphic bundle isomorphism between $(A(E), \bar{\partial}_{A(E)_B})$ and $(A(E), \bar{\partial}_{A(E)_{B'}})$. Since the curvature F_∇ differs by an exact $\mathrm{End}(E)$-valued 1-form if another metric was used, this shows that the holomorphic structure of $A(E)_B$ only depends on the class $[B]$ but not the metric. $\qquad\square$

Remark 2.4. *Under the Dolbeault isomorphism*

$$H^1(X, Hom(T_X, Q)) \cong H^{1,1}(X, Q),$$

the class $[B]$ corresponds to the Atiyah class $[F_\nabla]$. Hence the holomorphic structure of $A(E)$ depends only on the Atiyah class of E.

By abuse of notations, we will now write $\bar{\partial}_{A(E)_B}$ simply as $\bar{\partial}_{A(E)}$, keeping in mind that a hermitian metric on E has been chosen.

Definition 2.5. *The holomorphic vector bundle $\left(A(E), \bar{\partial}_{A(E)} \right)$, which is an extension of $Q = End(E)$ by T_X, is called the* Atiyah extension *of E.*

3. MAURER–CARTAN EQUATIONS

In this section, we start our study of the deformation theory of pairs (X, E). Our goal is to derive the DGLA and Maurer–Cartan equation which govern this deformation problem.

3.1. **Deformations of complex structures and holomorphic vector bundles.** We begin by a brief review of the classical theory of deformations of complex structures and holomorphic vector bundles; the textbooks [11] and [10] are classic references for these theories respectively.

We first recall that a family of deformations $\pi : \mathcal{X} \to \Delta$ of a compact complex manifold X can be represented by a family of sections $\varphi_t \in \Omega^{0,1}(T_X)$, where T_X is the holomorphic tangent bundle of X (or the i-eigenbundle of the almost complex structure defining X), satisfying the Maurer–Cartan equation

$$(1) \qquad\qquad \bar{\partial}_{T_X} \varphi_t + \frac{1}{2}[\varphi_t, \varphi_t] = 0.$$

An essential ingredient in the proof is the Newlander–Nirenberg Theorem [18] which states that any integrable almost complex structure comes from a complex structure.

Proposition 3.1. *Define an operator* $\bar{\partial} + \varphi_t \lrcorner \partial : \Omega^0 \to \Omega^{0,1}$ *by* $f \mapsto \bar{\partial}f + \varphi_t \lrcorner (\partial f)$, *where* $\lrcorner$ *denotes the contraction or interior product. Then a local smooth function* f *is holomorphic on* X_t *if and only if* $\left(\bar{\partial} + \varphi_t \lrcorner \partial \right) f = 0$, *i.e.*

$$\bar{\partial}_t f = 0 \iff \left(\bar{\partial} + \varphi_t \lrcorner \partial \right) f = 0,$$

where $\bar{\partial}_t$ *is the* $\bar{\partial}$-*operator of the complex manifold* X_t.

Proof. Let $z^1, \ldots, z^n$ be local holomorphic coordinates on X (where n is the complex dimension of X). Then φ_t is of the form [1]

$$\varphi_t = \varphi_i^j(z,t) d\bar{z}^i \otimes \frac{\partial}{\partial z^j}.$$

Hence $T_{X_t}^{0,1}$ is locally spanned by

$$\frac{\partial}{\partial \bar{z}^i} + \varphi_i^j(z,t) \frac{\partial}{\partial z^j}.$$

The result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next we recall the deformation theory of holomorphic vector bundles. Let $E \to X$ be a complex vector bundle over a complex manifold X. It is a standard fact in complex geometry that E admits a holomorphic structure if and only if there exists a linear operator $\bar{\partial}_E : \Omega^{0,q}(E) \to \Omega^{0,q+1}(E)$ satisfying $\bar{\partial}_E^2 = 0$ and the Leibniz rule

$$\bar{\partial}_E(\alpha \otimes s) = \bar{\partial}\alpha \otimes s + (-1)^{|\alpha|}\alpha \wedge \bar{\partial}_E(s),$$

for any $\alpha \in \Omega^{0,q}(E)$ and smooth section s of E (we call this the linearized version of the Newlander–Nirenberg Theorem; see e.g. [8, Theorem 2.6.26] or [17, Theorem 3.2]). Hence if we have a family of holomorphic vector bundles $\mathcal{E} \to \Delta$ (or $\{E_t\}_{t \in \Delta}$) on X, then we have a family of Dolbeault operators $\bar{\partial}_{E_t}$, whose squares are zero and all satisfy the Leibniz rule.

Proposition 3.2. *Given a family of deformations* $\{E_t\}_{t \in \Delta}$ *of* E, *the element* $A_t := \bar{\partial}_{E_t} - \bar{\partial}_E \in \Omega^{0,1}(End(E))$ *satisfies the Maurer–Cartan equation*

$$\bar{\partial}_{End(E)} A_t + \frac{1}{2}[A_t, A_t] = 0$$

for all $t \in \Delta$. *Conversely, if we are given a family* $\{A_t\}_{t \in \Delta} \subset \Omega^{0,1}(End(E))$ *which satisfies the Maurer–Cartan equation for each* t, *then* $\left\{ (E, \bar{\partial}_E + A_t) \right\}_{t \in \Delta}$ *defines a family of deformations of* E.

Proof. Note that

$$(\bar{\partial}_E + A_t)^2 = \bar{\partial}_E A_t + A_t \bar{\partial}_E + A_t \wedge A_t = \bar{\partial}_{\mathrm{End}(E)} A_t + \frac{1}{2}[A_t, A_t].$$

The result follows from the linearized version of the Newlander–Nirenberg Theorem. $\qquad\square$

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[1]The Einstein summation convention will be used throughout this article.

3.2. Deformations of holomorphic pairs and the operator $\bar{D}_t$.

Definition 3.3. *A* holomorphic pair (X, E) *consists of a compact complex manifold* X *together with a holomorphic vector bundle* E *over* X.

Definition 3.4. *Let* (X, E) *be a holomorphic pair. A* family of deformations *of* (X, E) *over a small ball* Δ *centered at the origin in* $\mathbb{C}^d$ *consists of a proper and submersive holomorphic map* $\pi : \mathcal{X} \to \Delta$ *(a family of deformations of* X *over* Δ*) and a holomorphic vector bundle* $\mathcal{E} \to \mathcal{X}$ *such that* $\pi^{-1}(0) = X$ *and* $\mathcal{E}|_{\pi^{-1}(0)} = E$. *For* $t \in \Delta$, *we denote by* (X_t, E_t) *the holomorphic pair parametrized by* t.

By the theorem of Ehresmann, if Δ is chosen to be small enough, the family $\mathcal{X}$ is smoothly trivial, i.e. one can find a diffeomorphism $F : \mathcal{X} \to \Delta \times X$. Restricting to a fiber $\mathcal{X}_t \subset \mathcal{X}$, one can push forward the complex structure on $\mathcal{X}_t$ to define J_t on $X_t := \{t\} \times X$ via F. One can also trivialize $\mathcal{E}$ as $\Delta \times E$ by a smooth bundle isomorphism P and the holomorphic structure on $E_t := \{t\} \times E$ is induced from that on $\mathcal{E}|_{\mathcal{X}_t}$ via the map P. Hence we can assume that our family is a smoothly trivial family $\Delta \times E \to \Delta \times X$ over a small ball Δ in $\mathbb{C}^d$ centered at the origin.

Now let $\{(X_t, E_t)\}_{t \in \Delta}$ be a family of deformations of (X, E). By definition, $\{X_t\}_{t \in \Delta}$ is a family of deformations of X, so it can be represented by an analytic family of sections $\varphi_t \in \Omega^{0,1}(T_X)$ satisfying the Maurer–Cartan equation (1). Define the operator $\bar{D}_t : \Omega^{0,q}(E) \to \Omega^{0,q+1}(E)$ by

$$\bar{D}_t(s^k e_k) = (\bar{\partial} + \varphi_t \lrcorner \partial)s^k \otimes e_k,$$

where $\{e_k\}$ is a local holomorphic frame of E_t.

Proposition 3.5. *The linear operator* $\bar{D}_t$ *is well-defined, that is, independent of the local holomorphic frame of* E_t. *Moreover, it satisfies the Leibniz rule*

$$\bar{D}_t(\alpha \otimes s) = (\bar{\partial} + \varphi_t \lrcorner \partial)\alpha \otimes s + (-1)^{|\alpha|}\alpha \wedge \bar{D}_t(s)$$

for any $s \in \Omega^0(E)$ *and* $\alpha \in \Omega^{0,*}(X)$. *Also,* $\bar{D}_t(s) = 0$ *if and only if* $\bar{\partial}_{E_t}(s) = 0$.

Proof. To prove well-definedness, we need to show that $\bar{D}_t$ is independent of the choice of a local holomorphic frame $\{e_k\}$ of E_t. So suppose $\{f_j\}$ is another local holomorphic frame of E_t. Let τ_j^k be local holomorphic functions on X_t such that $f_j = \tau_j^k e_k$. Then for a local section $s = s^k e_k = \widetilde{s}^j f_j$, we have $\widetilde{s}^j = s^k \tau_k^j$ and thus

$$\bar{D}_t(\widetilde{s}^j f_j) = (\bar{\partial} + \varphi_t \lrcorner \partial)\widetilde{s}^j \otimes f_j = (\bar{\partial} + \varphi_t \lrcorner \partial)(s^k \tau_k^j) \otimes f_j$$

$$= (\bar{\partial} + \varphi_t \lrcorner \partial)s^k \otimes \tau_k^j f_j = \bar{D}_t(s^k e_k).$$

Hence $\bar{D}_t$ is well-defined.

The Leibniz rule for $\bar{D}_t$ is clear since $\bar{\partial}$ and ∂ both satisfy the usual Leibniz rule. Finally, for a smooth section s of E, if we write $s = s^k e_k$ locally with $\{e_k\}$ a local holomorphic frame of E_t, then we have

$$\bar{D}_t(s) = 0 \iff (\bar{\partial} + \varphi_t \lrcorner \partial)s^k = 0 \iff \bar{\partial}_t s^k = 0 \iff \bar{\partial}_{E_t}(s) = 0.$$

$$\qquad \square$$

We claim that $\bar{D}_t^2 = 0$. By our definition of $\bar{D}_t$, for any smooth function $f : X \to \mathbb{C}$ and local nowhere vanishing holomorphic section e of E_t, we have

$$\bar{D}_t^2(fe) = (\bar{\partial} + \varphi_t \lrcorner \partial)^2 f \otimes e.$$

To compute the right hand side, we need the following

Lemma 3.6. *For any $\varphi \in \Omega^{0,p}(T_X)$ and $\alpha \in \Omega^1(E)$, we have the Leibniz rule*

$$\bar{\partial}_E(\varphi \lrcorner \alpha) = \bar{\partial}_{T_X}\varphi \lrcorner \alpha - (-1)^p \varphi \lrcorner \bar{\partial}_E \alpha.$$

Proof. Writing $\varphi = \varphi^i_J d\bar{z}^J \otimes \frac{\partial}{\partial z^i}$, we have

$$\varphi \lrcorner \alpha = \varphi^i_J d\bar{z}^J \otimes \alpha\left(\frac{\partial}{\partial z^i}\right).$$

Let $\alpha_i := \alpha(\frac{\partial}{\partial z^i}) \in \Omega^0(E)$. Then

$$\bar{\partial}_E(\varphi \lrcorner \alpha) = (\bar{\partial}\varphi^i_J \wedge d\bar{z}^J) \otimes \alpha_i + (-1)^p \varphi^i_J d\bar{z}^J \wedge \bar{\partial}_E \alpha_i$$
$$= \bar{\partial}_{T_X}\varphi \lrcorner \alpha + (-1)^p \varphi^i_J d\bar{z}^J \wedge \bar{\partial}_E \alpha_i.$$

To compute the last term, first note that the contraction of $\frac{\partial}{\partial z^i}$ with α is taken in the $(1,0)$-part, we can therefore assume $\alpha = \alpha^k_i dz^i \otimes e_k$, where $\{e_k\}$ is a local holomorphic frame of E. So we have

$$\varphi^i_J d\bar{z}^J \wedge \bar{\partial}_E \alpha_i = -\varphi^i_J d\bar{z}^J \wedge \left(\frac{\partial}{\partial z^i} \lrcorner \bar{\partial}\alpha^k_l \wedge dz^l\right) \otimes e_k$$

$$= -\varphi^i_J d\bar{z}^J \otimes \frac{\partial}{\partial z^i} \lrcorner \bar{\partial}_E(\alpha^k_l dz^l \otimes e_k) = -\varphi \lrcorner \bar{\partial}_E \alpha,$$

and hence the desired formula. $\qquad\square$

We can now compute $\bar{D}^2_t$.

