# HOLOMORPHIC LINE BUNDLES ON PROJECTIVE TORIC MANIFOLDS FROM LAGRANGIAN SECTIONS OF THEIR MIRRORS BY SYZ TRANSFORMATIONS

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Abstract. The mirror of a projective toric manifold $X_\Sigma$ is given by a Landau-Ginzburg model $(Y, W)$. We introduce a class of Lagrangian submanifolds in $(Y, W)$ and show that, under the SYZ mirror transformation, they can be transformed to torus-invariant hermitian metrics on holomorphic line bundles over $X_\Sigma$. Through this geometric correspondence, we also identify the mirrors of Hermitian-Einstein metrics, which are given by distinguished Lagrangian sections whose potentials satisfy certain Laplace-type equations.

Contents



## 1. Introduction

Let $X_\Sigma$ be a projective toric manifold defined by a fan $\Sigma$. The mirror of $X_\Sigma$ is given by a Landau-Ginzburg model $(Y, W)$, which consists of a noncompact Kähler manifold $Y$ and a holomorphic function $W : Y \to \mathbb{C}$ (the superpotential). Mirror symmetry relates the complex geometry of $X_\Sigma$ to the symplectic geometry of $(Y, W)$. In particular, holomorphic vector bundles (or more generally, coherent sheaves) over $X_\Sigma$ should correspond to Lagrangian cycles in $(Y, W)$. This is succinctly expressed by Kontsevich's Homological Mirror Symmetry Conjecture for toric manifolds [14], which states that the derived category of coherent sheaves $D^bCoh(X_\Sigma)$ is equivalent to the Fukaya-Kontsevich-Seidel category of $(Y, W)$. Since then, much work has been done [13], [17], [19], [4], [5], [1], [8], culminating in proofs of the conjecture for all projective toric manifolds in Abouzaid [2] and, more recently, in Fang-Liu-Treumann-Zaslow [9].[1]

In this paper, we will examine the correspondence between holomorphic line bundles on $X_\Sigma$ and Lagrangian cycles on $(Y, W)$ from a different angle, namely, by applying *SYZ mirror transformations* [6], [7]. Our goal is to put the correspondence

[1] Fang-Liu-Treumann-Zaslow [9] also proved an equivariant version of the conjecture.

in the toric case in the same footing as the semi-flat Calabi-Yau case as done in Leung-Yau-Zaslow [15]. This approach is also closely related to the works [1], [2], [8], [9], where T-duality was used implicitly or explicitly.

Let $N \cong \mathbb{Z}^n$ be a rank $n$ lattice, $M = \mathrm{Hom}(N, \mathbb{Z})$ the dual lattice and $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ the dual pairing, and let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $T_N$ and $T_M$ the real tori $N_{\mathbb{R}}/N$ and $M_{\mathbb{R}}/M$ respectively. A projective toric $n$-fold $X_\Sigma$ contains an open dense torus orbit $U = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$, which can also be written as

$$U = N_{\mathbb{R}} \times \sqrt{-1} T_N = TN_{\mathbb{R}}/N,$$

where we have, by abuse of notations, also used $N$ to denote the family of lattices $N_{\mathbb{R}} \times \sqrt{-1}N \subset TN_{\mathbb{R}}$. The projection map $U \to N_{\mathbb{R}}$ is a (trivial) torus bundle. According to the philosophy of the *Strominger-Yau-Zaslow Conjecture* [18], the mirror manifold $Y$ is given by the dual torus bundle (see [6], [7])

$$Y = N_{\mathbb{R}} \times \sqrt{-1} T_M = T^* N_{\mathbb{R}}/M,$$

with $M$ denoting the family of lattices $N_{\mathbb{R}} \times \sqrt{-1}M \subset T^*N_{\mathbb{R}}$. Using the semi-flat SYZ mirror transformation (or T-duality), $T_N$-invariant hermitian metrics on holomorphic line bundles over $X_\Sigma$ (when restricted to $U$) can be transformed to give Lagrangian sections of $Y \to N_{\mathbb{R}}$ as in [15].[2] Naturally, one would ask the following

**Question:** *Which Lagrangian sections of $Y \to N_{\mathbb{R}}$ can be transformed back, by the inverse SYZ mirror transformation, to $T_N$-invariant hermitian metrics on holomorphic line bundles over $X_\Sigma$?*

Put it in another way, the problem is to characterize the set of Lagrangian sections of $Y \to N_{\mathbb{R}}$ we get by transforming $T_N$-invariant hermitian metrics on holomorphic line bundles over $X_\Sigma$. One of our aims in this paper is to answer this question.

Recall that the superpotential $W$ is a Laurent polynomial (see, for example, [6], [7]). Write $W$ as a sum of monomials: $W = \sum_{i=1}^d W_i$. In a sense, the monomial $W_i$ (for $i = 1, \ldots, d$) is mirror to the toric prime divisor $D_i \subset \bar{X}$ associated to the primitive generator $v_i \in N$ of a 1-dimensional cone in $\Sigma$. Consider the embedding $\iota : M \hookrightarrow \mathbb{Z}^d$ defined by $\iota(u) = (\langle u, v_1 \rangle, \ldots, \langle u, v_d \rangle)$. By the theory of toric varieties, the quotient $\mathbb{Z}^d/\iota(M)$ is canonically identified with $H^2(X_\Sigma, \mathbb{Z})$. In Section 3, we will define, for each $[a] \in H^2(X_\Sigma, \mathbb{Z})$, a growth condition $(*_{[a]})$ for Lagrangian sections of $Y \to N_{\mathbb{R}}$. We can now state our main result as follows, which will be proved in Section 4.

**Theorem 1.1.** *Let $\mathcal{L}_{[a]}$ be the holomorphic line bundle over $X_\Sigma$ corresponding to $[a] \in H^2(X_\Sigma, \mathbb{Z})$. Then the SYZ mirror transformation gives a bijective correspondence between $T_N$-invariant hermitian metrics on $\mathcal{L}_{[a]}$ and Lagrangian sections of $Y \to N_{\mathbb{R}}$ satisfying the growth condition $(*_{[a]})$.*

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[2] More precisely, one should get Lagrangian sections equipped with flat $U(1)$-connections. But our Lagrangian sections are simply connected, so all flat $U(1)$-connections are gauge equivalent to the trivial one and we will ignore this data.

Notice that all Lagrangian sections of $Y \to N_{\mathbb{R}}$ are Hamiltonian isotopic to the zero section, i.e. they represent the same Hamiltonian class. To get a correspondence with the class of holomorphic line bundles on $X_\Sigma$, it is therefore necessary to find a finer equivalence relation. For this purpose, we define two Lagrangian sections of $(Y, W)$ to be equivalent if they can be deformed to each other through Hamiltonian isotopies which *preserve a growth condition* $(*_{[a]})$. It is easy to see that each equivalence class then consists of exactly those Lagrangian sections which satisfy the same growth condition $(*_{[a]})$.

Furthermore, by our main result, we can easily identify the Lagrangian sections which are *mirror to Hermitian-Einstein metrics* on holomorphic line bundles. These turn out to be Lagrangian sections whose potentials satisfy certain Laplace-type equations. We call these Lagrangian sections *harmonic*. Hence, as an immediate consequence of our main result, we have the following

**Corollary 1.1.**

1. *The SYZ mirror transformation provides a bijective correspondence between isomorphism classes of holomorphic line bundles over $X_\Sigma$ and equivalence classes of Lagrangian sections of $(Y, W)$.*
2. *Each equivalence class of Lagrangian sections of $(Y, W)$ is represented by a unique harmonic Lagrangian section.*

All of these will be discussed with more details in Section 4. The next section (Section 2) is a brief review of mirror symmetry for toric manifolds. Some further remarks and discussions are contained in the final section (Section 5).

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## 2. Projective toric manifolds and their mirrors

In this section, we briefly review the geometric aspects of the mirror symmetry for projective toric manifolds and fix our notations.

A projective toric manifold by $X_\Sigma$ is defined by a smooth, complete fan $\Sigma$ in $N_{\mathbb{R}}$. By the general theory of toric varieties [10], [11], any ample line bundle $\mathcal{L}$ on $X_\Sigma$ is determined by a lattice polytope $\bar{P} \subset M_{\mathbb{R}}$ dual to $\Sigma$. If $v_1, \ldots, v_d \in N$ are the primitive generators of the 1-dimensional cones of $\Sigma$, then there is a $d$-tuple of integers $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^d$ such that

$$\bar{P} = \{x = (x_1, \ldots, x_n) \in M_{\mathbb{R}} : \langle x, v_i \rangle + \lambda_i \geq 0 \text{ for } i = 1, \ldots, d\},$$

and $\mathcal{L}$ is then canonically identified with the divisor line bundle $\mathcal{O}(D_\lambda)$, where $D_\lambda = \sum_{i=1}^d \lambda_i D_i$ is an ample toric divisor. We fix such an ample line bundle $\mathcal{L}$ and equip $X_\Sigma$ with the Kähler structure $\omega_{X_\Sigma} = \iota^* \omega_{FS}$, where $\iota : X_\Sigma \hookrightarrow \mathbb{C}P^N$ is an embedding induced by $\mathcal{L}$ (note that since $X_\Sigma$ is smooth and projective, every ample line bundle $\mathcal{L}$ is in fact very ample; see Fulton [10]), and $\omega_{FS}$ is the Fubini-Study Kähler structure on $\mathbb{C}P^N$.