Lemma 3.7. *For any smooth function, $f : X \to \mathbb{C}$, we have the equality*

$$(\bar{\partial} + \varphi_t \lrcorner \partial)^2 f = \left(\bar{\partial}_{T_X}\varphi_t + \frac{1}{2}[\varphi_t, \varphi_t]\right) \lrcorner \partial f.$$

Proof. First, we have

$$(\bar{\partial} + \varphi_t \lrcorner \partial)^2 f = \bar{\partial}(\varphi_t \lrcorner \partial f) + \varphi_t \lrcorner \partial \bar{\partial} f + \varphi_t \lrcorner \partial(\varphi_t \lrcorner \partial f).$$

By Lemma 3.6, the first term is given by

$$\bar{\partial}(\varphi_t \lrcorner \partial f) = \bar{\partial}_{T_X}\varphi_t \lrcorner \partial f + \varphi_t \lrcorner \bar{\partial}\partial f.$$

Since $\partial\bar{\partial} = -\bar{\partial}\partial$, we have

$$\bar{\partial}(\varphi_t \lrcorner \partial f) + \varphi_t \lrcorner \partial\bar{\partial} f = \bar{\partial}_{T_X}\varphi_t \lrcorner \partial f.$$

For the last term, by writing $\varphi_t = \varphi^l_m d\bar{z}^m \otimes \frac{\partial}{\partial z^l}$ in local coordinates, we have

$$\varphi_t \lrcorner \partial f = \varphi^l_m \frac{\partial f}{\partial z^l} d\bar{z}^m,$$

and so

$$\varphi_t \lrcorner \partial(\varphi_t \lrcorner \partial f) = \varphi^i_j \frac{\partial \varphi^l_m}{\partial z^i} \frac{\partial f}{\partial z^l} d\bar{z}^j \wedge d\bar{z}^m + \varphi^i_j \varphi^l_m \frac{\partial^2 f}{\partial z^i \partial z^l} d\bar{z}^j \wedge d\bar{z}^m$$

But

$$\varphi^i_j \varphi^l_m \frac{\partial^2 f}{\partial z^i \partial z^l} d\bar{z}^j \wedge d\bar{z}^m = -\varphi^l_m \varphi^i_j \frac{\partial^2 f}{\partial z^l \partial z^i} d\bar{z}^m \wedge d\bar{z}^j,$$

so we obtain

$$\varphi_t \lrcorner \partial(\varphi_t \lrcorner \partial f) = \varphi^i_j \frac{\partial \varphi^l_m}{\partial z^i} \frac{\partial f}{\partial z^l} d\bar{z}^j \wedge d\bar{z}^m = \frac{1}{2}[\varphi_t, \varphi_t] \lrcorner \partial f.$$

The result follows. $\qquad\square$

As $\{X_t\}_{t \in \Delta}$ is an honest family of deformations of X, the Maurer–Cartan equation (1) for φ_t holds. Hence we have

Proposition 3.8. $\bar{D}_t^2 = 0$.

From the viewpoint of Proposition 3.1, it is natural to compare the operator $\bar{D}_t$ with $\bar{\partial}_E + \varphi_t \lrcorner \nabla$.

Proposition 3.9. $A_t := \bar{D}_t - \bar{\partial}_E - \varphi_t \lrcorner \nabla \in \Omega^{0,1}(End(E))$.

Proof. Let f be a smooth function and s a smooth section of E. Using the Leibniz rules, and the fact that the contraction is only taken in the $(1,0)$-part, we have

$$A_t(fs) = (\bar{\partial} + \varphi_t \lrcorner \partial)f \otimes s + f\bar{D}_t(s) - \bar{\partial}f \otimes s - f\bar{\partial}_E(s) - \varphi_t \lrcorner \nabla(fs)$$
$$= (\varphi_t \lrcorner \partial)f \otimes s + f\bar{D}_t(s) - f\bar{\partial}_E(s) - \varphi_t \lrcorner df \otimes s - f\varphi_t \lrcorner \nabla(s) = fA_t(s).$$

$\square$

In the other direction, suppose we are now given elements $A_t \in \Omega^{0,1}(\text{End}(E))$ and $\varphi_t \in \Omega^{0,1}(T_X)$, parameterized by $t \in \Delta$, we can then define an operator $\bar{D}_t : \Omega^0(E) \to \Omega^{0,1}(E)$ by

$$\bar{D}_t := \bar{\partial}_E + \varphi_t \lrcorner \nabla + A_t.$$

We extend $\bar{D}_t$ to $\Omega^{0,q}(E)$ in the obvious way, so that the Leibniz rule

$$\bar{D}_t(\alpha \otimes s) = (\bar{\partial} + \varphi_t \lrcorner \partial)\alpha \otimes s + (-1)^{|\alpha|}\alpha \wedge \bar{D}_t s$$

holds. We want to show that if $\bar{D}_t^2 = 0$, then (A_t, φ_t) defines a holomorphic pair (X_t, E_t). First of all, we have

Proposition 3.10. *If $\bar{D}_t^2 = 0$, then X_t is a complex manifold.*

Proof. Using the Leibniz rule, we have for any smooth function and sections of E that

$$0 = \bar{D}^2(fs) = (\bar{\partial} + \varphi_t \lrcorner \partial)^2 f \otimes s.$$

Hence $(\bar{\partial} + \varphi_t \lrcorner \partial)^2 = 0$, which is equivalent to the Maurer–Cartan equation (1) by Lemma 3.7. Therefore, the almost complex structure defined by φ_t is integrable. $\square$

We now need to show that E also admits a holomorphic structure over X_t. We will follow the approach of [17]. Let us first make the following assertion:

Any smooth sections of E can locally be written as $s^k e_k$,

where $\{e_k\} \subset \ker(\bar{D}_t)$. We can then define $\bar{\partial}_{E_t}$ by

$$\bar{\partial}_{E_t}(s^k e_k) := \bar{\partial}_t s^k \otimes e_k.$$

To check that it is well-defined, suppose we have another local basis $\{f_j\} \subset \ker(\bar{D}_t)$, then there exist (h_j^k) such that $f_j = h_j^k e_k$. Applying $\bar{D}_t$, we have

$$(\bar{\partial} + \varphi_t \lrcorner \partial)h_j^k \otimes e_k = 0.$$

Since $\{e_k\}$ is assumed to be a local basis, we have $(\bar{\partial} + \varphi_t \lrcorner \partial)h_j^k = 0$, which is equivalent to $\bar{\partial}_t h_j^k = 0$. Hence

$$\bar{\partial}_{E_t}(\widetilde{s}^j f_j) = \bar{\partial}_t \widetilde{s}^j \otimes f_j = \bar{\partial}_t(s^k h_k^j) \otimes f_j = \bar{\partial}_t(s^j) \otimes h_k^j f_j = \bar{\partial}_{E_t}(s^k e_k).$$

This proves well-definedness.

Clearly, it satisfies the Leibniz rule

$$\bar{\partial}_{E_t}(\alpha \otimes s) = \bar{\partial}_t \alpha \otimes s + (-1)^{|\alpha|} \alpha \wedge \bar{\partial}_{E_t} s$$

and $\bar{\partial}_{E_t}^2 = 0$ since φ_t defines an integrable complex structure on X. Hence by the linearized version of the Newlander–Nirenberg Theorem, $E_t = (E, \bar{\partial}_{E_t})$ is a holomorphic vector bundle over X_t.

It remains to prove that our assertion is correct:

Lemma 3.11. $\ker(\bar{D}_t)$ *generates* $\Omega^0(E)$ *locally.*

Proof. Let us first fix a smooth local frame $\{\sigma_k\}$ of E_t over a coordinate neighborhood $U \subset X_t$. What we need are coordinate changes $(f_j^i(z,t)) \in \Gamma_{sm}(U, GL_r(\mathbb{C}))$ such that $f_j^i \sigma_i \in \ker(\bar{D}_t)$. Writing $\bar{D}_t \sigma_i = \tau_i^k \otimes \sigma_k$ with $\tau_i^k \in \Omega^{0,1}(X)$, the existence of $(f_j^i(z,t))$ is equivalent to

$$0 = \bar{D}_t(f_j^i \sigma_i) = (\bar{\partial} f_j^i + \varphi_t \lrcorner \partial f_j^i) \otimes \sigma_i + f_j^i \tau_i^k \sigma_k.$$

This in turn is equivalent to the following system of PDEs

$$(\bar{\partial} + \varphi_t \lrcorner \partial) f_j^k + f_j^i \tau_i^k = 0$$

subject to the condition:

$$\bar{D}_t^2 = 0 \iff (\bar{\partial} + \varphi_t \lrcorner \partial) \tau_j^i = \tau_k^i \wedge \tau_j^k.$$

We will show that this system is solvable, following the line of proof in [17, Theorem 9.2] (linearized version of the Newlander–Nirenberg Theorem).