Recall that $X_\Sigma$ contains an open dense orbit $U = X_\Sigma \setminus \bigcup_{i=1}^d D_i = N \otimes_{\mathbb{Z}} \mathbb{C}^* = N_{\mathbb{R}} \times \sqrt{-1}T_N = TN_{\mathbb{R}}/N$, and we have a natural torus fibration $\nu_U : U = TN_{\mathbb{R}}/N \to N_{\mathbb{R}}$ given by projection to the first factor. If $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and $u_1, \ldots, u_n \in \mathbb{R}/2\pi\mathbb{Z}$ are the base coordinates on $N_{\mathbb{R}}$ and fiber coordinates on $T_N$ respectively, then the complex coordinates on $U = (\mathbb{C}^*)^n$ are given by $w_j = e^{\xi_j + \sqrt{-1}u_j}$, $j = 1, \ldots, n$, and the restriction of $\omega_{X_\Sigma}$ to $U$ can be explicitly written as

$$\omega_U = \omega_{X_\Sigma}|_U = 2\sqrt{-1}\partial\bar{\partial}\phi = \sum_{j,k=1}^n \frac{\partial^2 \phi}{\partial \xi_j \partial \xi_k} d\xi_j \wedge du_k,$$

where $\phi : N_{\mathbb{R}} \to \mathbb{R}$ is the function given by

$$\phi(\xi) = \frac{1}{2} \log \left( \sum_{u \in \bar{P} \cap M} c_u e^{2\langle u, \xi \rangle} \right),$$

for some nonnegative constants $c_u$, $u \in \bar{P} \cap M$, which depend on the embedding $\iota$. We use $\phi_j$ and $\phi_{jk}$ to denote the partial derivatives $\frac{\partial \phi}{\partial \xi_j}$ and $\frac{\partial^2 \phi}{\partial \xi_j \partial \xi_k}$ respectively, and let $(\phi^{jk})_{j,k=1}^n$ be the inverse matrix of $(\phi_{jk})_{j,k=1}^n$.

If $\mu : X_\Sigma \to \bar{P}$ is the moment map of the Hamiltonian $T_N$-action on $(X_\Sigma, \omega_{X_\Sigma})$, then the restriction of $\mu$ to $U \subset X_\Sigma$ is the map $\mu_U : U \to M_{\mathbb{R}}$ given by

$$\mu_U(w) = d\phi(\log|w_1|, \ldots, \log|w_n|) = \frac{\sum_{u \in \bar{P} \cap M} c_u |w^u|^2 \cdot u}{\sum_{u \in \bar{P} \cap M} c_u |w^u|^2},$$

for $w = (w_1, \ldots, w_n) \in U = (\mathbb{C}^*)^n$. The image of $\mu_U$ is the interior $P$ of the polytope $\bar{P}$. In fact, the Legendre transform of the function $\phi$ gives a diffeomorphism $\Phi = d\phi : N_{\mathbb{R}} \to P$ and $\mu_U = \Phi \circ \nu_U$. We also have a nowhere vanishing holomorphic $n$-form on $U$ given by

$$\Omega_U = \frac{dw_1}{w_1} \wedge \ldots \wedge \frac{dw_n}{w_n}.$$

With respect to $\omega_U$ and $\Omega_U$, $\nu_U : U \to N_{\mathbb{R}}$ and $\mu_U : U \to P$ are special Lagrangian torus fibrations, in the sense of Auroux [3] (and $U$ is an almost Calabi-Yau manifold).

The mirror of $X_\Sigma$ is the Landau-Ginzburg model $(Y, W)$ described as follows. The mirror manifold $Y$ is the dual torus fibration $Y = N_{\mathbb{R}} \times \sqrt{-1}T_M = T^*N_{\mathbb{R}}/M$. Written in this way, $Y$ is naturally a symplectic manifold, equipped with the standard symplectic structure $\omega_Y = \sum_{j=1}^n d\xi_j \wedge dy_j$, where $y_1, \ldots, y_n \in \mathbb{R}/2\pi\mathbb{Z}$ are the dual coordinates on the fiber $T_M$. The projection map $\mu_Y : Y = T^*N_{\mathbb{R}}/M \to N_{\mathbb{R}}$ is the moment map for the Hamiltonian $T_M$-action on $Y$. To describe the complex structure on $Y$ and write down the superpotential $W$, it is more convenient to change the coordinates on the base by the diffeomorphism $\Phi : N_{\mathbb{R}} \to P$ and rewrite $Y$ as $Y = P \times \sqrt{-1}T_M = TP/M$, where $M$ here denotes the (trivial) family of lattices $P \times \sqrt{-1}M$. Then $Y$ is naturally a complex manifold with complex coordinates given by $z_j = e^{-x_j + \sqrt{-1}y_j}$, where $x_1, \ldots, x_n$ are the coordinates on $P$. There is a nowhere vanishing holomorphic $n$-form on $Y$ given by

$$\Omega_Y = \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}.$$

The superpotential $W : Y \to \mathbb{C}$ is the Laurent polynomial

$$W(z) = e^{-\lambda_1} z^{v_1} + \ldots + e^{-\lambda_d} z^{v_d}$$

for $z = (z_1, \ldots, z_n) \in (\mathbb{C}^*)^n$, where $z^{v_i}$ denotes the monomial $z_1^{v_i^1} \ldots z_n^{v_i^n}$. $W$ can be obtained as the SYZ mirror transformation of a certain function on the geodesic loop space $L_U$ of $U \subset X_\Sigma$ (see Chan-Leung [6], [7] for details).

Notice that, as a complex manifold, $Y$ is biholomorphic to the bounded domain $\{z \in (\mathbb{C}^*)^n : |e^{-\lambda_i} z^{v_i}| < 1, \text{ for } i = 1, \ldots, d\}$ in $(\mathbb{C}^*)^n$. On the other hand, since $\Phi$ is a Legendre transform, there exists a function $\psi : P \to \mathbb{R}$ such that $\phi^{jk} = \psi_{jk} = \frac{\partial^2 \psi}{\partial x_j \partial x_k}$ and $(\psi^{jk}) := (\psi_{jk})^{-1} = (\phi_{jk})$. The Legendre transform $\Psi : P \to N_\mathbb{R}$ of $\psi$ is then the inverse of $\Phi : N_\mathbb{R} \to P$, i.e. $\Psi = \Phi^{-1}$. Now, the symplectic structure $\omega_Y$ is given in the $x_j, y_j$ coordinates by

$$\omega_Y = \sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial x_j \partial x_k} dx_j \wedge dy_k.$$

If we denote by $\nu_Y : Y = TP/M \to P$ the projection map to the base $P$, then we have $\mu_Y = \Psi \circ \nu_Y$. With respect to $\omega_Y$ and $\Omega_Y$, $\nu_Y : Y \to N_\mathbb{R}$ and $\mu_Y : Y \to P$ are special Lagrangian torus fibrations, which are dual to $\nu_U : U \to N_\mathbb{R}$ and $\mu_U : U \to P$ respectively.

Physical arguments predict that the complex (respectively, symplectic) geometry of $X_\Sigma$ is interchanged with the symplectic (respectively, complex) geometry of $(Y, W)$ under mirror symmetry. For precise mathematical statements and how SYZ mirror transformations are applied to explain the geometry underlying this mirror symmetry, we refer the reader to [6], [7].

## 3. A class of Lagrangian submanifolds in Landau-Ginzburg models

In this section, we introduce a class of Lagrangian submanifolds in $(Y, W)$, which are sections of the torus fibration $\mu_Y : Y \to N_\mathbb{R}$ (or $\nu_Y : Y \to P$), satisfying certain growth conditions at infinity.

Let $(Y, W)$ be a Landau-Ginzburg model mirror to a projective toric manifold $X_\Sigma$. Recall that the superpotential $W \in \mathcal{O}(Y)$ is a Laurent polynomial of the form $\sum_{i=1}^{d} b_i z^{v_i}$, for some $v_1, \ldots, v_d \in N$. Define $A(W)$ to be the quotient group $\mathbb{Z}^d / \iota(M)$, where $\iota : M \hookrightarrow \mathbb{Z}^d$, $u \mapsto (\langle u, v_1 \rangle, \ldots, \langle u, v_d \rangle)$ is the homomorphism defined in the introduction. As we have mentioned before, $A(W)$ is canonically identified with the second cohomology group $H^2(X_\Sigma, \mathbb{Z})$ of $X_\Sigma$. Moreover, if we let $\mathrm{Log} : TM_\mathbb{R}/M = (\mathbb{C}^*)^n \to M_\mathbb{R} = \mathbb{R}^n$ be the map defined by

$$\mathrm{Log}(z_1, \ldots, z_n) = (\log |z_1|, \ldots, \log |z_n|),$$

then (the closure of) the image of $Y$ under Log, i.e. $P := \mathrm{Log}(Y) = \mathrm{Log}(\{z \in (\mathbb{C}^*)^n : |b_v z^v| < 1 \text{ for all } v \in A\})$, is a polytope in $M_\mathbb{R}$, and this determines a fan $\Sigma$ in $N_\mathbb{R}$. These are exactly the polytope and fan defining the projective toric manifold $X_\Sigma$.

Now, we write $Y = N_\mathbb{R} \times \sqrt{-1} T_M = T^* N_\mathbb{R} / M$ and equip $Y$ with the standard symplectic form $\omega_Y = \sum_{j=1}^{n} d\xi_j \wedge dy_j$. Since $N_\mathbb{R}$ is simply connected, any section $L$ of $\mu_Y : Y \to N_\mathbb{R}$ can be lifted to a section $\tilde{L} = \{(\xi, y(\xi)) : \xi = (\xi_1, \ldots, \xi_n) \in N_\mathbb{R}\}$

of $T^*N_{\mathbb{R}}$, where $y : N_{\mathbb{R}} \to M_{\mathbb{R}}$ should be regarded as a 1-form on $N_{\mathbb{R}}$; moreover, if $\{(\xi, y_1(\xi)) : \xi \in N_{\mathbb{R}}\}, \{(\xi, y_2(\xi)) : \xi \in N_{\mathbb{R}}\} \subset T^*N_{\mathbb{R}}$ are two lifts of $L \subset Y$, then $y_1 - y_2 \equiv u$ for some constant $u \in M$. By the standard argument as shown in [15], a section $L$ of $\mu_Y : Y \to N_{\mathbb{R}}$ is Lagrangian if and only if some lift $\tilde{L} = \{(\xi, y(\xi)) : \xi = (\xi_1, \ldots, \xi_n) \in N_{\mathbb{R}}\}$ of $L$ to $T^*N_{\mathbb{R}}$ is the graph of an exact 1-form, i.e. if and only if

$$y(\xi) = dg(\xi) = \left(\frac{\partial g}{\partial \xi_1}, \ldots, \frac{\partial g}{\partial \xi_n}\right),$$

for some function $g$ on $N_{\mathbb{R}}$, which is unique up to adding a constant. $g$ is called a *potential* of the lift $\tilde{L}$ of the Lagrangian section $L$. For our purpose, we need $g$ to be of class $C^2$.

**Definition 3.1.** *Let $a = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ be a d-tuple of integers. A Lagrangian section $\tilde{L} = \{(\xi, y(\xi)) : \xi \in N_{\mathbb{R}}\}$ of $T^*N_{\mathbb{R}} \to N_{\mathbb{R}}$ is said to satisfy the growth condition $(*_a)$ if a potential $g \in C^2(N_{\mathbb{R}})$ of $\tilde{L}$ satisfies the following conditions: Given any n-dimensional cone $\sigma \in \Sigma$. Suppose that, without loss of generality, $\sigma$ is generated by $v_1, \ldots, v_n$; and let $\xi(t) = \xi(t_1, \ldots, t_n) = t_1 v_1 + \ldots + t_n v_n$, for $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$. Then, we have,*

1. *the functions $2e^{-2t_j}(\langle dg(\xi(t)), v_j\rangle + a_j)$ and $e^{-2t_j}(v_j^T Hess(g) v_j)(\xi(t))$ have the same limit as $t_j \to -\infty$, for $j = 1, \ldots, n$;*
2. *for any $j, k, l \in \{1, \ldots, n\}$, the function $(v_j^T Hess(g) v_k)(\xi(t))$ has a limit as $t_l \to -\infty$; and,*
3. *for any distinct $j, k \in \{1, \ldots, n\}$, the function $e^{-t_j - t_k}(v_j^T Hess(g) v_k)(\xi(t))$ goes to zero when $t_j \to -\infty$ or $t_k \to -\infty$.*

*Let $[a] \in A(W)$. A Lagrangian section $L$ of $\mu_Y : Y \to N_{\mathbb{R}}$ is said to satisfy the growth condition $(*_{[a]})$ if some lift $\tilde{L}$ of $L$ from $Y$ to $T^*N_{\mathbb{R}}$ satisfies $(*_a)$ for some representative $a = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ of $[a]$.*

We denote the set of Lagrangian sections of $\mu_Y : Y \to N_{\mathbb{R}}$ satisfying $(*_{[a]})$ for some $[a] \in A(W)$ by $\mathbb{L}(Y, W)$.

**Remark 3.1.** *The condition that a Lagrangian section $L$ of $\mu_Y : Y \to N_{\mathbb{R}}$ satisfies $(*_{[a]})$ is well-defined because if $\{(\xi, y_1(\xi)) : \xi \in N_{\mathbb{R}}\}, \{(\xi, y_2(\xi)) : \xi \in N_{\mathbb{R}}\} \subset T^*N_{\mathbb{R}}$ are two lifts of $L \subset Y$, then their potentials $g_1, g_2$ will differ by a linear function of the form $\langle u, \xi\rangle + \alpha$, for some $u \in M$ and $\alpha \in \mathbb{R}$. Thus, when one of the lifts satisfies $(*_a)$, the other will satisfy $(*_{a'})$, where $a' = a + (\langle u, v_1\rangle, \ldots, \langle u, v_d\rangle)$, and note that we have $[a] = [a']$.*

We give a couple of examples to illustrate our definitions.

**Example 1.** The simplest example is given by $X_\Sigma = \mathbb{C}P^1$. The fan $\Sigma$ in $N_{\mathbb{R}} = \mathbb{R}$ is generated by two primitive vectors $v_1 = 1, v_2 = -1$ (see Figure 1 below). The

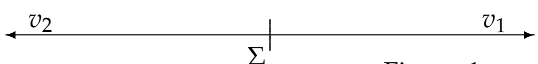


Figure 1

mirror manifold $Y$, as a symplectic manifold, is the cylinder $Y = \mathbb{R} \times \sqrt{-1}S^1$.

Any Lagrangian section $L$ of $\mu_Y : Y \to \mathbb{R}$ lifts to the universal cover $T^*N_{\mathbb{R}} = \mathbb{R}^2$. A lift $\tilde{L}$ of $L$ is given by a graph

$$\tilde{L} = \{(\xi, y(\xi)) : \xi \in \mathbb{R}\} \subset \mathbb{R}^2,$$

where $y(\xi) = g'(\xi)$ is the derivative of a function $g = g(\xi) \in C^2(\mathbb{R})$. Given $(a, b) \in \mathbb{Z}^2$, the conditions in Definition 3.1 reduces to the following two equalities of limits:

$$\begin{aligned} \lim_{\xi \to -\infty} 2e^{-2\xi}(y(\xi) + a) &= \lim_{\xi \to -\infty} e^{-2\xi} y'(\xi), \\ \lim_{\xi \to \infty} 2e^{2\xi}(b - y(\xi)) &= \lim_{\xi \to \infty} e^{2\xi} y'(\xi). \end{aligned}$$

This implies that, geometrically, we have $y(\xi) \to -a$ as $\xi \to -\infty$ and $y(\xi) \to b$ as $\xi \to \infty$, and the slope of the graph goes to zero as $\xi \to \pm\infty$; there are no restrictions on the graph for finite values of $\xi$. The equalities of limits place further restrictions on the growth rates of $y(\xi)$ and its derivative as $\xi$ tends to $\pm\infty$.

**Example 2.** Consider the case when $X_\Sigma = \mathbb{C}P^2$. The fan $\Sigma$ in $N_{\mathbb{R}} = \mathbb{R}^2$ is generated by $v_1 = (1, 0), v_2 = (0, 1), v_3 = (-1, -1)$ (see Figure 2 below). The

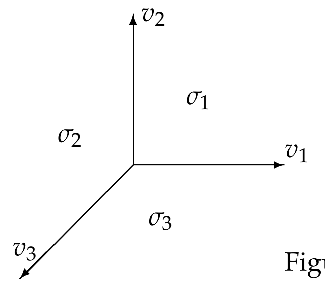


Figure 2

mirror manifold $Y$ is given by $Y = \mathbb{R}^2 \times \sqrt{-1}T^2$ equipped with the standard symplectic structure. Any Lagrangian section $L$ of $\mu_Y : Y \to \mathbb{R}^2$ can be lifted to a graph

$$\tilde{L} = \{(\xi_1, \xi_2, y_1(\xi_1, \xi_2), y_2(\xi_1, \xi_2)) : (\xi_1, \xi_2) \in \mathbb{R}^2\}$$

in the universal cover $T^*N_{\mathbb{R}} = \mathbb{R}^4$, where $y_1(\xi_1, \xi_2) = \frac{\partial g}{\partial \xi_1}, y_2(\xi_1, \xi_2) = \frac{\partial g}{\partial \xi_2}$ are the partial derivatives of a function $g = g(\xi_1, \xi_2) \in C^2(\mathbb{R}^2)$. Let $(a_1, a_2, a_3) \in \mathbb{Z}^3$. Consider the maximal cone $\sigma_1$. Then the conditions in Definition 3.1 can be restated as

$$\begin{aligned} \lim_{\xi_1 \to -\infty} 2e^{-2\xi_1}(y_1(\xi_1, \xi_2) + a_1) &= \lim_{\xi_1 \to -\infty} e^{-2\xi_1} y_{1,1}(\xi_1, \xi_2), \\ \lim_{\xi_2 \to -\infty} 2e^{-2\xi_2}(y_2(\xi_1, \xi_2) + a_2) &= \lim_{\xi_2 \to -\infty} e^{-2\xi_2} y_{2,2}(\xi_1, \xi_2), \\ \lim_{\xi_1 \to -\infty} e^{-\xi_1 - \xi_2} y_{1,2}(\xi_1, \xi_2) &= \lim_{\xi_1 \to -\infty} e^{-\xi_1 - \xi_2} y_{1,2}(\xi_1, \xi_2) = 0, \end{aligned}$$

where we denote by $y_{i,j}$ the partial derivative $\frac{\partial y_i}{\partial \xi_j}$. In particular, we must have $y_1 \to -a_1$ as $\xi_1 \to -\infty$, $y_2 \to -a_2$ as $\xi_2 \to -\infty$, and various partial derivatives of $y_1, y_2$ go to zero as $t_1, t_2$ tends to $-\infty$.

For another maximal cone, say, $\sigma_2$, the conditions can similarly be rewritten as

$$\begin{aligned}
\lim_{\xi_2\to-\infty} 2e^{-2\xi_2}(y_2(\xi_1,\xi_1+\xi_2)+a_2) &= \lim_{\xi_2\to-\infty} e^{-2\xi_2}y_{2,2}(\xi_1,\xi_1+\xi_2),\\
\lim_{\xi_1\to\infty} 2e^{2\xi_1}((-y_1-y_2)(\xi_1,\xi_1+\xi_2)+a_3) &= \lim_{\xi_1\to\infty} e^{2\xi_1}(y_{1,1}+2y_{1,2}+y_{2,2})(\xi_1,\xi_1+\xi_2),\\
\lim_{\xi_1\to\infty} e^{\xi_1-\xi_2}(y_{1,2}+y_{2,2})(\xi_1,\xi_1+\xi_2) &= \lim_{\xi_2\to-\infty} e^{\xi_1-\xi_2}(y_{1,2}+y_{2,2})(\xi_1,\xi_1+\xi_2)=0.
\end{aligned}$$

Geometrically, this means that we should also have $-y_1-y_2\to-a_3$ as $\xi$ goes to $-\infty$ in the $(-1,-1)$ direction, and various combinations of the partial derivatives of $y_1,y_2$ go to zero as $\xi$ tends to $-\infty$ in the $(0,1)$ and $(-1,-1)$ directions. Again, the equalities of limits indicate the growth rates of $y_1,y_2$ and combinations of their partial derivatives as $\xi$ tends to $-\infty$ in the $(1,0),(0,1),(-1,-1)$ directions.