First of all we set

$$N := U \times \mathbb{C}^r, \ T := span\{dz^\alpha - \varphi_t \lrcorner dz^\alpha, dw_i - \tau_i^k w_k\}.$$

We want to show that $d(T) \subset \Omega^0(\bigwedge_{\mathbb{C}}^1 N) \wedge T$. First we have

$$d(dz^\alpha - \varphi_t \lrcorner dz^\alpha) = \frac{\partial \varphi_\beta^\alpha}{\partial z^\gamma} d\bar{z}^\beta \wedge dz^\gamma - \bar{\partial}_{T_X} \varphi_t \lrcorner dz^\alpha.$$

Then applying the Maurer–Cartan equation (1) gives

$$d(dz^\alpha - \varphi_t \lrcorner dz^\alpha) = \frac{\partial \varphi_\beta^\alpha}{\partial z^\gamma} d\bar{z}^\beta \wedge \left(dz^\gamma - \varphi_\eta^\mu d\bar{z}^\eta \otimes \frac{\partial}{\partial z^\mu} \lrcorner dz^\gamma \right)$$

$$= \frac{\partial \varphi_\beta^\alpha}{\partial z^\gamma} d\bar{z}^\beta \wedge (dz^\gamma - \varphi_t \lrcorner dz^\gamma).$$

Secondly,

$$d(dw_i - \tau_i^l w_l) = -\partial \tau_i^l w_l - \bar{\partial} \tau_i^l w_l + \tau_i^l \wedge dw_l$$

$$= -(\partial - \varphi_t \lrcorner \partial) \tau_i^l w_l - (\bar{\partial} + \varphi_t \lrcorner \partial) \tau_i^l w_l + \tau_i^l \wedge dw_l$$

$$= -(\partial - \varphi_t \lrcorner \partial) \tau_i^l w_l - \tau_i^k \wedge \tau_k^l w_l + \tau_i^l \wedge dw_l$$

$$= -(\partial - \varphi_t \lrcorner \partial) \tau_i^l w_l - \tau_i^k \wedge (dw_k - \tau_k^l w_l).$$

Hence by the Newlander–Nirenberg Theorem, we obtain holomorphic coordinates (ζ_t^α, u_i^t) on N and smooth functions $F_\beta^\alpha(\zeta_t) = F_\beta^\alpha(z,t)$, $F_i^l(\zeta_t, u_t) = F_i^l(z, u, t)$, and $F_\alpha^l(\zeta_t, u_t) = F_{l\alpha}(z, w, t)$ such that

$$\begin{cases} d\zeta_t^\alpha = F_\beta^\alpha(z,t)(dz^\beta - \varphi_t \lrcorner dz^\beta), \\ du_l^t = F_l^i(z, w, t)(dw_i - \tau_i^k w_k) + F_{l\alpha}(z, w, t)(dz^\alpha - \varphi_t \lrcorner dz^\alpha). \end{cases}$$

Since $\{dz^\alpha - \varphi_t \lrcorner dz^\alpha, dw_i - \tau_i^k w_k\}$ and $\{d\zeta_t^\alpha, du_t^i\}$ are basis of T, we see that the $(n+r) \times (n+r)$-matrix

$$\begin{pmatrix} (F_\beta^\alpha) & (F_{i\alpha}) \\ O_{r\times n} & (F_l^i) \end{pmatrix}$$

is invertible for all (z, w, t). It follows that (F_l^i) is also invertible for all (z, w, t).

Applying the exterior differential on N and evaluating at $w = 0$, we have

$$0 = dF_l^i \wedge dw_i + F_l^i \tau_i^k \wedge dw_k + dF_{l\alpha} \wedge dz^\alpha - dF_{l\alpha} \wedge \varphi_t \lrcorner dz^\alpha - F_{l\alpha} d(\varphi_t \lrcorner dz^\alpha).$$

Comparing the $dz \wedge dw$-component on both sides gives $\partial_z F_l^i \wedge dw_i + \partial_w F_{l\alpha} \wedge dz^\alpha = 0$, which implies, by contracting with φ_t, that

$$\varphi_t \lrcorner dF_l^i \wedge dw_i + \partial_w F_{l\alpha} \wedge \varphi_t \lrcorner dz^\alpha = 0.$$

Then by comparing the $d\bar{z} \wedge dw$-component, we have

$$\bar{\partial} F_l^i \wedge \partial_z w_i + F_l^i \tau_i^k \wedge dw_k - \partial_w F_{l\alpha} \wedge \varphi_t \lrcorner dz^\alpha = 0.$$

Together with the formula we just obtained, we arrive at

$$(\bar{\partial} + \varphi_t \lrcorner \partial) F_l^i(z, 0) + F_l^k(z, 0) \tau_k^i = 0.$$

The result now follows by setting $f_i^j(z, t) := F_i^j(z, 0, t)$. $\qquad\qquad\square$

In summary, we have proved the following

Theorem 3.12. *Given $A_t \in \Omega^{0,1}(End(E))$ and $\varphi_t \in \Omega^{0,1}(T_X)$. If the induced differential operator $\bar{D}_t : \Omega^{0,q}(E) \to \Omega^{0,q+1}(E)$ satisfies $\bar{D}_t^2 = 0$ and the Leibniz rule*

$$\bar{D}_t(\alpha \otimes s) = (\bar{\partial} + \varphi_t \lrcorner \partial)\alpha \otimes s + (-1)^{|\alpha|} \alpha \wedge \bar{D}_t(s),$$

then E admits a holomorphic structure over the complex manifold X_t, which we will denote by $E_t \to X_t$ or just E_t.

The operator $\bar{D}_t$ gives a cochain complex

$$(\Omega^{0,\bullet}(E), \bar{D}_t).$$

It is then natural to compare the cohomologies $H^\bullet(X_t, E_t)$ and $H^\bullet(\Omega^{0,\bullet}(E), \bar{D}_t)$. But $\bar{D}_t$ captures only the holomorphicity of the pair (X_t, E_t), so we would not expect $H^\bullet(\Omega^{0,\bullet}(E), \bar{D}_t)$ to be something new.

Proposition 3.13. *For any $t \in \Delta$, we have the isomorphism*

$$H^q(X_t, E_t) \cong H^q(\Omega^{0,\bullet}(E), \bar{D}_t)$$

for any $q \geq 0$.

Proof. We first prove the case when $E = \mathcal{O}_X$ and $q = 0$. Let $P_t : \Omega_X^{0,1} \to \Omega_{X_t}^{0,1}$ be the restriction of the projection $\Omega_X^1 \to \Omega_{X_t}^{0,1}$. Since P_t is an isomorphism for $|t|$ small, it suffices to prove that

$$\bar{\partial}_t P_t = P_t(\bar{\partial} + \varphi_t \lrcorner \partial)$$

at every point $x \in X$. So let us fix $x \in X$ and let $\{z^j\}$ be local complex coordinates around x. Let $\bar{v}_j := \frac{\partial}{\partial \bar{z}^j} + \varphi_j^k \frac{\partial}{\partial z^k}$ and $\bar{\epsilon}^j$ be its dual vector. Then Maurer–Cartan equation of φ_t implies

$$[\bar{v}_j, \bar{v}_k] = 0.$$

By the Newlander-Nirenberg theorem, we have complex coordinates $\{\zeta^j\}$ on X_t such that
$$\frac{\partial}{\partial \bar{\zeta}^j} = \bar{v}^j \text{ and } d\bar{\zeta}^j = \bar{\epsilon}^j$$
at the point x. Then at x,
$$P_t(\bar{\partial} + \varphi \lrcorner \partial)f = \left(\frac{\partial f}{\partial \bar{z}^j} + \varphi_j^k \frac{\partial f}{\partial z^k}\right)P_t(d\bar{z}^j).$$
We need to show that $P_t(d\bar{z}^j) = \bar{\epsilon}^j$. We write
$$d\bar{z}^j = c_k^j \bar{\epsilon}^j + d_k^j \epsilon^k.$$
Then
$$c_k^j = d\bar{z}^j(\bar{v}_k) = d\bar{z}^j\left(\frac{\partial}{\partial \bar{z}^j} + \varphi_j^k \frac{\partial}{\partial z^k}\right) = \delta_k^i.$$
Hence $P(d\bar{z}^j) = \bar{\epsilon}^j$. Therefore,
$$P_t(\bar{\partial} + \varphi \lrcorner \partial)f = (\bar{v}_j f)\bar{\epsilon}^j = \frac{\partial f}{\partial \bar{\zeta}^j} d\bar{\zeta}^j = \bar{\partial}_t P_t f$$
at x. Since x is arbitrary, the case $q = 0$ is done.

For $q > 0$. By abusing the notation, we still denote the induced projection $\Omega_X^{0,q} \to \Omega_{X_t}^{0,q}$ by P_t. Let $\alpha = \alpha_J d\bar{z}^J$. Then at the point x,
$$P_t(\bar{\partial} + \varphi_t \lrcorner \partial)\alpha = P_t(\bar{\partial} + \varphi_t \lrcorner \partial)(\alpha_J) \wedge P_t(d\bar{z}^J) = \bar{\partial}_t P\alpha_J \wedge \bar{\epsilon}^J.$$
We need to show that $\bar{\partial}_t(\bar{\epsilon}^j) = 0$ for all j. Since $\{\bar{\epsilon}^j\}$ is a local frame of $(T_{X_t}^{0,1})^*$, $d\bar{\epsilon}^j \in \Omega_{X_t}^{1,1} \oplus \Omega_{X_t}^{0,2}$. Hence, in order to prove $\bar{\partial}_t(\bar{\epsilon}^j) = 0$, it suffices to show that $d\bar{\epsilon}^j(\bar{v}_k, \bar{v}_l) = 0$ for all k, l. This follows from
$$d\bar{\epsilon}^j(\bar{v}_k, \bar{v}_l) = \bar{v}_k \bar{\epsilon}^j(\bar{v}_l) - \bar{v}_l \bar{\epsilon}^j(\bar{v}_k) - \bar{\epsilon}^j([\bar{v}_k, \bar{v}_l]) = 0.$$
This completes the case $E = \mathcal{O}_X$ and $q \geq 0$.

For a general holomorphic vector bundle E, $\bar{D}_t$ is locally given by
$$\bar{D}_t(\alpha^j \otimes e_j) = (\bar{\partial} + \varphi_t \lrcorner \partial)\alpha^j \otimes e_j,$$
where $\{e_j\}$ are local holomorphic frame of E_t over X_t. In this case, P_t is extended by
$$\alpha^j \otimes e_j \mapsto P_t(\alpha^j) \otimes e_j.$$
The required relation follows immediately from the $E = \mathcal{O}_X$ case. $\qquad\square$

3.3. DGLA and the Maurer–Cartan equation.

We are now ready to derive the Maurer–Cartan equation governing the deformations of pairs. Given $A_t \in \Omega^{0,1}(\mathrm{End}(E))$, $\varphi_t \in \Omega^{0,1}(T_X)$ such that the induced differential operator $\bar{D}_t$ satisfies $\bar{D}_t^2 = 0$, we have
$$(\bar{\partial}_E + \varphi_t \lrcorner \nabla + A_t)^2 = \bar{D}_t^2 = 0.$$

Let us expand the left hand side:
$$
\begin{aligned}
(\bar{\partial}_E + \varphi_t \lrcorner \nabla + A_t)^2 =& \bar{\partial}_E(\varphi_t \lrcorner \nabla) + \varphi_t \lrcorner \nabla \bar{\partial}_E + \varphi_t \lrcorner \nabla(\varphi_t \lrcorner \nabla) \\
& + \bar{\partial}_E A_t + A_t \bar{\partial}_E + \varphi_t \lrcorner \nabla A_t + A_t(\varphi_t \lrcorner \nabla) + A_t \wedge A_t \\
=& \bar{\partial}_E(\varphi_t \lrcorner \nabla) + \varphi_t \lrcorner \nabla \bar{\partial}_E + \varphi_t \lrcorner \nabla(\varphi_t \lrcorner \nabla) \\
& + \bar{\partial}_Q A_t + \varphi_t \lrcorner \nabla^Q A_t + A_t \wedge A_t.
\end{aligned}
$$

Applying Lemma 3.6 to the term $\bar{\partial}_E(\varphi_t \lrcorner \nabla)$, we get

$$(\bar{\partial}_E + \varphi_t \lrcorner \nabla + A_t)^2 = \bar{\partial}_{T_X} \varphi_t \lrcorner \nabla + \varphi_t \lrcorner (\bar{\partial}_E \nabla + \nabla \bar{\partial}_E) + \varphi_t \lrcorner \nabla(\varphi_t \lrcorner \nabla)$$
$$+ \bar{\partial}_Q A_t + \varphi_t \lrcorner \nabla^Q A_t + A_t \wedge A_t.$$

Since ∇ is the Chern connection, we have $F_\nabla = \bar{\partial}_E \nabla + \nabla \bar{\partial}_E$, and so

$$(\bar{\partial}_E + \varphi_t \lrcorner \nabla + A_t)^2 = (\bar{\partial}_{T_X} \varphi_t \lrcorner \nabla + \varphi_t \lrcorner \nabla(\varphi_t \lrcorner \nabla)) + \bar{\partial}_Q A_t + \varphi_t \lrcorner F_\nabla + \varphi_t \lrcorner \nabla^Q A_t + A_t \wedge A_t.$$