In general, given a lift $\tilde{L}=\{(\xi,y(\xi)):\xi\in N_{\mathbb{R}}\}$ of a Lagrangian section $L$ of from $Y$ to $T^*N_{\mathbb{R}}$, the conditions in Definition 3.1 specify the values and growth rates of the functions $y_1(\xi),\ldots,y_n(\xi)$ and combinations of their partial derivatives as $\xi$ tends to $-\infty$ in the directions of $v_1,\ldots,v_d$. In particular, for $i=1,\ldots,d$, $\langle y(\xi),v_i\rangle$ goes to $-a_i$ as $\xi$ tends to $-\infty$ in the direction of $v_i$.

We may also regard $L$ and any lift $\tilde{L}$ of $L$ as Lagrangian sections over $P$, the interior of the polytope $\bar{P}$, so that we can write $\tilde{L}=\{(x,y(x)):x\in P\}$. Then the conditions in Definition 3.1 can be viewed as boundary conditions for the functions $y_1(x),\ldots,y_n(x)$ and combinations of their partial derivatives over the boundary $\partial\bar{P}$. For example, if $\tilde{L}$ satisfies $(*_a)$, where $a=(a_1,\ldots,a_d)\in\mathbb{Z}^d$, then the function $\langle y(x),v_k\rangle$ tends to $-a_k$ as $x$ approaches the facet of $\bar{P}$ with normal vector $v_k$.

**Remark 3.2.** *Our Lagrangian sections are closely related to the tropical Lagrangian sections defined and used by Abouzaid in his proof* [1], [2] *of the Homological Mirror Symmetry Conjecture for toric varieties. This relation is similar to the one explained in Appendix C of Fang-Liu-Treumann-Zaslow* [9].[3] *Let us describe the relation briefly as follows. In* [1], [2], *Abouzaid considered the family of superpotentials*

$$W_t=\sum_{i=1}^{d} c_i t^{-\lambda_i} z^{v_i},$$

*and the smooth hypersurfaces $M_t=W_t^{-1}(0)$ in $TM_{\mathbb{R}}/M=(\mathbb{C}^*)^n$. The amoeba of $M_t$ is the image under the logarithm map, i.e. $\mathcal{A}_t=Log(M_t)\subset M_{\mathbb{R}}$, and the tropical amoeba is the limit $\Pi=\lim_{t\to\infty}(\mathcal{A}_t/\log t)\subset M_{\mathbb{R}}$. Abouzaid showed that there is a distinguished connected component $Q$ of $M_{\mathbb{R}}\setminus\Pi$ which is a copy of the moment polytope $\bar{P}$ of $(X_\Sigma,\omega_{X_\Sigma})$. Abouazid then defined his tropical Lagrangian sections to be the Lagrangian sections over $Q$ with boundary in $M_\infty=\lim_{t\to\infty}M_t$. Now, given a Lagrangian section $L$ in $Y$ satisfying $(*_{[a]})$ for some $[a]\in A(W)$, we may regard $L$ as a Lagrangian section over $P$ (by writing $Y$ as $P\times\sqrt{-1}T_M$) and hence over the interior of $Q\subset M_{\mathbb{R}}\setminus\Pi$. Then $L$ is in the equivalence class of Abouzaid's Lagrangian section associated to the line bundle $\mathcal{L}_{[a]}$.*

[3] Indeed, we believe that the boundary conditions for Lagrangian sections used in [9] are equivalent to those we use here.

We now return to the general discussion of the set $\mathbb{L}(Y, W)$ of Lagrangian sections.

**Proposition 3.1.** *Any two Lagrangian sections $L_1, L_2 \in \mathbb{L}(Y, W)$ satisfying the same growth condition $(*_{[a]})$ can be deformed to each other through Hamiltonian isotopies which preserve $(*_{[a]})$.*

*Proof.* Choose lifts $\tilde{L}_1, \tilde{L}_2$ of $L_1, L_2$ respectively, such that they satisfy the same growth condition $(*_a)$, for some representative $a \in \mathbb{Z}^d$ of $[a]$. Let $g_1, g_2$ be the potentials of $\tilde{L}_1, \tilde{L}_2$ respectively. Regard $H := g_1 - g_2$ as a $T_M$-invariant function on $Y$. Then the Hamiltonian flow $\rho_t : Y \to Y$ associated to $H$ moves $L_1$ to $L_2$ at time $t = 1$, and $\rho_t(L_1)$ satisfies $(*_{[a]})$ for all $t$ because $H$, as a function on $N_{\mathbb{R}}$, satisfies $(*_0)$. $\square$

In view of this proposition, we define two Lagrangian sections $L_1, L_2 \in \mathbb{L}(Y, W)$ to be equivalent, denoted $L_1 \sim L_2$, if they satisfy the same growth condition $(*_{[a]})$; and we denote the equivalence class to which $L \in \mathbb{L}(Y, W)$ belongs by $[L]$.

Now rewrite $Y$ as $Y = P \times \sqrt{-1}T_M = TP/M$ and use the coordinates $x_j$'s and $y_j$'s to express a lift $\tilde{L} = \{(\xi, dg(\xi)) : \xi \in N_{\mathbb{R}}\}$ of the Lagrangian section $L$ as the graph of the gradient of the function $\Psi^* g$, with respect to the metric $\sum_{j,k=1}^n \psi_{jk} dx_j \otimes dx_k$ on $P$. In other words, we have $\tilde{L} = \{(x, y(x)) : x \in P, y(x) = \nabla(\Psi^* g)(x)\}$, or in coordinates,

$$y_j(x) = \sum_{k=1}^n \psi^{jk} \frac{\partial(\Psi^* g)}{\partial x_k}.$$

For any Lagrangian section $L$ of $\nu_Y : Y \to P$, define the *normalized slope* of $L$ by

$$\lambda(L) = \frac{1}{\mathrm{Vol}(P)} \int_P \sum_{j=1}^n \frac{\partial y_j(x)}{\partial x_j} dx_1 \wedge \ldots \wedge dx_n,$$

where $\tilde{L} = \{(x, y(x)) : x \in P\} \subset T^* N_{\mathbb{R}}$ is any lift of $L$ to $T^* N_{\mathbb{R}}$. $\lambda(L)$ is clearly independent of the choice of the lift $\tilde{L}$.

**Proposition 3.2.** *If $L_1 \sim L_2$, then $\lambda(L_1) = \lambda(L_2)$. Hence $\lambda$ is an invariant on the set of equivalence classes $\mathbb{L}(Y, W)/\sim$.*

*Proof.* As in the proof of the above proposition, we choose lifts $\tilde{L}_1, \tilde{L}_2$ of $L_1, L_2$ respectively such that they satisfy the same growth condition $(*_a)$, for some $a \in \mathbb{Z}^d$ representing $[a]$. Let $g_1, g_2$ be the potentials of $\tilde{L}_1, \tilde{L}_2$ respectively, and let $H := g_1 - g_2$. Set $y_j(x) = \sum_{k=1}^n \psi^{jk} \frac{\partial(\Psi^* H)}{\partial x_k}$. Then, for $j = 1, \ldots, n$, we have

$$\begin{aligned} \int_P \sum_{j=1}^n \frac{\partial y_j}{\partial x_j} dx_1 \wedge \ldots \wedge dx_n &= \int_P d\Big(\sum_{j=1}^n (-1)^{j-1} y_j dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n\Big) \\ &= \int_{\partial \bar{P}} \sum_{j=1}^n (-1)^{j-1} y_j dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n, \end{aligned}$$

by Stokes theorem. Consider a facet $F_k = \{x \in \bar{P} : l_k(x) = 0\}$ of $\bar{P}$. Without loss of generality, suppose that $v_k^n \neq 0$. Then use $x_1, \ldots, x_n$ as the coordinates on $F_k$, so that $x_n = -\frac{\lambda_k}{v_k^n} - \sum_{p=1}^{n-1} \frac{v_k^p}{v_k^n} x_p$. We have

$$\sum_{j=1}^{n} (-1)^{j-1} y_j dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n = \frac{(-1)^{n-1}}{v_k^n} \langle y(x), v_k \rangle dx_1 \wedge \ldots \wedge dx_{n-1}.$$

Now, since $H$ satisfies $(*_0)$, $\langle y(x), v_k \rangle = 0$ for $x \in F_k$. Hence $\int_P \sum_{j=1}^n \frac{\partial y_j}{\partial x_j} dx_1 \wedge \ldots \wedge dx_n = 0$, and we have $\lambda(L_1) = \lambda(L_2)$. □

**Definition 3.2.** *A Lagrangian section $L \in \mathbb{L}(Y, W)$ is said to be harmonic if the following Laplace-type equation is satisfied*

$$\sum_{j=1}^{n} \frac{\partial y_j(x)}{\partial x_j} = \lambda(L), \tag{3.1}$$

*for some lift $\tilde{L} = \{(x, y(x)) : x \in P\} \subset T^* N_{\mathbb{R}}$ of $L$.*

The equation (3.1) is equivalent to the following equation

$$\sum_{j,k=1}^{n} \psi^{jk} \left( \frac{\partial^2 (\Psi^* g)}{\partial x_j \partial x_k} - \sum_{p,q=1}^{n} \psi^{pq} \psi_{pjk} \frac{\partial (\Psi^* g)}{\partial x_q} \right) = \lambda(L)$$

on $P$, where $\psi_{pjk}$ denotes $\frac{\partial^3 \psi}{\partial x_p \partial x_j \partial x_k}$. If we regard $L = \{(\xi, dg(\xi)) : \xi \in N_{\mathbb{R}}\}$ as a section of $\mu_Y : Y \to N_{\mathbb{R}}$, then $L$ is harmonic if and only of $g$ is a solution to the equation

$$\sum_{j,k=1}^{n} \phi^{jk} \frac{\partial^2 g}{\partial \xi_j \partial \xi_k} = \lambda(L)$$

on $N_{\mathbb{R}}$. In the next section, we will see that in each equivalence class $[L] \in \mathbb{L}(Y, W)/\sim$ of Lagrangian sections, there exists a unique harmonic representative. This is mirror to the existence of a unique Hermitian-Einstein metric on each holomorphic line bundle over $X_\Sigma$, and $\lambda(L)$ is the mirror analogue of the (normalized) slope of a line bundle.