Note that the curvature F_∇ is given by

$$F_\nabla(\varphi, \psi) = \varphi \lrcorner \nabla(\psi \lrcorner \nabla) - (-1)^{|\varphi||\psi|} \psi \lrcorner \nabla(\varphi \lrcorner \nabla) + [\varphi, \psi] \lrcorner \nabla$$

for $\varphi, \psi \in \Omega^{0,*}(T_\mathbb{C} X)$. Hence

$$2\varphi_t \lrcorner \nabla(\varphi_t \lrcorner \nabla) = F_\nabla(\varphi_t, \varphi_t) + [\varphi_t, \varphi_t] \lrcorner \nabla.$$

However, $\varphi_t \in \Omega^{0,1}(T_X)$ and F_∇ is of type-$(1,1)$, we must have $F_\nabla(\varphi_t, \varphi_t) = 0$. Therefore,

$$\varphi_t \lrcorner \nabla(\varphi_t \lrcorner \nabla) = \frac{1}{2}[\varphi_t, \varphi_t] \lrcorner \nabla.$$

As a whole we obtain

$$(\bar{\partial}_E + \varphi_t \lrcorner \nabla + A_t)^2 = \left(\bar{\partial}_{T_X} \varphi_t + \frac{1}{2}[\varphi_t, \varphi_t] \right) \lrcorner \nabla + \bar{\partial}_Q A_t + \varphi_t \lrcorner F_\nabla + \varphi_t \lrcorner \nabla^Q A_t + \frac{1}{2}[A_t, A_t].$$

But since X_t is integrable, φ_t satisfies the Maurer–Cartan equation (1), and so

$$(\bar{\partial}_E + \varphi_t \lrcorner \nabla + A_t)^2 = \bar{\partial}_Q A_t + \varphi_t \lrcorner F_\nabla + \varphi_t \lrcorner \nabla^Q A_t + \frac{1}{2}[A_t, A_t].$$

Hence we conclude that $\bar{D}_t^2 = 0$ is equivalent to the following two equations

(2)
$$\begin{cases} \bar{\partial}_Q A_t + \varphi_t \lrcorner F_\nabla + \varphi_t \lrcorner \nabla^Q A_t + \frac{1}{2}[A_t, A_t] = 0, \\ \bar{\partial}_{T_X} \varphi_t + \frac{1}{2}[\varphi_t, \varphi_t] = 0. \end{cases}$$

Recall that $A(E) = Q \oplus T_X$ as smooth vector bundle. Define a bracket $[-, -] : \Omega^{0,p}(A(E)) \times \Omega^{0,q}(A(E)) \to \Omega^{0,p+q}(A(E))$ by

$$[(A, \varphi), (B, \psi)] := (\varphi \lrcorner \nabla^Q B - (-1)^{pq} \psi \lrcorner \nabla^Q A + [A, B], [\varphi, \psi]).$$

The following proposition can be proven by straightforward, but tedious, computations which we omit:

Proposition 3.14. *The bracket* $[-, -] : \Omega^{0,p}(A(E)) \times \Omega^{0,q}(A(E)) \to \Omega^{0,p+q}(A(E))$ *defined by*

$$[(A, \varphi), (B, \psi)] := (\varphi \lrcorner \nabla^Q B - (-1)^{pq} \psi \lrcorner \nabla^Q A + [A, B], [\varphi, \psi])$$

satisfies

    (1) $[(A, \varphi), (B, \psi)] = -(-1)^{pq}[(B, \psi), (A, \varphi)]$,
    (2) $[(A, \varphi), [(B, \psi), (C, \tau)]] = [[(A, \varphi), (B, \psi)], (C, \tau)] + (-1)^{pq}[(B, \psi), [(A, \varphi), (C, \tau)]]$,
for $(A, \varphi) \in \Omega^{0,p}(A(E))$, $(B, \psi) \in \Omega^{0,q}(A(E))$ *and* $(C, \tau) \in \Omega^{0,r}(A(E))$.

We also recall the differential operator $\bar{\partial}_{A(E)}$ defined in Section 2.

Again by direct computations, one can prove that the bracket $[-, -]$ and the Dolbeault operator $\bar{\partial}_{A(E)}$ are compatible with each other:

Proposition 3.15. *We have*

$$\bar{\partial}_{A(E)}[(A,\varphi),(B,\psi)] = [\bar{\partial}_{A(E)}(A,\varphi),(B,\psi)] + (-1)^p[(A,\varphi),\bar{\partial}_{A(E)}(B,\psi)]$$

for $(A,\varphi) \in \Omega^{0,p}(A(E))$ and $(B,\psi) \in \Omega^{0,\bullet}(A(E))$.

Propositions 3.14 and 3.15 together say that $(\Omega^{0,\bullet}(A(E)), \bar{\partial}_{A(E)}, [-,-])$ defines a differential graded Lie algebra (DGLA).

Remark 3.16. *In the appendix, we will prove that there exists a natural isomorphism between the complex $(\Omega^{0,\bullet}(A(E)), \bar{\partial}_{A(E)})$ and the one obtained using algebraic methods [21, 16] intertwining our bracket $[-,-]$ with the algebraic one. This gives alternative proofs of Propositions 3.14 and 3.15, and shows that our DGLA is naturally isomorphic to the one derived using algebraic methods. In particular, the isomorphism class of our DGLA is independent of the choice of the hermitian metric we used to define the Chern connection ∇.*

Using the bracket $[-,-]$ and the Dolbeault operator $\bar{\partial}_{A(E)}$, we can now rewrite the two equations (2) as the following Maurer–Cartan equation:

$$\bar{\partial}_{A(E)}(A_t,\varphi_t) + \frac{1}{2}[(A_t,\varphi_t),(A_t,\varphi_t)] = 0,$$

which governs the deformation of pairs. We summarize our results by the following

Theorem 3.17. *Given a holomorphic pair (X,E) and a smooth family of elements $\{(A_t,\varphi_t)\}_{t\in\Delta} \subset \Omega^{0,1}(A(E))$. Then (A_t,φ_t) defines a holomorphic pair (X_t,E_t) (namely, an integrable complex structure J_t on X together with a holomorphic bundle structure on E over (X,J_t)) if and only if the Maurer–Cartan equation*

$$(3) \qquad\qquad \bar{\partial}_{A(E)}(A_t,\varphi_t) + \frac{1}{2}[(A_t,\varphi_t),(A_t,\varphi_t)] = 0$$

is satisfied.

4. First order deformations

The Maurer–Cartan equation (3) implies that a first order deformation (A_1,φ_1) (the linear term of the Taylor series expansion of a family (A_t,φ_t)) is $\bar{\partial}_{A(E)}$-closed:

$$\bar{\partial}_{A(E)}(A_1,\varphi_1) = 0,$$

and hence defines a cohomology class in the Dolbeault cohomology group $H^{0,1}_{\bar{\partial}_{A(E)}} \cong H^1(X,A(E))$. To determine the space of first order deformations of a holomorphic pair (X,E), it remains to identify isomorphic deformations.

Definition 4.1. *Two deformations $\mathcal{E} \to \mathcal{X}$, $\mathcal{E}' \to \mathcal{X}'$ of (X,E) are said to be isomorphic if there exists a biholomorphism $F : \mathcal{X} \to \mathcal{X}'$ and a holomorphic bundle isomorphism $\Phi : \mathcal{E} \to \mathcal{E}'$ covering F such that $F|_X = id_X$ and $\Phi|_E = id_E$.*

Proposition 4.2. *Suppose $\mathcal{E} \to \mathcal{X}$ and $\mathcal{E}' \to \mathcal{X}'$ are isomorphic 1-real parameter family of deformations of (X,E). If we denote by (A_t,φ_t) and (A'_t,φ'_t) the elements that represent the families $\mathcal{E} \to \mathcal{X}$ and $\mathcal{E}' \to \mathcal{X}'$ respectively, then there exists $(\Theta_1,v) \in \Omega^0(A(E))$ such that*

$$(A'_t,\varphi'_t) = (A_t,\varphi_t) + t\bar{\partial}_{A(E)}(-\Theta_1,v) + R((A_t,\varphi_t),t(\Theta_1,v)),$$

where the error R depends smoothly on $t, A(t), \varphi(t), \Theta_1, v$ and first partial derivatives of Θ, v. Moreover, R is of order s^2 in the sense that

$$R(s(A, \varphi), s(\Theta, v)) = s^2 R_1((A, \varphi), (\Theta, v), s),$$

for some map R_1 which depends smoothly (with respect to the Sobolev norm; see Section 6 for its precise definition) in $s, (A, \varphi) \in \Omega^{0,1}(A(E))$ and $(\Theta, v) \in \Omega^0(A(E))$.

Proof. As before, let $v \in \Omega^0(T_X)$ be the vector field which generates the 1-parameter family of diffeomorphisms $F_t : X \to X$ of the underlying smooth manifold X. Since

$$dF_t(\text{Graph}(\varphi_t : T_X^{0,1} \to T_X^{1,0})) = \text{Graph}(\varphi'_t : T_X^{0,1} \to T_X^{1,0}),$$

we already have

$$\varphi'_t = \varphi_t + t\bar{\partial}_{T_X} v + R(\varphi_t, tv).$$

Hence it remains to show that

$$A'_t = A_t + t(\bar{\partial}_Q(-\Theta_1) - v \lrcorner F_\nabla) + R((A_t, \varphi_t), t(\Theta_1, v)),$$

for some $\Theta_1 \in \Omega^0(Q)$.

We define an endomorphism of E as follows: Fix $p \in X$ and the fiber E_p of E. Let $P_{\gamma_p(t)} : E_p \to E_{F_t(p)}$ be the parallel transport along $t \longmapsto \gamma_p(t) := F_t(p)$. Define $\Theta_t := P_{\gamma_p(t)}^{-1} \Phi_t : E_p \to E_p$. Then Θ_t defines a bundle endomorphism of E. Let us write

$$\Theta_t = \Theta_0 + t\Theta_1 + O(t^2),$$
$$A'_t = A'_0 + tA'_1 + O(t^2),$$

Since $\Phi_0 = \text{id}_E$, we have $\Theta_0 = \text{id}_E$ and $A'_0 = 0$. We need to compute A'_1.

Now let e_t be a local holomorphic section of $E_t \subset \mathcal{E}$. Since Φ_t is holomorphic, $\Phi_t(e_t)$ is a holomorphic section of $E'_t \subset \mathcal{E}'$, so that $\bar{D}'_t \Phi_t(e_t) = 0$, i.e.

$$\bar{\partial}_E \Phi_t(e_t) = -\varphi'(t) \lrcorner \nabla \Phi_t(e_t) - A'_t \Phi_t(e_t).$$

We want to compute the first derivatives of both sides of this equation with respective to t at $t = 0$.

First note that $\Phi_t = P_{\gamma(t)} \Theta_t$, so we have

$$\frac{\partial}{\partial t} \Phi_t(e_t)|_{t=0} = -v \lrcorner \nabla e + \Theta_1(e) + \frac{\partial}{\partial t} e_t|_{t=0},$$

where $e = e_0$. Hence

$$\frac{\partial}{\partial t} \bar{\partial}_E \Phi_t(e_t)|_{t=0} = \bar{\partial}_E \left(\frac{\partial}{\partial t} \Phi_t(e_t)|_{t=0} \right)$$

$$= -\bar{\partial}_E(v \lrcorner \nabla e) + \bar{\partial}_E(\Theta_1(e)) + \bar{\partial}_E \left(\frac{\partial}{\partial t} e_t|_{t=0} \right).$$

For the term, $-\varphi'_t \lrcorner \nabla \Phi_t(e_t)$, we have

$$\frac{\partial}{\partial t}(-\varphi'_t \lrcorner \nabla \Phi_t(e_t))|_{t=0} = -\varphi'_1 \lrcorner \nabla e - \bar{\partial}_{T_X} v \lrcorner \nabla e$$

$$= -\varphi'_1 \lrcorner \nabla e - \bar{\partial}_E(v \lrcorner \nabla e) - v \lrcorner \bar{\partial}_E \nabla e.$$

Since $e = e_0$ is holomorphic with respective to $\mathcal{E}|_{\pi^{-1}(0)}$, we have

$$\bar{\partial}_E \nabla e = F_\nabla(e).$$

Moreover, since e_t is holomorphic with respective to E_t, we have $\bar{D}_t e_t = 0$, that is,

$$\bar{\partial}_E e_t = -\varphi_t \lrcorner \nabla e_t - A_t e_t.$$

Differentiate with respective to t and set $t = 0$, we obtain

$$-\varphi_1 \lrcorner \nabla e = \bar{\partial}_E \left(\frac{\partial}{\partial t} e(t)|_{t=0} \right) + A_1 e.$$

Hence

$$\frac{\partial}{\partial t}(-\varphi'_t \lrcorner \nabla \Phi_t(e_t))|_{t=0} = \bar{\partial}_E \left(\frac{\partial}{\partial t} e_t|_{t=0} \right) - \bar{\partial}_E(v \lrcorner \nabla e) - v \lrcorner F_\nabla e + A_1 e.$$

For the term $-A'_t \Phi_t(e_t)$, we have

$$\frac{\partial}{\partial t}(-A'_t \Phi_t(e_t))|_{t=0} = -A'_1 e.$$

As a whole we obtain the formula

$$\bar{\partial}_E(\Theta_1(e)) = -v \lrcorner F_\nabla(e) + (A_1 - A'_1)(e).$$

Since e is holomorphic with respective to $\mathcal{E}|_{\pi^{-1}(0)}$, $\bar{\partial}_E(\Theta_1(e)) = (\bar{\partial}_Q \Theta_1)(e)$, so that

$$A'_1 = A_1 + \bar{\partial}_Q(-\Theta_1) - v \lrcorner F_\nabla.$$

Therefore we have

$$\frac{\partial}{\partial t}(A'_t - A_t)|_{t=0} = A'_1 - A_1 = \bar{\partial}_Q(-\Theta_1) - v \lrcorner F_\nabla,$$

or in other words,

$$A'_t = A_t + t(\bar{\partial}_Q(-\Theta_1) - v \lrcorner F_\nabla) + O(t^2).$$

Since A'_t is completely determined by (A_t, φ_t) and $t(\Theta_1, v)$, we have

$$A'_t = A_t + t(\bar{\partial}_Q(-\Theta_1) - v \lrcorner F_\nabla) + R((A_t, \varphi_t), t(\Theta_1, v)),$$

where R is of order t^2 and depends smoothly on $t, A_t, \varphi_t, \Theta_1, v$. Moreover, since the equation

$$\bar{\partial}_E \Phi_t(e_t) = -\varphi'_t \lrcorner \nabla \Phi_t(e_t) - A'_t \Phi_t(e_t)$$

depends smoothly on first order partial derivatives of Θ_1 and v, we see that the error R also depends smoothly on first order partial derivatives of Θ_1 and v.

Finally, R satisfies

$$R(s(A, \varphi), st(\Theta, v)) = s^2 R_1((A, \varphi), (\Theta_1, v), s),$$

for some map R_1 which depends smoothly in $s, (A, \varphi) \in \Omega^{0,1}(A(E))$ and $(\Theta, v) \in \Omega^0(A(E))$. This follows from the fact that

$$R((A, \varphi), (\Theta, v)) \to 0 \quad \text{as } (\Theta, v) \to 0, \text{ and}$$
$$R((A, \varphi), (\Theta, v)) \to R((\Theta, v)) \quad \text{as } (A, \varphi) \to 0,$$

with $R(s(\Theta, v)) = s^2 R((\Theta, v))$. $\qquad \square$

Corollary 4.3. *If $\mathcal{E} \to \mathcal{X}$ and $\mathcal{E}' \to \mathcal{X}'$ are isomorphic deformations of (X, E), then the first order terms (A_1, φ_1) and (A'_1, φ'_1) of the corresponding families (A_t, φ_t) and (A'_t, φ'_t) respectively differ by an $\bar{\partial}_{A(E)}$-exact form.*

Proof. We have $A'_1 - A_1 = \bar{\partial}_Q(-\Theta_1) - v \lrcorner F_\nabla$ and $\varphi'_1 - \varphi_1 = \bar{\partial}_{T_X} v$, whence

$$(A'_1, \varphi'_1) - (A_1, \varphi_1) = \bar{\partial}_{A(E)}(-\Theta_1, v).$$

$\qquad \square$

We conclude that the space of first order deformations of a holomorphic pair (X, E) is precisely given by the Dolbeault cohomology group $H^{0,1}_{\bar{\partial}_{A(E)}} \cong H^1(X, A(E))$

5. Obstructions and Kuranishi family

Now given a first order deformation $[(A_1, \varphi_1)] \in H^{0,1}_{\bar{\partial}_{A(E)}} \cong H^1(X, A(E))$, it is standard in deformation theory to ask whether one can find a family (A_t, φ_t) integrating (A_1, φ_1) to give an actual family of deformations. To study this problem, we use a method due to Kuranishi [14].

We need to review several operators commonly used in Hodge theory. We first choose a hermitian metric g on X and h on $A(E)$, so that we can define a hermitian product $(\cdot, \cdot)$ on $\Omega^{0,\bullet}(A(E))$. Define the formal adjoint of $\bar{\partial}_{A(E)}$ with respective to $(\cdot, \cdot)$ by

$$\left(\bar{\partial}_{A(E)}\alpha, \beta\right) = \left(\alpha, \bar{\partial}^*_{A(E)}\beta\right).$$

Then the *Laplacian* is defined by

$$\Delta_{A(E)} := \bar{\partial}_{A(E)}\bar{\partial}^*_{A(E)} + \bar{\partial}^*_{A(E)}\bar{\partial}_{A(E)}.$$

This is an elliptic self-adjoint operator and thus has a finite dimensional kernel $\mathbb{H}^p(X, A(E))$, consisting of *harmonic forms*. We have the standard isomorphism from Hodge theory:

$$H^p(X, A(E)) \cong H^{0,p}_{\bar{\partial}_{A(E)}} \cong \mathbb{H}^p(X, A(E)).$$

Take a completion of $\Omega^{0,\bullet}(A(E))$ with respective to $(\cdot, \cdot)$ to get a Hilbert space L^*, and let $H : L^* \to \mathbb{H}^*(X, A(E))$ be the harmonic projection. The Green's operator $G : L^* \to L^*$ is defined by

$$I = H + \Delta_{A(E)}G = H + G\Delta_{A(E)}.$$

It commutes with $\bar{\partial}_{A(E)}$ and $\bar{\partial}^*_{A(E)}$.

Now let $\eta_1, \ldots, \eta_m \in \mathbb{H}^1(X, A(E))$ be a basis and $\epsilon_1(t) := \sum_{j=1}^m t_j \eta_j \in \mathbb{H}^1(X, A(E))$. Consider the equation

$$\epsilon(t) = \epsilon_1(t) - \frac{1}{2}\bar{\partial}^*_{A(E)}G[\epsilon(t), \epsilon(t)].$$

We denote the Hölder norm by $\|\cdot\|_{k,\alpha}$. The following estimates are obvious:

$$\|\bar{\partial}^*_{A(E)}\epsilon\|_{k,\alpha} \leq C_1\|\epsilon\|_{k+1,\alpha}$$
$$\|[\epsilon, \delta]\|_{k,\alpha} \leq C_2\|\epsilon\|_{k+1,\alpha}\|\delta\|_{k+1,\alpha}.$$

In [2], Douglis and Nirenberg proved the following nontrivial a priori estimate:

$$\|\epsilon\|_{k,\alpha} \leq C_3(\|\Delta_{A(E)}\epsilon\|_{k-2,\alpha} + \|\epsilon\|_{0,\alpha}).$$

Applying these and following the proof of [11, Chapter 4, Proposition 2.3], one can deduce an estimate for the Green's operator G:

$$\|G\epsilon\|_{k,\alpha} \leq C_4\|\epsilon\|_{k-2,\alpha},$$

where all C_i's are positive constants which depend only on k and α.

Then by the same argument as in [11, Chapter 4, Proposition 2.4], or alternatively using an implicit function theorem for Banach spaces [14], we obtain a unique solution $\epsilon(t)$ which satisfies the equation

$$\epsilon(t) = \epsilon_1(t) - \frac{1}{2}\bar{\partial}^*_{A(E)}G[\epsilon(t), \epsilon(t)],$$

and is analytic in the variable t. Note that the solution $\epsilon(t)$ is always smooth. Indeed, by applying the Laplacian to the above equation, we get

$$\Delta_{A(E)}\epsilon(t) + \frac{1}{2}\bar{\partial}^*_{A(E)}[\epsilon(t), \epsilon(t)] = 0.$$

Also, the solution $\epsilon(t)$ is holomorphic in t, so we have

$$\sum_j \frac{\partial^2 \epsilon(t)}{\partial t_j \partial \bar{t}_j} = 0.$$

Now since the operator

$$\Delta_{A(E)} + \sum_j \frac{\partial^2}{\partial t_j \partial \bar{t}_j}$$

is elliptic, we see that $\epsilon(t)$ is smooth by elliptic regularity.

Following Kuranishi [14] (see also [11, Chapter 4]), we have the following

Proposition 5.1. *The solution $\epsilon(t)$ that satisfies*

$$\epsilon(t) = \epsilon_1(t) - \frac{1}{2}\bar{\partial}^*_{A(E)}G[\epsilon(t), \epsilon(t)]$$

solves the Maurer–Cartan equation if and only if $H[\epsilon(t), \epsilon(t)] = 0$, where H is the harmonic projection.