On the other hand, we may also choose special Lagrangian sections as representatives. According to the definition of Auroux [3], a Lagrangian submanifold $L \subset Y$ is special with phase $\theta \in \mathbb{R}$ if $\mathrm{Im}(e^{\sqrt{-1}\theta} \Omega_Y)|_L = 0$. In terms of the $x_j, y_j$ coordinates,

$$\begin{aligned}
\Omega_Y|_L &= \bigwedge_{j=1}^{n} \left( -dx_j + \sqrt{-1} dy_j(x) \right) \\
&= \bigwedge_{j=1}^{n} \left( \sum_{k=1}^{n} \left( -\delta_{jk} + \sqrt{-1} \frac{\partial y_j(x)}{\partial x_k} \right) dx_k \right) \\
&= \det \left( -I_n + \sqrt{-1} \Big( \frac{\partial y_j(x)}{\partial x_k} \Big)_{j,k=1}^{n} \right) dx_1 \wedge \ldots \wedge dx_n,
\end{aligned}$$

where $I_n$ denotes the $n \times n$ identity matrix. So $L = \{(x, y(x) : x \in P\}$ is special Lagrangian with phase $\theta \in \mathbb{R}$ if and only if the following equation is satisfied

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\det\left(I_n - \sqrt{-1}\Big(\frac{\partial y_j(x)}{\partial x_k}\Big)^n_{j,k=1}\right)\right) = 0. \tag{3.2}$$

Equivalently, this means $\Psi^* g$ satisfies the equation

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\det\left(I_n - \sqrt{-1}\left[\sum_{l=1}^{n}\psi^{jl}\left(\frac{\partial^2(\Psi^*g)}{\partial x_l \partial x_k} - \sum_{p,q=1}^{n}\psi^{pq}\psi_{plk}\frac{\partial(\Psi^*g)}{\partial x_q}\right)\right]^n_{j,k=1}\right)\right) = 0.$$

or, in the $\xi_j, y_j$ coordinates, $g$ satisfies the equation

$$\operatorname{Im}\left(e^{\sqrt{-1}\theta}\det\left(I_n - \sqrt{-1}\left(\sum_{l=1}^{n}\phi^{kl}\frac{\partial^2 g}{\partial\xi_j\partial\xi_l}\right)^n_{j,k=1}\right)\right) = 0.$$

Our harmonic Lagrangians are closely related to special Lagrangians, at least in the large radius limit: If we rescale the fiber coordinates by replacing $y_j$ by $\epsilon y_j$, then, for small $\epsilon$, the leading term of equation (3.2) will give

$$\sum_{j=1}^{n}\frac{\partial y_j(x)}{\partial x_j} = \frac{1}{\epsilon}\tan\theta,$$

which is nothing but equation (3.1) if we choose $\theta$ such that $\tan\theta = \epsilon\lambda(L)$.

## 4. The SYZ mirror transformation as a geometric correspondence

In this section, we first recall the definition of the SYZ mirror transformation. Then we proceed to prove our main result.

For $[a] \in H^2(X_\Sigma, \mathbb{Z})$, let $\mathcal{L}_{[a]}$ be the corresponding holomorphic line bundle over $X_\Sigma$. Choose a $T_N$-equivariant meromorphic section $s$ of $\mathcal{L}_{[a]}$. Then $\operatorname{div}(s) = \sum_{i=1}^{d} a_i D_i$, for some integers $a_1, \ldots, a_d \in \mathbb{Z}$ such that $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ gives a representative of the class $a$. Note that $s$ is holomorphic and nowhere vanishing over $U \subset X_\Sigma$, so it is a holomorphic frame of $\mathcal{L}_{[a]}|_U$.

Let $h$ be a $T_N$-invariant hermitian metric of class $C^2$ on $\mathcal{L}_{[a]}$. The Chern connection $\nabla_h$ is given by $\nabla_h = d + \partial \log h(s,s)$ over $U$. If we define a function $g_h : N_\mathbb{R} \to \mathbb{R}$ by setting

$$g_h(\xi) = -\frac{1}{2}\log h(s(e^{\xi+\sqrt{-1}u}), s(e^{\xi+\sqrt{-1}u})),$$

then the restriction of $\nabla_h$ to a fiber $F_\xi := \nu_U^{-1}(\xi) \cong T_N$ gives a flat $U(1)$-connection

$$d + \frac{\sqrt{-1}}{2}\sum_{j=1}^{n}\frac{\partial \log h(s,s)}{\partial \xi_j}du_j = d - \sqrt{-1}\sum_{j=1}^{n}\frac{\partial g_h}{\partial \xi_j}du_j$$

on the trivial line bundle $\underline{\mathbb{C}}$ over $T_N$. Recall that the dual torus $T_M = (T_N)^*$ can be interpreted as the space of flat $U(1)$-connections on the trivial line bundle $\underline{\mathbb{C}}$ over $T_N$ modulo gauge equivalence.[4] In our situation, the connection

[4]This is in fact the starting point of the SYZ conjecture [18]

$d - \sqrt{-1}\sum_{j=1}^{n} \frac{\partial g_h}{\partial \xi_j} du_j$ corresponds to the point $(\frac{\partial g_h}{\partial \xi_1}, \ldots, \frac{\partial g_h}{\partial \xi_n}) \in T_M$. Hence, the hermitian metric $h$, or the Chern connection $\nabla_h$, determines a section

$$\tilde{L}_h = \{(\xi, dg_h(\xi)) = (\xi_1, \ldots, \xi_n, \frac{\partial g_h}{\partial \xi_1}, \ldots, \frac{\partial g_h}{\partial \xi_n}) : \xi \in N_{\mathbb{R}}\}$$

of $T^* N_{\mathbb{R}} = N_{\mathbb{R}} \times \sqrt{-1} M_{\mathbb{R}} \to N_{\mathbb{R}}$, which is Lagrangian since $\nabla_h$ is holomorphic (see [15]). $\tilde{L}_h$ descends to give a Lagrangian section $L_h$ of $\mu_Y : Y \to N_{\mathbb{R}}$.

If $s'$ is another $T_N$-equivariant meromorphic section of $\mathcal{L}_a$, then $s' = cw^u \cdot s$, for some constant $c \in \mathbb{C}^*$ and $u \in M$, where $w^u$ is the monomial $w_1^{u^1} \ldots w_n^{u^n}$. Since $h(s'(w), s'(w)) = |cw^u|^2 h(s(w), s(w)) = |c|^2 e^{2\langle u, \xi \rangle} h(s(w), s(w))$, we have $g'_h(\xi) = -\log|c| - \langle u, \xi \rangle + g_h(\xi)$, where $g'_h := -\frac{1}{2}\log h(s', s')$. So $dg'_h(\xi) = dg_h(\xi) - u$. This gives a different Lagrangian section $\tilde{L}'_h = \{(\xi, dg'_h(\xi)) : \xi \in N_{\mathbb{R}}\}$ in $T^* N_{\mathbb{R}}$, but it descends to the same Lagrangian section $L_h$ in $Y$.

Thus we have a well-defined transformation

$$\mathcal{F} : h \mapsto L_h$$

from the set of $T_N$-invariant hermitian metrics on holomorphic line bundles over $X_\Sigma$ to the set of Lagrangian sections of $\mu_Y : Y \to N_{\mathbb{R}}$. This is called the *SYZ mirror transformation*. This is (fiberwise) a real version of the Fourier-Mukai transform in algebraic geometry. We can invert the construction and define the inverse SYZ mirror transformation $\mathcal{F}^{-1}$, which produces, from a Lagrangian section $L$ of $\mu_Y : Y \to N_{\mathbb{R}}$, a $T_N$-invariant hermitian metric $h_L := \mathcal{F}^{-1}(L)$ on a holomorphic line bundle *over* $U$. However, $h_L$ may not be extended to a hermitian metric on a holomorphic line bundle over $X_\Sigma$. The question we raised in the introduction is to characterize the set of Lagrangian sections $L$ for which $h_L$ can be extended over $X_\Sigma$. Our main result says that this set is precisely $\mathbb{L}(Y, W)$, which we introduced in the last section.

**Theorem 4.1.** *The image of the SYZ mirror transformation $\mathcal{F}$ is $\mathbb{L}(Y, W)$, i.e. for a Lagrangian section $L$ of $\mu_Y : Y \to N_{\mathbb{R}}$, there exists a $T_N$-invariant hermitian metric $h$ on a holomorphic line bundle over $X_\Sigma$ such that $L = L_h = \mathcal{F}(h)$ if and only if $L$ satisfies the growth condition $(*_{[a]})$ for some $[a] \in A(W)$.*

Before we prove the theorem, we need a couple of lemmas. Let $[a]$ be an element in $A(W) = H^2(X_\Sigma, \mathbb{Z})$ and $\mathcal{L}_{[a]}$ the corresponding holomorphic line bundle over $X_\Sigma$. We first consider a particular $T_N$-invariant hermitian metric $h_0$ on $\mathcal{L}_{[a]}$ defined as follows. Choose a representative $(a_1, \ldots, a_n) \in \mathbb{Z}^d$ of $a$, and fix a $T_N$-equivariant meromorphic section $s$ of $\mathcal{L}_{[a]}$ such that $\mathrm{div}(s) = D_a = \sum_{i=1}^{d} a_i D_i$, so that we can canonically identify $\mathcal{L}_{[a]}$ with the toric divisor line bundle $\mathcal{O}(D_a)$. Recall that the moment map $\mu_U : U \to P$ is given by

$$\mu_U(w) = d\phi(\log|w_1|, \ldots, \log|w_n|) = \frac{\sum_{u \in \bar{P} \cap M} c_u |w^u|^2 \cdot u}{\sum_{u \in \bar{P} \cap M} c_u |w^u|^2},$$

for $w = (w_1, \ldots, w_n) \in U = (\mathbb{C}^*)^n$. For $i = 1, \ldots, d$, let $l_i : M_{\mathbb{R}} \to \mathbb{R}$ be the function defined by $l_i(x) = \langle x, v_i \rangle + \lambda_i$. In [12], Guillemin showed that there is a $T_N$-invariant hermitian metric $h_0$ on $\mathcal{L}_{[a]}$ such that

$$h_0(s, s) = \prod_{i=1}^{d} (l_i \circ \mu_U)^{a_i}.$$

**Lemma 4.1.** *$\tilde{L}_{h_0}$ satisfies the growth condition $(*_a)$*

The proof of this lemma, which is a straightforward but lengthy calculation, will be given in the appendix.