Proof. Suppose the Maurer–Cartan equation holds. Then

$$H[\epsilon(t), \epsilon(t)] = 2H\bar{\partial}_{A(E)}\epsilon(t) = 0.$$

Conversely, suppose that $H[\epsilon(t), \epsilon(t)] = 0$. We must show that

$$\delta(t) := \bar{\partial}_{A(E)}\epsilon(t) + \frac{1}{2}[\epsilon(t), \epsilon(t)] = 0.$$

Recall that $\epsilon(t)$ is a solution to

$$\epsilon(t) = \epsilon_1(t) - \frac{1}{2}\bar{\partial}^*_{A(E)}G[\epsilon(t), \epsilon(t)]$$

and $\epsilon_1(t)$ is $\bar{\partial}_{A(E)}$-closed. By applying $\bar{\partial}_{A(E)}$ to this equation, we get

$$\bar{\partial}_{A(E)}\epsilon(t) = -\frac{1}{2}\bar{\partial}_{A(E)}\bar{\partial}^*_{A(E)}G[\epsilon(t), \epsilon(t)].$$

Hence

$$2\delta(t) = \bar{\partial}_{A(E)}\bar{\partial}^*_{A(E)}G[\epsilon(t), \epsilon(t)] - [\epsilon(t), \epsilon(t)].$$

Using the Hodge decomposition on forms, we can write

$$[\epsilon(t), \epsilon(t)] = H[\epsilon(t), \epsilon(t)] + \Delta_{A(E)}G[\epsilon(t), \epsilon(t)] = \Delta_{A(E)}G[\epsilon(t), \epsilon(t)].$$

Therefore, we have

$$2\delta(t) = (\Delta_{A(E)}G - \bar{\partial}_{A(E)}\bar{\partial}^*_{A(E)}G)[\epsilon(t), \epsilon(t)] = \bar{\partial}^*_{A(E)}\bar{\partial}_{A(E)}G[\epsilon(t), \epsilon(t)] = 2\bar{\partial}^*_{A(E)}G[\bar{\partial}_{A(E)}\epsilon(t), \epsilon(t)],$$

and hence

$$\delta(t) = \bar{\partial}^*_{A(E)}G[\bar{\partial}_{A(E)}\epsilon(t), \epsilon(t)] = \bar{\partial}^*_{A(E)}G[\delta(t) - \frac{1}{2}[\epsilon(t), \epsilon(t)], \epsilon(t)] = \bar{\partial}^*_{A(E)}G[\delta(t), \epsilon(t)],$$

where we have used the Jacobi identity in the last equality. Using the estimate

$$\|[\xi, \eta]\|_{k,\alpha} \leq C_{k,\alpha}\|\xi\|_{k+1,\alpha}\|\eta\|_{k+1,\alpha},$$

we get

$$\|\delta(t)\|_{k,\alpha} \leq C_{k,\alpha}\|\delta(t)\|_{k,\alpha}\|\epsilon(t)\|_{k,\alpha}.$$

By choosing $|t|$ to be small enough so that $C_{k,\alpha}\|\epsilon(t)\|_{k,\alpha} < 1$, we must have $\delta(t) = 0$ for all $|t|$ small enough. This finishes the proof. $\square$

In the case when $H[\epsilon(t), \epsilon(t)]$ vanishes identically (which always holds if $H^2(X, A(E)) = 0$), we have the following

Corollary 5.2. *If $H[\epsilon(t), \epsilon(t)] = 0$ for all t, then we have a complex analytic family $\mathcal{E} \to \mathcal{X}$.*

Proof. If $H[\epsilon(t), \epsilon(t)] = 0$ for all t, then $\epsilon(t) = (A_t, \varphi_t)$ satisfies the Maurer–Cartan equation and so (X_t, E_t) is holomorphic for each t. In particular, we obtain a deformation $\mathcal{X}$ of X. Let $\mathcal{E} := \Delta \times E$. A smooth section $\sigma : \mathcal{X} \to \mathcal{E}$ of $\mathcal{E}$ on $\mathcal{X}$ can be written as

$$\sigma : (t, x) \longmapsto (t, s(x, t)),$$

for some smooth map $s : \mathcal{X} \to E$. We define a Dolbeault operator $\bar{\partial}_{\mathcal{E}} : \Omega^0_{\mathcal{X}}(\mathcal{E}) \to \Omega^{0,1}_{\mathcal{X}}(\mathcal{E})$ on $\mathcal{E}$ by

$$\bar{\partial}_{\mathcal{E}}\sigma(t, x) = (t, \bar{\partial}_{E_t}s(t, x)).$$

Note that $\bar{\partial}_{\mathcal{E}}$ is well-defined for, if $\{e_k(t, x)\}$ are local holomorphic frame of $E_t \to X_t$, then we can write

$$\bar{\partial}_{\mathcal{E}}\sigma(t, x) = (t, \bar{\partial}_{\mathcal{E}}(s^k(t, x)e_k(t, x)) = \bar{\partial}_t s^k(t, x) \otimes e_k(t, x)),$$

which is a smooth section of $\Omega^{0,1}_{\mathcal{X}}(\mathcal{E})$. Clearly, $\bar{\partial}^2_{\mathcal{E}} = 0$ and hence $\mathcal{E}$ is a holomorphic vector bundle over $\mathcal{X}$. $\square$

In general, the condition $H^2(X, A(E)) = 0$ may not be satisfied. But we can define the (singular) analytic space

$$\mathcal{S} := \{t \in \Delta : H[\epsilon(t), \epsilon(t)] = 0\}$$

and form a family $\mathcal{E} \to \mathcal{X}$ over $\mathcal{S}$, which is called the *Kuranishi family* of (X, E). In particular, we see that the obstruction space is precisely given by the Dolbeault cohomology group $H^{0,2}_{\bar{\partial}_{A(E)}} \cong H^2(X, A(E))$, and the obstructions to deformations of a holomorphic pair (X, E) is captured by the Kuranishi map

$$Ob_{(X,E)} : U \subset H^1(X, A(E)) \to H^2(X, A(E)), \ \sum_{i=1}^{m} t_j \eta_j \mapsto H[\epsilon(t), \epsilon(t)],$$

where U is a small open subset around the origin $0 \in H^1(X, A(E))$ whose diameter is less than twice of the radius of convergence of $\epsilon(t)$.

6. A PROOF OF COMPLETENESS

The goal of this section is to give a proof of the local completeness of a Kuranishi family for the deformation of the pair (X, E). Existence of a locally complete (or versal) family for deformations of pairs was first proved by Siu-Trautmann [22]. Here we give another proof using Kuranishi's method.

Definition 6.1. *A family $\mathcal{E} \to \mathcal{X}$ over an analytic space $\mathcal{S}$ is said to be* locally complete (or versal) *if for any family $\mathcal{E}' \to \mathcal{X}'$ over a sufficiently small ball Δ, there exists an analytic map $f : \Delta \to \mathcal{S}$ such that the family $\mathcal{E}' \to \mathcal{X}'$ is the pull-back of $\mathcal{E} \to \mathcal{X}$ via f.*

Recall that for given $\epsilon_1(t) \in \mathbb{H}^1(X, A(E))$, we have existence of solutions $\epsilon(t)$ to

$$\epsilon(t) = \epsilon_1(t) - \frac{1}{2}\bar{\partial}^*_{A(E)}G[\epsilon(t), \epsilon(t)]$$

and $\epsilon(t)$ satisfies the Maurer–Cartan equation if and only if $H[\epsilon(t), \epsilon(t)] = 0$. We then obtain an analytic family $\mathcal{E} \to \mathcal{X}$ over

$$\mathcal{S} := \{t \in \Delta : H[\epsilon(t), \epsilon(t)] = 0\}.$$

The main theorem is as follows:

Theorem 6.2. *The Kuranishi family $\mathcal{E} \to \mathcal{X}$ over $\mathcal{S}$ is locally complete.*

Before going into the details of the proof, we first introduce the Sobolev norm: One can endow $A(E)$ a hermitian metric H, induced from that of E and X, and define the inner product

$$(\alpha, \beta)_k := \sum_{|I| \leq k} \int_X H(D^I\alpha, D^I\beta),$$

$\alpha, \beta \in \Omega^{0,*}(A(E))$. The Sobolev norm is defined by

$$|\alpha| := (\alpha, \alpha)^{\frac{1}{2}}.$$

One has the estimate

$$|[\alpha, \beta]|_k \leq C_k|\alpha|_{k+1}|\beta|_{k+1},$$

for some constant $C_k > 0$.

We take a completion of $\Omega^{0,\bullet}(A(E))$ with respective to $(\cdot, \cdot)_k$ to get a Hilbert space L^*_k. The harmonic projection $H : L^\bullet_k \to \mathbb{H}^\bullet(X, A(E))$ and the Green's operator $G : L^\bullet_k \to L^\bullet_{k+2}$ satisfy the estimates

$$|H\alpha|_k \leq C_k|\alpha|_k,$$

$$|\bar{\partial}^*_{A(E)}G\alpha|_k \leq C_k|\alpha|_{k-1}.$$

The following lemma will be useful in the proof of the completeness theorem.

Lemma 6.3. *For fixed $\epsilon_1(t) \in \mathbb{H}^1(X, A(E))$, $t \in \mathcal{S}$, the equation*

$$\epsilon(t) = \epsilon_1(t) - \frac{1}{2}\bar{\partial}^*_{A(E)}G[\epsilon(t), \epsilon(t)]$$

has only one small solution.

Proof. Suppose ϵ is another solution. Let $\delta := \epsilon - \epsilon(t)$. Then

$$\delta = -\frac{1}{2}\bar{\partial}^*_{A(E)}G([\epsilon, \epsilon] - [\epsilon(t), \epsilon(t)])$$

$$= -\frac{1}{2}\bar{\partial}^*_{A(E)}G([\delta, \epsilon(t)] + [\epsilon(t), \delta] + [\delta, \delta])$$

$$= -\frac{1}{2}\bar{\partial}^*_{A(E)}G(2[\delta, \epsilon(t)] + [\delta, \delta]).$$

Hence

$$|\delta|_k \leq C_k(|\delta|_k|\epsilon(t)|_k + |\delta|^2_k) \leq C_k|\delta|_k(|\epsilon(t)|_k + |\delta|_k).$$

For $|\epsilon(t)|_k$ and $|\epsilon|_k$ small, we can only have $|\delta|_k = 0$. $\qquad\square$

We are now ready to prove the local completeness theorem.

Proof of Theorem 6.2. Let $\mathcal{E}' \to \mathcal{X}'$ be a deformation of (X, E). Let ϵ', be the element representing this deformation. We first prove that if $\bar{\partial}^*_{A(E)}\epsilon' = 0$, then there exists $t \in \mathcal{S}$ such that $\epsilon' = \epsilon(t)$.

Note that ϵ' satisfies the Maurer–Cartan equation:

$$\bar{\partial}_{A(E)}\epsilon' + \frac{1}{2}[\epsilon', \epsilon'] = 0.$$

Applying $\bar{\partial}^*_{A(E)}$, we get

$$\bar{\partial}^*_{A(E)}\bar{\partial}_{A(E)}\epsilon' + \frac{1}{2}\bar{\partial}^*_{A(E)}[\epsilon', \epsilon'] = 0.$$

Since $\bar{\partial}^*_{A(E)}\epsilon' = 0$, we have

$$\Delta_{A(E)}\epsilon' + \frac{1}{2}\bar{\partial}^*_{A(E)}[\epsilon', \epsilon'] = 0.$$

Then using $I = H + G\Delta_{A(E)}$, we get

$$\epsilon' = H\epsilon' - \frac{1}{2}\bar{\partial}^*_{A(E)}G[\epsilon', \epsilon'].$$

Note that $H\epsilon' \in \mathbb{H}^1(X, A(E))$ and by the estimate $|H\epsilon'|_k \leq C_k|\epsilon'|$, we see that $|H\epsilon'|_k$ is small if $|\epsilon'|_k$ is small. Hence $H\epsilon' = \epsilon_1(t)$ for some $t \in \mathcal{S}$. Therefore, if the ball Δ is small enough, ϵ' is a solution to

$$\epsilon' = \epsilon_1(t) - \frac{1}{2}\bar{\partial}^*_{A(E)}G[\epsilon', \epsilon'].$$

Therefore, $\epsilon' = \epsilon(t)$ for some $t \in \mathcal{S}$ by Lemma 6.3.

Now we prove that for any given small deformation $\mathcal{E}' \to \mathcal{X}'$, one can find an isomorphic deformation $\mathcal{E}'' \to \mathcal{X}''$ such that the element ϵ'' which represents the family $\mathcal{E}'' \to \mathcal{X}''$ is $\bar{\partial}^*_{A(E)}$-closed. This will prove the local completeness. Indeed, we will prove the following: Given a deformation ϵ', there exists $\eta \in Im(\bar{\partial}^*_{A(E)}) \subset \Omega^0(A(E))$ such that the element ϵ'', which represents the deformation $\mathcal{E}'' \to \mathcal{X}''$, is $\bar{\partial}^*_{A(E)}$-closed.

Let $\eta = (\Theta, v) \in \Omega^0(A(E))$, then the elements ϵ', ϵ'', which represent the deformations $\mathcal{E}' \to \mathcal{X}'$ and $\mathcal{E}'' \to \mathcal{X}''$ respectively, satisfy

$$\epsilon'' = \epsilon' + \bar{\partial}_{A(E)}\eta + R(\epsilon', \eta),$$

where the error term R satisfies $R(s\epsilon', s\eta) = s^2 R_1(\epsilon', \eta, s)$ as in Proposition 4.2. Hence $\bar{\partial}^*_{A(E)}\epsilon'' = 0$ if and only if

$$\bar{\partial}^*_{A(E)}\epsilon' + \bar{\partial}^*_{A(E)}\bar{\partial}_{A(E)}\eta + \bar{\partial}^*_{A(E)}R(\epsilon', \eta) = 0.$$

If $\eta \in Im(\bar{\partial}^*_{A(E)})$, then

$$\Delta_{A(E)}(\eta) + \bar{\partial}^*_{A(E)}R(\epsilon'(s), \eta) + \bar{\partial}^*_{A(E)}\epsilon' = 0.$$

Applying G, we get

$$\eta + \bar{\partial}^*_{A(E)}GR(\epsilon', \eta) + \bar{\partial}^*_{A(E)}G\epsilon' = 0.$$

Let $U_1 \subset L^1_k$ and $V_1 \subset L^0_k$ be neighborhoods around 0 such that $R(\epsilon', \eta)$ is defined. Define $F : U_1 \times V_1 \to L^0_k$ by

$$F(\epsilon', \eta) := \eta + \bar{\partial}^*_{A(E)}GR(\epsilon', \eta) + \bar{\partial}^*_{A(E)}G\epsilon'.$$

By the order condition on the error term R, the derivative of F with respective to η at $(0,0)$ is the identity map. Hence by the implicit function theorem, there is a C^∞ function g such that $F(\epsilon', \eta) = 0$ if and only if $\eta = g(\epsilon')$. By the error condition again, the (second order) operator $|\bar{\partial}^*_{A(E)} R(\epsilon', -)|_k$ is small if $|\epsilon'|_k$ is small. Hence

$$\Delta_{A(E)} + \bar{\partial}^*_{A(E)} R(\epsilon', -) + \bar{\partial}^*_{A(E)} \epsilon'$$

is still a quasi-linear elliptic operator. By elliptic regularity, η is smooth. This completes our proof. $\square$

7. UNOBSTRUCTED DEFORMATIONS

In this section, we investigate various circumstances under which deformations of holomorphic pairs are unobstructed.

To begin with, note that we have an exact sequence of holomorphic vector bundles

$$0 \longrightarrow \operatorname{End}(E) \longrightarrow A(E) \longrightarrow T_X \longrightarrow 0$$

by the construction of $A(E)$ (which shows that $A(E)$ is an extension of $Q = \operatorname{End}(E)$ by T_X). This induces a long exact sequence in cohomology groups:

$$\cdots \longrightarrow H^1(X, Q) \longrightarrow H^1(X, A(E)) \longrightarrow H^1(X, T_X) \longrightarrow$$
$$\longrightarrow H^2(X, Q) \longrightarrow H^2(X, A(E)) \longrightarrow H^2(X, T_X) \longrightarrow \cdots,$$

and the first order term (A_1, φ_1) defines a class $[(A_1, \varphi_1)] \in H^1(X, A(E))$.

The following proposition, which first appeared in [7, Appendix] without proof, describes the relations between the deformations of a pair (X, E) and that of X and E.

Proposition 7.1. *Denote the Kuranishi obstruction maps of the deformation theories of X, E and (X, E) by Ob_X, Ob_E and $Ob_{(X,E)}$ respectively. Then, wherever the obstruction maps are defined, we have the following commutative diagram:*

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & H^1(X, Q) & \xrightarrow{\iota^*} & H^1(X, A(E)) & \xrightarrow{\pi^*} & H^1(X, T_X) & \xrightarrow{\delta} \cdots \\
& \Big\downarrow{\scriptstyle Ob_E} & & \Big\downarrow{\scriptstyle Ob_{(X,E)}} & & \Big\downarrow{\scriptstyle Ob_X} & \\
\cdots \longrightarrow & H^2(X, Q) & \xrightarrow{\iota^*} & H^2(X, A(E)) & \xrightarrow{\pi^*} & H^2(X, T_X) & \xrightarrow{\delta} \cdots
\end{array}
$$

Here, the connecting homomorphism δ is given by contracting with the Atiyah class:

$$\delta(\varphi) = \varphi \lrcorner [F_\nabla].$$

Proof. By definition,

$$\iota^*([A]) = [(A, 0)], \quad \pi^*([(A, \varphi)]) = [\varphi].$$

The commutative diagram follows directly from the definitions of the maps Ob_X, Ob_E and $Ob_{(X,E)}$. $\square$

Remark 7.2. *We remark that since $tr[A, A] = 0$ for any $A \in \Omega^{0,1}(Q)$, the obstruction of deforming E (with X fixed) actually lies in $H^2(X, \operatorname{End}_0(E))$, where $\operatorname{End}_0(E) \subset \operatorname{End}(E)$ is the trace-free part of $\operatorname{End}(E)$.*

Remark 7.3. *For any $[(A, \varphi)] \in H^1(X, A(E))$ such that $Ob_{(X,E)}(A, \varphi) = 0$, we have*

$$0 = Ob_X \circ \pi^*[(A, \varphi)] = Ob_X([\varphi]).$$

In this case, the map $(A, \varphi) \mapsto \varphi$ induces a map of Kuranishi slices*, i.e. every deformation of the pair (X, E) induces a deformation of the manifold X.*

An immediate consequence of this proposition is the following slight generalization of a result in [20]:

Proposition 7.4. *Suppose that $Ob_X \circ \pi^* = 0$ and the connecting homomorphism $\delta : H^1(X, T_X) \to H^2(X, Q)$ is surjective, then deformations of the pair (X, E) are unobstructed.*

Proof. Surjectivity of δ implies that the map $\iota^* : H^2(X, Q) \to H^2(X, A(E))$ is a zero map, and hence the map $\pi^* : H^2(X, A(E)) \to H^2(X, T_X)$ is injective. But $\pi^* \circ Ob_{(X,E)} = Ob_X \circ \pi^* = 0$, so we have $Ob_{(X,E)} = 0$. $\qquad\square$

In the case when $E = L$ is a line bundle, we recover the following

Corollary 7.5 ([20], Lemma 2.4)**.** *Let X be a compact complex manifold with unobstructed deformations and L be a holomorphic line bundle over X such that the map*

$$\cup c_1(L) : H^1(X, T_X) \to H^2(X, \mathcal{O}_X)$$

is surjective. Then deformations of the pair (X, L) are unobstructed.

For example, if X is an n-dimensional compact Kähler manifold with trivial canonical line bundle, then X admits unobstructed deformations. If we further assume that $H^{0,2}(X) = 0$ (e.g. when the holonomy of X is precisely $SU(n)$), then deformations of (X, L) for any line bundle L are unobstructed.

Definition 7.6. *A holomorphic vector bundle E over a compact complex manifold X is said to be* good *if $H^2(X, Q_0) = 0$, where Q_0 is the trace-free part of $Q = End(E)$.*

Proposition 7.7. *Let X be a compact complex surface with trivial canonical line bundle (e.g. a K3-surface), and let E be a good bundle over X with $c_1(E) \neq 0$. Then deformations of the pair (X, E) are unobstructed.*

Proof. By the theorem of Tian and Todorov [23, 24], we have $Ob_X = 0$. Hence the condition $Ob_X \circ \pi^* = 0$ is automatic.

On the other hand, note that $Q^* = (E^* \otimes E)^* = E^* \otimes E = Q$ and similarly $Q_0^* = Q_0$. By Serre duality and the fact that $K_X \cong \mathcal{O}_X$, we have $H^0(X, Q_0) \cong (H^2(X, Q_0))^* = 0$ since E is good. This implies that $H^0(X, Q) \cong H^0(X, \mathcal{O}_X) \cong \mathbb{C}$. Then applying Serre duality again gives

$$H^2(X, Q) \cong (H^0(X, Q^* \otimes K_X))^* \cong (H^0(X, Q))^* \cong \mathbb{C}.$$

In this case, the connecting homomorphism $\delta : H^1(X, T_X) \cong H^{1,1}(X) \to H^2(X, Q) \cong \mathbb{C}$ is simply given by

$$\delta(\varphi) = \int_X \varphi \cup [tr F_\nabla] = -2\pi i \int_X \varphi \cup c_1(E).$$

When $c_1(E) \neq 0$, δ is a nonzero map and hence surjective. Proposition 7.4 then says that any deformation of (X, E) is unobstructed. $\qquad\square$

Appendix A. Comparison with the algebraic approach

The aim of this appendix is to give an explicit comparison between the analytic approach we adopt here and the classical algebraic approach (see the book [21] for the case when $E = L$ is a holomorphic line bundle on X, and the thesis [16] for the general case).

We start with a definition

Definition A.1. *A* differential operator of order 1 *on a vector bundle E is a linear map $P : \Omega^0(E) \to \Omega^0(E)$ such that locally,*

$$P = (g_{ij}) + \sum_k h_{ij}^k \frac{\partial}{\partial x^k},$$

with (g_{ij}) be a matrix with entries in $\mathcal{O}_X(U_\alpha)$ and $h_{ij}^k \in \mathcal{O}_X(U_\alpha)$.

A differential operator of order 1 is said to be with scalar principle symbol *if $h_{ij}^k = h^k \cdot I$.*

In the algebraic approach, the role of the Atiyah extension $A(E)$ is replaced by the sheaf $D^1(E)$ of scalar differential operators of order less than 1 on E, namely, we have an exact sequence

$$0 \longrightarrow \operatorname{End}(E) \longrightarrow D^1(E) \longrightarrow T_X \longrightarrow 0,$$

where the surjective map $\sigma : D^1(E) \to T_X$ is locally defined by the symbol

$$\sigma(P) = \sum_k h^k \frac{\partial}{\partial x^k}.$$

There is an obvious identification of $D^1(E)$ with $A(E)$ as smooth vector bundles, but we will see that in fact $D^1(E)$ can be given a holomorphic structure so that $D^1(E)$ and $A(E)$ are isomorphic as holomorphic vector bundles.

First of all, locally on an open set U_α, we can write

$$P|_{U_\alpha} = g_\alpha + d_\alpha.$$

Let e_α be local sections of E, $\{f_{\alpha\beta}\}$ be holomorphic transition functions of E and $P_\alpha := P|_{U_\alpha}(e_\alpha)$. To get a global differential operator, we must have

$$f_{\alpha\beta} P_\beta = P_\alpha f_{\alpha\beta}.$$

Hence

$$g_\beta = f_{\beta\alpha} g_\alpha f_{\alpha\beta} + f_{\beta\alpha}(d_\alpha f_{\beta\alpha}), \quad d_\alpha = d_\beta.$$

Set

$$\tau_\alpha := g_\alpha - d_\alpha \lrcorner (\bar{h}_\alpha^{-1} \partial \bar{h}_\alpha),$$

where h_α is the Hermitian metric on $E|_{U_\alpha}$. Define a map

$$\Phi : g_\alpha + d_\alpha \longmapsto (\tau_\alpha, d_\alpha).$$

Straightforward computations give the identities

$$f_{\alpha\beta} \tau_\beta = \tau_\alpha f_{\alpha\beta}, \quad \bar{\partial}_{A(E)}(\tau_\alpha, d_\alpha) = 0.$$

It then follows that Φ defines an isomorphism between $D^1(E)$ and $A(E)$. So we can give $D^1(E)$ a holomorphic structure by pulling back that on $A(E)$ via Φ. Hence we obtain

Proposition A.2. $D^1(E)$ *carries a natural holomorphic structure so that it is iso-morphic to the Atiyah extension* $A(E)$. *In particular,* $H^p(X, D^1(E)) \cong H^p(X, A(E))$ *for any* p.