To describe the other lemma we require, consider the diagonal $T^n$-action on $\mathbb{C}^n$. If $F : \mathbb{C}^n \to \mathbb{R}$ is a $T^n$-invariant function, then we can define a function $f : \mathbb{R}^n \to \mathbb{R}$, by $f(\xi_1, \ldots, \xi_n) = F(e^{\xi_1+\sqrt{-1}u_1}, \ldots, e^{\xi_n+\sqrt{-1}u_n})$, where $w_j = e^{\xi_j+\sqrt{-1}u_j}$, $j = 1, \ldots, n$, are the complex coordinates on $\mathbb{C}^n$. But not all functions on $\mathbb{R}^n$ come from this way.

**Lemma 4.2.** *Given a function $f \in C^2(\mathbb{R}^n)$. Define $F : (\mathbb{C}^*)^n \to \mathbb{R}$ by $F(w_1, \ldots, w_n) = f(\log|w_1|, \ldots, \log|w_n|)$. Then $F$ can be extended to a $T^n$-invariant $C^2$ function on $\mathbb{C}^n$ if and only if the following three conditions are satisfied*

1. *For $j = 1, \ldots, n$, $e^{-2\xi_j}\frac{\partial^2 f}{\partial \xi_j^2}$ and $2e^{-2\xi_j}\frac{\partial f}{\partial \xi_j}$ go to the same limit as $\xi_j \to -\infty$.*
2. *For any $j, k, l \in \{1, \ldots, n\}$, the limit of $\frac{\partial^2 f}{\partial \xi_j \partial \xi_k}$ exists as $\xi_l \to -\infty$.*
3. *For any distinct $j, k \in \{1, \ldots, n\}$, $e^{-\xi_j-\xi_k}\frac{\partial^2 f}{\partial \xi_j \partial \xi_k}$ goes to zero as $\xi_j \to -\infty$ or $\xi_k \to -\infty$.*

*Proof.* Write $e^{\xi_j+\sqrt{-1}u_j} = w_j = x_j + \sqrt{-1}y_j$. Then, by the chain rule, we have, for $j = 1, \ldots, n$,

$$\begin{aligned}
\frac{\partial F}{\partial x_j} &= e^{-\xi_j}\cos u_j \frac{\partial f}{\partial \xi_j}, \ \frac{\partial F}{\partial y_j} = e^{-\xi_j}\sin u_j \frac{\partial f}{\partial \xi_j}, \\
\frac{\partial^2 F}{\partial x_j^2} &= e^{-2\xi_j}\cos^2 u_j (\frac{\partial^2 f}{\partial \xi_j^2} - 2\frac{\partial f}{\partial \xi_j}) + e^{-2\xi_j}\frac{\partial f}{\partial \xi_j}, \\
\frac{\partial^2 F}{\partial x_j \partial y_j} &= e^{-2\xi_j}\cos u_j \sin u_j (\frac{\partial^2 f}{\partial \xi_j^2} - 2\frac{\partial f}{\partial \xi_j}), \\
\frac{\partial^2 F}{\partial y_j^2} &= e^{-2\xi_j}\sin^2 u_j (\frac{\partial^2 f}{\partial \xi_j^2} - 2\frac{\partial f}{\partial \xi_j}) + e^{-2\xi_j}\frac{\partial f}{\partial \xi_j},
\end{aligned}$$

and, for $j \neq k$,

$$\begin{aligned}
\frac{\partial^2 F}{\partial x_j \partial x_k} &= e^{-\xi_j-\xi_k}\cos u_j \cos u_k \frac{\partial^2 f}{\partial \xi_j \partial \xi_k}, \\
\frac{\partial^2 F}{\partial x_j \partial y_k} &= e^{-\xi_j-\xi_k}\cos u_j \sin u_k \frac{\partial^2 f}{\partial \xi_j \partial \xi_k}, \\
\frac{\partial^2 F}{\partial y_j \partial y_k} &= e^{-\xi_j-\xi_k}\sin u_j \sin u_k \frac{\partial^2 f}{\partial \xi_j \partial \xi_k}.
\end{aligned}$$

It is then not hard to see that the conditions (1)-(3) are necessary and sufficient conditions for extending $F$ to $\mathbb{C}^n$. □

*Proof of Theorem 4.1.* Let $h$ be any other $T_N$-invariant $C^2$ hermitian metric of $\mathcal{L}_{[a]}$. Then there is a function $F \in C^2(X_\Sigma)$ such that $h = e^{-2F}h_0$. Restrict $F$ to $U \subset X_\Sigma$, and define $f : N_\mathbb{R} \to \mathbb{R}$ by $f(\xi_1, \ldots, \xi_n) = F(e^{\xi_1+\sqrt{-1}u_1}, \ldots, e^{\xi_n+\sqrt{-1}u_n})$. Let $\sigma \in \Sigma$ be an $n$-dimensional cone, and $U_\sigma = \operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M]$ the corresponding affine toric variety. $X_\Sigma$ is covered by these $U_\sigma$'s, and since $X_\Sigma$ is nonsingular, $U_\sigma \cong \mathbb{C}^n$. Without loss of generality, suppose that the generators of $\sigma$ are $v_1, \ldots, v_n \in N$. They give a $\mathbb{Z}$-basis of $N$. Let $\tilde{w}_1 = e^{\tilde{\xi}_1+\sqrt{-1}\tilde{u}_1}, \ldots, \tilde{w}_n = e^{\tilde{\xi}_n+\sqrt{-1}\tilde{u}_n}$ be the corresponding (inhomogeneous) complex coordinates on $U_\sigma$. This gives coordinates $\tilde{\xi}_1, \ldots, \tilde{\xi}_n$ on $N_\mathbb{R}$, and the transformation from these coordinates to the original coordinates $\xi_1, \ldots, \xi_n$ is given by

$$\xi = (\xi_1, \ldots, \xi_n) = v_1\tilde{\xi}_1 + \ldots + v_n\tilde{\xi}_n.$$

Apply the chain rule, we get

$$\begin{aligned}\frac{\partial f}{\partial \tilde{\xi}_i} &= \sum_{k=1} \frac{\partial f}{\partial \xi_k}\frac{\partial \xi_k}{\partial \tilde{\xi}_i} = \sum_{k=1} v_i^k \frac{\partial f}{\partial \xi_k} = \langle df, v_i \rangle, \\ \frac{\partial^2 f}{\partial \tilde{\xi}_j \partial \tilde{\xi}_k} &= \sum_{p,q=1}^n v_j^p v_k^q \frac{\partial^2 f}{\partial \xi_p \partial \xi_q} = v_j^T \mathrm{Hess}(f) v_k.\end{aligned}$$

Hence, by Lemma 4.2, we conclude that the function $f$ satisfies the growth condition $(*_0)$. Now, by Lemma 4.1, $\tilde{L}_{h_0}$ satisfies the growth condition $(*_a)$. Since $g_h = g_{h_0} + f$, we see that $\tilde{L}_h$ also satisfies $(*_a)$.

Conversely, let $L$ be a Lagrangian section in $Y$ satisfying $(*_{[a]})$. Choose a lift $\tilde{L} = \{(\xi, dg(\xi)) : \xi \in N_\mathbb{R}\} \subset T^*N_\mathbb{R}$ of $L$ which satisfies the same growth condition $(*_a)$ as $\tilde{L}_{h_0}$. Then the $C^2$ function $f := g - g_{h_0} : N_\mathbb{R} \to \mathbb{R}$ satisfies the growth condition $(*_0)$. By the above argument and Lemma 4.2, $f$ extends to a function $F \in C^2(X_\Sigma)$. So $h := e^{-2F}h_0$ defines a $T_N$-invariant hermitian metric on $\mathcal{L}_{[a]}$. This completes the proof of the theorem. □

Theorem 4.1 establishes a bijective correspondence between $T_N$-invariant hermitian metrics on the holomorphic line bundle $\mathcal{L}_{[a]}$ over $X_\Sigma$ and Lagrangian sections of $(Y, W)$ satisfying the growth condition $(*_{[a]})$, for any $[a] \in H^2(X_\Sigma, \mathbb{Z}) = A(W)$. In addition, by our definition in Section 3, two Lagrangian sections $L_1, L_2 \in \mathbb{L}(Y, W)$ are equivalent, denoted $L_1 \sim L_2$, if and only if they satisfies the same growth condition. Hence, an immediate consequence of our main result is the following

**Corollary 4.1.** *The SYZ mirror transformation $\mathcal{F}$ induces a bijective map*

$$\mathcal{F} : Pic(X_\Sigma) \xrightarrow{\cong} (\mathbb{L}(Y, W)/\sim).$$

Recall that a hermitian metric $h$ on the line bundle $\mathcal{L}_{[a]}$ is Hermitian-Einstein, with respect to the Kähler metric $\omega_{X_\Sigma}$ on $X_\Sigma$, if and only if the following equation is satisfied

$$\sqrt{-1}F_h \wedge \omega_{X_\Sigma}^{n-1} = \frac{\lambda(\mathcal{L}_{[a]})}{n} \cdot \omega_{X_\Sigma}^n,$$

where $F_h$ is the curvature of the Chern connection $\nabla_h$, and $\lambda(\mathcal{L}_{[a]})$ is the *normalized slope* of $\mathcal{L}_{[a]}$ defined by

$$\lambda(\mathcal{L}_{[a]}) := \frac{n \cdot \int_{X_\Sigma} \sqrt{-1}F_h \wedge \omega_{X_\Sigma}^{n-1}}{\int_{X_\Sigma} \omega_{X_\Sigma}^n} = \frac{2\pi n \mu(\mathcal{L}_{[a]})}{\int_{X_\Sigma} \omega_{X_\Sigma}^n}.$$

Now let $y_j = \frac{\partial g_h}{\partial \xi_j} = \sum_{k=1}^n \psi^{jk} \frac{\partial \Psi^* g_h}{\partial x_k}$, then, restricting to $U \subset X_\Sigma$, we have

$$\sqrt{-1}F_h = \bar{\partial}\partial \log h = \sum_{j=1}^n dy_j \wedge du_j.$$

Hence,

$$\begin{aligned}
\sqrt{-1}F_h \wedge \omega_{X_\Sigma}^{n-1} &= (\sum_{j=1}^n dy_j \wedge du_j) \wedge (\sum_{j=1}^n dx_j \wedge du_j)^{n-1} \\
&= (n-1)!(\sum_{j=1}^n \frac{\partial y_j}{\partial x_j}) \bigwedge_{k=1}^n (dx_k \wedge du_k) \\
\omega_{X_\Sigma}^n &= n! \bigwedge_{k=1}^n (dx_k \wedge du_k).
\end{aligned}$$

From this, we see that

**Corollary 4.2.** *$\lambda(\mathcal{L}_{[a]}) = \lambda(L_h)$ and $h$ is Hermitian-Einstein if and only the Lagrangian section $L_h$ is harmonic. In particular, each equivalence class $[L] \in \mathbb{L}(Y, W)/\sim$ is represented by a unique harmonic Lagrangian section.*

On the other hand, the condition for preserving supersymmetry is given by the following *MMMS equation*, introduced by Marino-Minasian-Moore-Strominger in [16] (see also [15]):

$$\mathrm{Im}\, e^{\sqrt{-1}\theta}(F_h + \omega_{X_\Sigma})^n = 0,$$

for some $\theta \in \mathbb{R}$. Since

$$(F_h + \omega_{X_\Sigma})^n = (\sum_{j=1}^n (dx_j - \sqrt{-1}dy_j(x)) \wedge du_j)^n = \pm(\Omega_Y|_L) \wedge du_1 \wedge \ldots \wedge du_n,$$

$h$ satisfies the MMMS equation with $\theta \in \mathbb{R}$ if and only if $L_h$ is special Lagrangian with phase $\theta$.

## 5. Further remarks

We end this paper by several remarks.

**1.** For our purposes, we consider $C^2$ hermitian metrics and Lagrangian sections whose potential are $C^2$ functions. One can certainly consider metrics and Lagrangians in other differentiability classes, but then the growth conditions should be suitably modified.

In particular, when we only require the metrics to be $C^0$, *singular* Lagrangians can arise as follows. Given a divisor $\sum_{i=1}^d a_i D_i$ in $X_\Sigma$. Then for every $n$-dimensional cone $\sigma \in \Sigma[n]$, we can find a unique $u_\sigma \in M$ such that $\langle u_\sigma, v_i \rangle = -a_i$ for all $v_i \in \sigma$. This defines a piecewise linear function $\varphi : N_\mathbb{R} \to \mathbb{R}$ by $\varphi(\xi) = \langle u_\sigma, \xi \rangle$, for $\xi \in \sigma$. Let $[a] \in H^2(X_\Sigma, \mathbb{Z})$ be the class represented by $a = (a_1, \ldots, a_d)$. Then there is a $T_N$-invariant $C^0$ hermitian metric $h$ on the line bundle $\mathcal{L}_{[a]}$ such that $g_h(\xi) = \varphi(-\xi)$, and $dg_h : N_\mathbb{R} \to M_\mathbb{R}$ is the piecewise constant map given by $dg_h(\xi) = -u_\sigma$ for all $\xi \in \sigma$. Applying the SYZ mirror transformation, we get a singular Lagrangian $L_h = \mathcal{F}(h) \subset Y$. This satisfies the following boundary condition at infinity: for $\xi(t) = t_i v_i + \ldots$, $\langle dg_h(\xi(t)), v_i \rangle + a_i = 0$ for $t_i$ sufficiently negative. In this case, different line bundles may give rise to the same Lagrangian subspace. For example, $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ both transformed to the Lagrangian $L$, which is the zero section plus the fiber over $\xi = 0 \in \mathbb{R}$. One can distinguish the Lagrangian cycles corresponding to $\mathcal{O}(1)$ and $\mathcal{O}(-1)$ by equipping the circle fiber with different orientations. In this way, the SYZ mirror transformation would still give a bijective correspondence between isomorphism classes of holomorphic line bundles over $X_\Sigma$ and equivalence classes of Lagrangian sections of $(Y, W)$.

**2.** The SYZ mirror transformation we discuss in this note only gives a bijective correspondence between holomorphic line bundles over $X_\Sigma$ and Lagrangian sections of $(Y, W)$. But it should be extended to an equivalence between the derived category of coherent sheaves $D^bCoh(X_\Sigma)$ and a suitable variant of the Fukaya-Kontsevich-Seidel category of $(Y, W)$. In particular, it is interesting to see how higher rank holomorphic vector bundles over $X_\Sigma$ can be transformed to Lagrangian multi-sections of $(Y, W)$ equipped with certain extra data. We plan to address this in the future.

**3.** Since we equip $Y$ with the dual of the toric metric, it is not always possible to represent an equivalence class $[L] \in \mathbb{L}(Y, W)/\sim$ by a *minimal* Lagrangian section. The mirror of $\mathbb{C}P^1$ provides the simplest example of this. Our way out is to introduce the notion of harmonic Lagrangians, and as a corollary to our main result, we saw that each equivalence class $[L]$ is indeed represented by a unique harmonic representative. However, it would also interesting to look at the variational theory of Lagrangian sections of $(Y, W)$ *directly*. For example, one may attempt to prove the existence and uniqueness of harmonic Lagrangian sections by directly solving the PDE (3.1). On the other hand, the existence and uniqueness of the solutions of the MMMS equation and the special Lagrangian equation are largely unexplored. The toric case we considered here should be the first nontrivial case for one to investigate these equations.

## Appendix A

In this appendix, we give a proof of Lemma 4.1, which is restated as follows:

**Lemma A.1** (=Lemma 4.1)**.** *$\tilde{L}_{h_0}$ satisfies the growth condition $(*_a)$, i.e. the function $g_{h_0} : N_\mathbb{R} \to \mathbb{R}$ defined by $g_{h_0} = -\frac{1}{2}\log h_0(s,s)$ satisfies the following conditions: Given any $n$-dimensional cone $\sigma \in \Sigma$. Suppose that, without loss of generality, $\sigma$ is generated by $v_1, \ldots, v_n$; and let $\xi(t) = \xi(t_1, \ldots, t_n) = t_1v_1 + \ldots + t_nv_n$, for $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$. Then, we have,*

1. *the functions $2e^{-2t_j}(\langle dg(\xi(t)), v_j\rangle + a_j)$ and $e^{-2t_j}(v_j^T Hess(g)v_j)(\xi(t))$ have the same limit as $t_j \to -\infty$, for $j = 1, \ldots, n$;*
2. *for any $j, k, l \in \{1, \ldots, n\}$, the function $(v_j^T Hess(g)v_k)(\xi(t))$ has a limit as $t_l \to -\infty$; and,*
3. *for any distinct $j, k \in \{1, \ldots, n\}$, the function $e^{-t_j-t_k}(v_j^T Hess(g)v_k)(\xi(t))$ goes to zero when $t_j \to -\infty$ or $t_k \to -\infty$.*

*Proof.* By definition, we have $g_{h_0} = -\frac{1}{2}\log h_0(s,s) = -\frac{1}{2}\sum_{i=1}^d a_i \log(l_i \circ \mu_U)$, so that

$$g_{h_0}(\xi) = -\frac{1}{2}\sum_{i=1}^d a_i \log\left(\frac{\sum_{u\in\bar{P}\cap M} c_u l_i(u) e^{2\langle u,\xi\rangle}}{\sum_{u\in\bar{P}\cap M} c_u e^{2\langle u,\xi\rangle}}\right).$$

The first-order partial derivatives are given by

$$\frac{\partial g_{h_0}}{\partial \xi_j} = \sum_{i=1}^{d} a_i \left( \frac{\sum_{u\in\bar{P}\cap M} c_u u^j e^{2\langle u,\xi\rangle}}{\sum_{u\in\bar{P}\cap M} c_u e^{2\langle u,\xi\rangle}} - \frac{\sum_{u\in\bar{P}\cap M} c_u l_i(u) u^j e^{2\langle u,\xi\rangle}}{\sum_{u\in\bar{P}\cap M} c_u l_i(u) e^{2\langle u,\xi\rangle}} \right),$$

for $j = 1, \ldots, n$. Then, for $k = 1, \ldots, n$,

$$\begin{aligned} e^{-2t_k}(\langle dg_{h_0}(\xi(t)), v_k\rangle + a_k) \quad = \quad & \sum_{i=1}^{d} a_i \Bigg[ \frac{\sum_{l_k(u)\geq 1} l_k(u) b_u e^{2(l_k(u)-1)t_k}}{\sum_{u\in\bar{P}\cap M} b_u e^{2l_k(u)t_k}} \\ & - \frac{\sum_{l_k(u), l_i(u)\geq 1} l_k(u) l_i(u) b_u e^{2(l_k(u)-1)t_k}}{\sum_{l_i(u)\geq 1} l_i(u) b_u e^{2l_k(u)t_k}} + \delta_{ik} e^{-2t_k} \Bigg], \end{aligned}$$

where

$$\begin{aligned} b_u &= b_u(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n) \\ &= c_u e^{2(l_1(u)t_1 + \ldots + l_{k-1}(u)t_{k-1} + l_{k+1}(u)t_{k+1} + \ldots + l_n(u)t_n)}. \end{aligned}$$