Together with the Lie bracket [16]

$$[\omega \otimes P, \eta \otimes Q] := \omega \wedge \eta \otimes [P, Q] + \omega \wedge \mathcal{L}_{\sigma(P)} \eta \otimes Q - (-1)^{|\omega||\eta|} \eta \wedge \mathcal{L}_{\sigma(Q)} \omega \otimes P,$$

the triple $(\Omega^{0,*}(D^1(E)), \bar{\partial}, [-, -])$ forms a DGLA. Note that the Lie derivative acts by

$$\mathcal{L}_X \omega = d(i_X \omega) + i_X d\omega = i_X \partial \omega,$$

for any $\omega \in \Omega^{0,*}(X)$ and $X \in \Omega^0(T_X)$.

Theorem A.3. *The isomorphism* $\Phi : D^1(E) \to A(E)$

$$\Phi : g_\alpha + d_\alpha \longmapsto g_\alpha - \bar{h}_\alpha^{-1} d_\alpha \bar{h}_\alpha$$

intertwines with the brackets $[-, -]$ *and* $[-, -]_h$, *i.e.*

$$\Phi[\varphi \otimes P, \psi \otimes Q] = [\varphi \otimes \Phi(P), \psi \otimes \Phi(Q)]_h.$$

Proof. We first prove that

$$\Phi[P, Q] = [\Phi(P), \Phi(Q)]_h.$$

Write $P = g + d$ and $Q = g' + d'$ locally in a coordinate neighborhood $U \subset X$. Then

$$[P, Q] = [g, g'] + dg' - d'g + [d, d']$$

and so

$$\Phi[P, Q] = ([g, g'] + dg' - d'g - \bar{h}^{-1}[d, d']\bar{h}, [d, d']).$$

On the other hand,

$$[\Phi(P), \Phi(Q)]_h = (\nabla_d^Q(g' - \bar{h}^{-1} d'\bar{h}) - \nabla_{d'}^Q(g - \bar{h}^{-1} d\bar{h}) + [g - \bar{h}^{-1} d'\bar{h}, g - \bar{h}^{-1} d'\bar{h}], [d, d'])$$

Now, we compute

$$\nabla_d^Q(g' - \bar{h}^{-1} d'\bar{h}) - \nabla_{d'}^Q(g - \bar{h}^{-1} d\bar{h})$$
$$= d(g' - \bar{h}^{-1} d'\bar{h}) + [\bar{h}^{-1} d\bar{h}, g' - \bar{h}^{-1} d'\bar{h}] - d'(g + \bar{h}^{-1} d\bar{h}) - [\bar{h}^{-1} d'\bar{h}, g - \bar{h}^{-1} d\bar{h}]$$
$$= dg' - d'g + [\bar{h}^{-1} d\bar{h}, g'] - [\bar{h}^{-1} d'\bar{h}, g] - d\bar{h}^{-1} d'\bar{h} + d'\bar{h}^{-1} d\bar{h} - 2[\bar{h}^{-1} d\bar{h}, \bar{h}^{-1} d'\bar{h}].$$

and

$$[g - \bar{h}^{-1} d\bar{h}, g' - \bar{h}^{-1} d'\bar{h}] = [g, g'] - [g, \bar{h}^{-1} d'\bar{h}] - [\bar{h}^{-1} d\bar{h}, g'] + [\bar{h}^{-1} d\bar{h}, \bar{h}^{-1} d'\bar{h}].$$

Therefore, their sum equals to

$$[g, g'] + dg' - d'g - d\bar{h}^{-1} d'\bar{h} + d'\bar{h}^{-1} d\bar{h} - [\bar{h}^{-1} d\bar{h}, \bar{h}^{-1} d'\bar{h}].$$

Finally,

$$[\bar{h}^{-1} d\bar{h}, \bar{h}^{-1} d'\bar{h}] = \bar{h}^{-1}(d\bar{h})\bar{h}^{-1}(d'\bar{h}) - \bar{h}^{-1}(d'\bar{h})\bar{h}^{-1}(d\bar{h})$$
$$= -(d\bar{h}^{-1})(d'\bar{h}) + (d'\bar{h}^{-1})(d\bar{h})$$
$$= -d(\bar{h}^{-1} d'\bar{h}) + \bar{h}^{-1} dd'\bar{h} + d'(\bar{h}^{-1} d\bar{h}) - \bar{h}^{-1} d' d\bar{h}$$
$$= -d(\bar{h}^{-1} d'\bar{h}) + d'(\bar{h}^{-1} d\bar{h}) + \bar{h}^{-1}[d, d']\bar{h}.$$

Hence

$$\nabla_d^Q(g' - \bar{h}^{-1}d'\bar{h}) - \nabla_{d'}^Q(g - \bar{h}^{-1}d\bar{h}) + [g - \bar{h}^{-1}d'\bar{h}, g - \bar{h}^{-1}d'\bar{h}]$$
$$= [g, g'] + dg' - d'g - \bar{h}^{-1}[d, d']\bar{h},$$

which is the required equality.

To prove the general case, we have, by linearity and the case that we have proved, the $\mathrm{End}(E)$-part of $\Phi[\omega \otimes P, \eta \otimes Q]$ is equal to

$$\omega \wedge \eta \otimes [\tau(P), \tau(Q)]_h - \omega \wedge \eta \otimes [\tau(P), \sigma(Q)]_h + \omega \wedge \eta \otimes [\sigma(P), \tau(Q)]_h$$

$$+ \omega \wedge \mathcal{L}_{\sigma(P)}\eta \otimes \tau(Q) - (-1)^{|\omega||\eta|}\eta \wedge \mathcal{L}_{\sigma(Q)}\omega \otimes \tau(P),$$

where $\tau(P) := pr_{\mathrm{End}(E)} \circ \Phi(P)$. On the other hand, the $\mathrm{End}(E)$-part of $[\Phi(\omega \otimes P), \Phi(\eta \otimes Q)]_h$ is equal to

$$((\omega \otimes \sigma(P))\lrcorner\nabla^Q(\eta \otimes \tau(Q)) - (-1)^{|\omega||\eta|}(\eta \otimes \sigma(Q))\lrcorner\nabla^Q(\omega \otimes \tau(P)) + \omega \wedge \eta \otimes [\tau(P), \tau(Q)]_h.$$

The Leibniz rule for connections implies that

$$((\omega \otimes \sigma(P))\lrcorner\nabla^Q(\eta \otimes \tau(Q)) = \omega \wedge \mathcal{L}_{\sigma(P)}\eta \otimes \tau(Q) + \omega \wedge \eta \otimes \nabla_{\sigma(P)}^Q\tau(Q)$$
$$= \omega \wedge \mathcal{L}_{\sigma(P)}\eta \otimes \tau(Q) + \omega \wedge \eta \otimes [\sigma(P), \tau(Q)].$$

Similarly, we have

$$((\eta \otimes \sigma(Q))\lrcorner\nabla^Q(\omega \otimes \tau(P)) = \eta \wedge \mathcal{L}_{\sigma(Q)}\omega \otimes \tau(P) + (-1)^{|\omega||\tau|}\omega \wedge \eta \otimes [\sigma(Q), \tau(P)].$$

Putting these back into $[\Phi(\omega \otimes P), \Phi(\eta \otimes Q)]_h$, we get

$$\Phi[\omega \otimes P, \eta \otimes Q] = [\Phi(\omega \otimes P), \Phi(\eta \otimes Q)]_h.$$

This proves our theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark A.4. *This theorem gives a proof of the required identities in Propositions 3.14 and 3.15, and the fact that the DGLA $(\Omega^{0,\bullet}(A(E)), \bar{\partial}_{A(E)}, [-,-]_h)$ is independent of the choice of the hermitian metric h.*

REFERENCES

1. M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957), 181–207. MR 0086359 (19,172c)

2. A. Douglis and L. Nirenberg, *Interior estimates for elliptic systems of partial differential equations*, Comm. Pure Appl. Math. **8** (1955), 503–538. MR 0075417 (17,743b)

3. W. Goldman and J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 43–96. MR 972343 (90b:32041)

4. ———, *The homotopy invariance of the Kuranishi space*, Illinois J. Math. **34** (1990), no. 2, 337–367. MR 1046568 (91d:32025)

5. P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994, Reprint of the 1978 original. MR 1288523 (95d:14001)

6. L. Huang, *On joint moduli spaces*, Math. Ann. **302** (1995), no. 1, 61–79. MR 1329447 (96d:32021)

7. D. Huybrechts, *The tangent bundle of a Calabi-Yau manifold—deformations and restriction to rational curves*, Comm. Math. Phys. **171** (1995), no. 1, 139–158. MR 1341697 (96g:14032)

8. ———, *Complex geometry*, Universitext, Springer-Verlag, Berlin, 2005, An introduction. MR 2093043 (2005h:32052)

9. D. Huybrechts and R. P. Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes*, Math. Ann. **346** (2010), no. 3, 545–569. MR 2578562 (2011b:14030)

10. S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, vol. 15, Princeton University Press, Princeton, NJ; Iwanami Shoten, Tokyo, 1987, Kanô Memorial Lectures, 5. MR 909698 (89e:53100)

11. K. Kodaira and J. Morrow, *Complex manifolds*, AMS Chelsea Publishing, Providence, RI, 2006, Reprint of the 1971 edition with errata. MR 2214741 (2006j:32001)

12. K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures. I, II*, Ann. of Math. (2) **67** (1958), 328–466. MR 0112154 (22 #3009)

13. _____ , *On deformations of complex analytic structures. III. Stability theorems for complex structures*, Ann. of Math. (2) **71** (1960), 43–76. MR 0115189 (22 #5991)

14. M. Kuranishi, *New proof for the existence of locally complete families of complex structures*, Proc. Conf. Complex Analysis (Minneapolis, 1964), Springer, Berlin, 1965, pp. 142–154. MR 0176496 (31 #768)

15. S. Li, *On the deformation theory of pair* (X, E), preprint, arXiv:0809.0344.

16. E. Martinengo, *Higher brackets and moduli space of vector bundles*, Ph.D. thesis, Sapienza Università di Roma, 2009.

17. A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts, vol. 69, Cambridge University Press, Cambridge, 2007. MR 2325093 (2008f:32028)

18. A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), 391–404. MR 0088770 (19,577a)

19. A. Nijenhuis and R. W. Richardson, Jr., *Cohomology and deformations in graded Lie algebras*, Bull. Amer. Math. Soc. **72** (1966), 1–29. MR 0195995 (33 #4190)

20. X. Pan, *Deformations of polarized manifolds with torsion canonical bundles*, preprint, arXiv:1310.7162.

21. E. Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften, vol. 334, Springer-Verlag, Berlin, 2006. MR 2247603 (2008e:14011)

22. Y.-T. Siu and G. Trautmann, *Deformations of coherent analytic sheaves with compact supports*, Mem. Amer. Math. Soc. **29** (1981), no. 238, iii+155. MR 597091 (82c:32023)

23. G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, pp. 629–646. MR 915841

24. A. Todorov, *The Weil-Petersson geometry of the moduli space of* $\mathrm{SU}(n \geq 3)$ *(Calabi-Yau) manifolds. I*, Comm. Math. Phys. **126** (1989), no. 2, 325–346. MR 1027500 (91f:32022)

DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG

E-mail address: kwchan@math.cuhk.edu.hk

DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG

E-mail address: yhsuen@math.cuhk.edu.hk