Since, for each $i = 1, \ldots, d$, there exists $u \in \bar{P} \cap M$ with $l_i(u) = 1$ and $l_1(u) = \ldots = l_{k-1}(u) = l_{k+1}(u) = \ldots = l_n(u) = 0$, the limit of the function $e^{-2t_k}(\langle dg_{h_0}(\xi(t)), v_k\rangle + a_k)$ always exists as $t_m$ goes to $-\infty$ for any $m = 1, \ldots, k-1, k+1, \ldots, n$. Similar arguments apply to other functions below. Now, as $t_k \to -\infty$, the terms with the lowest powers of $e^{t_k}$ dominate. Also note that, for $i = 1, \ldots, d$, there exists $u \in \bar{P} \cap M$ such that $l_i(u) = 1$. So the function $e^{-2t_k}(\langle dg_{h_0}(\xi(t)), v_k\rangle + a_k)$ has a limit given by

$$\Big(\sum_{i=1}^{d} a_i\Big) \left( \frac{\sum_{l_k(u)=1} b_u}{\sum_{l_k(u)=0} b_u} \right) - \sum_{i\neq k} a_i \left( \frac{\sum_{l_k(u)=1, l_i(u)\geq 1} l_i(u) b_u}{\sum_{l_k(u)=0, l_i(u)\geq 1} l_i(u) b_u} \right) - 2a_k \left( \frac{\sum_{l_k(u)=2} b_u}{\sum_{l_k(u)=1} b_u} \right).$$

The second-order partial derivatives are given by

$$\begin{aligned} \frac{\partial^2 g_{h_0}}{\partial \xi_p \partial \xi_q} \quad = \quad & 2\sum_{i=1}^{d} a_i \Bigg[ \frac{\sum_{u\in\bar{P}\cap M} c_u u^p u^q e^{2\langle u,\xi\rangle}}{\sum_{u\in\bar{P}\cap M} c_u e^{2\langle u,\xi\rangle}} \\ & - \frac{\big(\sum_{u\in\bar{P}\cap M} c_u u^p e^{2\langle u,\xi\rangle}\big)\big(\sum_{u\in\bar{P}\cap M} c_u u^q e^{2\langle u,\xi\rangle}\big)}{\big(\sum_{u\in\bar{P}\cap M} c_u e^{2\langle u,\xi\rangle}\big)^2} \\ & - \frac{\sum_{u\in\bar{P}\cap M} c_u l_i(u) u^p u^q e^{2\langle u,\xi\rangle}}{\sum_{u\in\bar{P}\cap M} c_u l_i(u) e^{2\langle u,\xi\rangle}} \\ & + \frac{\big(\sum_{u\in\bar{P}\cap M} c_u l_i(u) u^p e^{2\langle u,\xi\rangle}\big)\big(\sum_{u\in\bar{P}\cap M} c_u l_i(u) u^q e^{2\langle u,\xi\rangle}\big)}{\big(\sum_{u\in\bar{P}\cap M} c_u l_i(u) e^{2\langle u,\xi\rangle}\big)^2} \Bigg], \end{aligned}$$

for $p, q = 1, \ldots, n$. From this we compute, for $j, k \in \{1, \ldots, n\}$,

$$
\begin{aligned}
& v_j^T \mathrm{Hess}(g_{h_0}) v_k \\
= \quad & \sum_{p,q=1}^{n} v_j^p v_k^q \frac{\partial^2 g_{h_0}}{\partial \xi_p \partial \xi_q} \\
= \quad & 2 \sum_{i=1}^{d} a_i \Bigg[ \frac{\sum_{l_j(u), l_k(u) \geq 1} c_u l_j(u) l_k(u) e^{2\langle u, \xi \rangle}}{\sum_{u \in \bar{P} \cap M} c_u e^{2\langle u, \xi \rangle}} \\
& - \frac{\left( \sum_{l_j(u) \geq 1} c_u l_j(u) e^{2\langle u, \xi \rangle} \right) \left( \sum_{l_k(u) \geq 1} c_u l_k(u) e^{2\langle u, \xi \rangle} \right)}{\left( \sum_{u \in \bar{P} \cap M} c_u e^{2\langle u, \xi \rangle} \right)^2} \\
& - \frac{\sum_{l_i(u), l_j(u), l_k(u) \geq 1} c_u l_i(u) l_j(u) l_k(u) e^{2\langle u, \xi \rangle}}{\sum_{l_i(u) \geq 1} c_u l_i(u) e^{2\langle u, \xi \rangle}} \\
& + \frac{\left( \sum_{l_i(u), l_j(u) \geq 1} c_u l_i(u) l_j(u) e^{2\langle u, \xi \rangle} \right) \left( \sum_{l_i(u), l_k(u) \geq 1} c_u l_i(u) l_k(u) e^{2\langle u, \xi \rangle} \right)}{\left( \sum_{l_i(u) \geq 1} c_u l_i(u) e^{2\langle u, \xi \rangle} \right)^2} \Bigg].
\end{aligned}
$$

It is easy to see that, for any $l = 1, \ldots, n$, as $t_l \to -\infty$, the limit of the function $(v_j^T \mathrm{Hess}(g_{h_0}) v_k)(\xi(t))$ always exists. When $j \neq k$, let

$$
\begin{aligned}
b_u &= b_u(t_1, \ldots, \widehat{t_j}, \ldots, \widehat{t_k}, \ldots, t_n) \\
&= c_u e^{2(l_1(u)t_1 + \ldots + \widehat{l_j(u)t_j} + \ldots + \widehat{l_k(u)t_k} + \ldots + l_n(u)t_n)}.
\end{aligned}
$$

Then the function $(v_j^T \mathrm{Hess}(g_{h_0}) v_k)(\xi(t))$ is equal to the following expression

$$
\begin{aligned}
& 2 \sum_{i=1}^{d} a_i \Bigg[ \frac{\sum_{l_j(u), l_k(u) \geq 1} l_j(u) l_k(u) b_u e^{2(l_j(u)t_j + l_k(u)t_k)}}{\sum_{u \in \bar{P} \cap M} b_u e^{2(l_j(u)t_j + l_k(u)t_k)}} \\
& - \frac{\left( \sum_{l_j(u) \geq 1} l_j(u) b_u e^{2(l_j(u)t_j + l_k(u)t_k)} \right) \left( \sum_{l_k(u) \geq 1} l_k(u) b_u e^{2(l_j(u)t_j + l_k(u)t_k)} \right)}{\left( \sum_{u \in \bar{P} \cap M} b_u e^{2(l_j(u)t_j + l_k(u)t_k)} \right)^2} \\
& - \frac{\sum_{l_i(u), l_j(u), l_k(u) \geq 1} l_i(u) l_j(u) l_k(u) b_u e^{2(l_j(u)t_j + l_k(u)t_k)}}{\sum_{l_i(u) \geq 1} l_i(u) b_u e^{2(l_j(u)t_j + l_k(u)t_k)}} \\
& + \frac{\left( \sum_{l_i(u), l_j(u) \geq 1} l_i(u) l_j(u) b_u e^{2(l_j(u)t_j + l_k(u)t_k)} \right) \left( \sum_{l_i(u), l_k(u) \geq 1} l_i(u) l_k(u) b_u e^{2(l_j(u)t_j + l_k(u)t_k)} \right)}{\left( \sum_{l_i(u) \geq 1} l_i(u) b_u e^{2(l_j(u)t_j + l_k(u)t_k)} \right)^2} \Bigg].
\end{aligned}
$$

Notice that in each term, the numerator is $O(e^{2t_j + 2t_k})$, while the denominator is $O(1)$. Thus, the function $e^{-t_j - t_k}(v_j^T \mathrm{Hess}(g_{h_0}) v_k)(\xi(t))$ goes to zero as $t_j \to -\infty$ or $t_k \to -\infty$.

For $j = k$, let

$$
\begin{aligned}
b_u &= b_u(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n) \\
&= c_u e^{2(l_1(u)t_1 + \ldots + l_{k-1}(u)t_{k-1} + l_{k+1}(u)t_{k+1} + \ldots + l_n(u)t_n)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
e^{-2t_k}(v_k^T \mathrm{Hess}(g_{h_0})v_k)(\xi(t)) \quad = \quad & 2e^{-2t_k}\sum_{i=1}^{d} a_i \left[ \frac{\sum_{l_k(u)\geq 1} l_k(u)^2 b_u e^{2l_k(u)t_k}}{\sum_{u\in\bar{P}\cap M} b_u e^{2l_k(u)t_k}} \right. \\
& - \left( \frac{\sum_{l_k(u)\geq 1} l_k(u) b_u e^{2l_k(u)t_k}}{\sum_{u\in\bar{P}\cap M} b_u e^{2l_k(u)t_k}} \right)^2 \\
& - \frac{\sum_{l_i(u),l_k(u)\geq 1} l_i(u) l_k(u)^2 b_u e^{2l_k(u)t_k}}{\sum_{l_i(u)\geq 1} l_i(u) b_u e^{2l_k(u)t_k}} \\
& \left. + \left( \frac{\sum_{l_i(u),l_k(u)\geq 1} l_i(u) l_k(u) b_u e^{2l_k(u)t_k}}{\sum_{l_i(u)\geq 1} l_i(u) b_u e^{2l_k(u)t_k}} \right)^2 \right].
\end{aligned}
$$

As $t_k \to -\infty$, the function $e^{-2t_k}(v_k^T \mathrm{Hess}(g_{h_0})v_k)(\xi(t))$ has a limit given by

$$
2\Big(\sum_{i=1}^{d} a_i\Big)\left(\frac{\sum_{l_k(u)=1} b_u}{\sum_{l_k(u)=0} b_u}\right) - 2\sum_{i\neq k} a_i \left(\frac{\sum_{l_k(u)=1, l_i(u)\geq 1} l_i(u) b_u}{\sum_{l_k(u)=0, l_i(u)\geq 1} l_i(u) b_u}\right) - 4a_k \left(\frac{\sum_{l_k(u)=2} b_u}{\sum_{l_k(u)=1} b_u}\right).
$$

This completes the proof of Lemma 4.1. □

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