CIRCULANT INTEGRAL OPERATORS AS PRECONDITIONERS FOR WIENER-HOPF EQUATIONS [1]

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In this paper, we study the solutions of finite-section Wiener-Hopf equations by the preconditioned conjugate gradient method. Our main aim is to give an easy and general scheme of constructing good circulant integral operators as preconditioners for such equations. The circulant integral operators are constructed from sequences of conjugate symmetric functions $\{C_\tau\}_\tau$. Let $k(t)$ denote the kernel function of the Wiener-Hopf equation and $\hat{k}(t)$ be its Fourier transform. We prove that for sufficiently large τ if $\{C_\tau\}_\tau$ is uniformly bounded on the real line $\mathbf{R}$ and the convolution product of the Fourier transform of C_τ with $\hat{k}(t)$ converges to $\hat{k}(t)$ uniformly on $\mathbf{R}$, then the circulant preconditioned Wiener-Hopf operator will have a clustered spectrum. It follows that the conjugate gradient method, when applied to solving the preconditioned operator equation, converges superlinearly. Several circulant integral operators possessing the clustering and fast convergence properties are constructed explicitly. Numerical examples are also given to demonstrate the performance of different circulant integral operators as preconditioners for Wiener-Hopf operators.

1. INTRODUCTION.

In this paper, we study the solutions of Wiener-Hopf equations

$$\sigma x(t) + \int_0^\infty k(t-s)x(s)ds = g(t), \quad 0 \le t < \infty, \tag{1}$$

where $\sigma > 0$, $k(t)$ and $g(t)$ are given functions in $L_1(-\infty,\infty)$ and $L_2[0,\infty)$ respectively. For simplicity, we denote

$$(\mathcal{K}x)(t) = \int_0^\infty k(t-s)x(s)ds, \quad 0 \le t < \infty,$$

and $\mathcal{I}$ the identity operator. In the following, we assume that the operator $\mathcal{K}$ is self-adjoint, i.e. the kernel function $k(t)$ of the Wiener-Hopf equation is conjugate symmetric,

$$k(-t) = \overline{k(t)},$$

and also that the operator $\mathcal{K}$ is positive definite.

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Since the half-line Wiener-Hopf equation cannot be solved explicitly, we use the projection method [8,p.71] to approximate the solution of the Wiener-Hopf equation. More precisely, the solution x of the Wiener-Hopf equation is approximated by the solution x_τ of the operator equation

$$[(\sigma\mathcal{I}+\mathcal{K}_\tau)x_\tau](t) = g(t), \quad 0 \le t \le \tau. \tag{2}$$

Here $\mathcal{K}_\tau$ is given by

$$(\mathcal{K}_\tau x_\tau)(t) = \begin{cases} \displaystyle\int_0^\tau k(t-s)x_\tau(s)ds, & 0 \le t \le \tau, \\ 0, & t > \tau. \end{cases} \tag{3}$$

Recently, Gohberg, Hanke and Koltracht [10] employed the conjugate gradient algorithm as an iterative method for solving finite-section Wiener-Hopf equations (2). In order to speed up the convergence rate of the method, they used *circulant integral operators* $\mathcal{C}_\tau$ to precondition $\mathcal{K}_\tau$. Circulant integral operators are operators of the form

$$(\mathcal{C}_\tau y)(t) = \begin{cases} \displaystyle\int_0^\tau c_\tau(t-s)y(s)ds, & 0 \le t \le \tau, \\ 0, & t > \tau, \end{cases} \tag{4}$$

where c_τ is a τ-periodic conjugate symmetric function in $L_1[-\tau,\tau]$. Instead of solving the original finite-section Wiener-Hopf equation, we solve the preconditioned equation

$$[(\sigma\mathcal{I}+\mathcal{C}_\tau)^{-1}(\sigma\mathcal{I}+\mathcal{K}_\tau)x_\tau](t) = [(\sigma\mathcal{I}+\mathcal{C}_\tau)^{-1}g](t), \quad 0 \le t \le \tau.$$

For each iteration step of the conjugate gradient method, we need to solve the operator equation

$$(\sigma\mathcal{I}+\mathcal{C}_\tau)y = z,$$

which can be solved by using the spectral decomposition of the operator, see [9,p.106]. The convergence rate of the method has been analyzed by Gohberg *et. al.* [10]. They proved that if

$$\lim_{\tau\to\infty} ||k - c_\tau||_{L_1[0,\tau/2]} = 0, \tag{5}$$

then the finite-section Wiener-Hopf integral operator can be approximated by circulant integral operators within a sum of a small and a finite rank operators. It follows that the spectra of the corresponding circulant preconditioned operators are clustered around 1 for large τ and the method converges superlinearly. More precisely, for all $\epsilon > 0$, there exists a constant $\nu > 0$ such that the error $x_\tau - x_\tau^{(j)}$ of the preconditioned conjugate gradient method at the jth iteration satisfies

$$||x_\tau - x_\tau^{(j)}||_{L_2[0,\infty)} \le \nu \left(\frac{\epsilon}{1+\sqrt{1-\epsilon^2}}\right)^j$$

when τ is sufficiently large, see [10]. Hence the circulant integral operators are good preconditioners for solving the finite-section Wiener-Hopf equations.

We remark that there is a close relationship between Wiener-Hopf integral equations and semi-infinite Toeplitz operators, see [8,p.5]. The idea of using the preconditioned conjugate gradient method with circulant preconditioners for solving Toeplitz systems has been studied recently by various researchers [5,7,11,13] and applied in different fields of applied mathematics and engineering such as numerical partial differential equations [3,12] and signal and image processing [4,14]. In Chan and Yeung [6], they considered circulant preconditioners from the viewpoint of convolution products and proved that most of the circulant preconditioners proposed for Toeplitz matrices can be derived by taking the convolution products of the generating function p of the Toeplitz matrix with some well-known convolution kernels such as the Dirichlet and the Fejér convolution kernels. They proved that if the convolution product converges to p uniformly, then the spectrum of the corresponding circulant preconditioned matrix will be clustered around 1.

In this paper, we will consider the circulant integral operators from the same viewpoint. The main aim of the paper is to give an easy and general scheme for constructing circulant integral operators for Wiener-Hopf equations. We will also study their convergence property by using convolution products. For ease of presentation, let us denote by $\hat{q}$ the Fourier transform of any function q. Our first step is to relate $c_\tau(t)$ in (4) to a sequence of conjugate symmetric functions $\{C_\tau(t)\}_\tau$ and translate the convergence requirement (5) on $c_\tau(t)$ to conditions on $\{C_\tau(t)\}_\tau$. Basically we want $C_\tau(t)$ to be such that the convolution product $\hat{C}_\tau * \hat{k}$ will converge uniformly to $\hat{k}$. We will see that the "wrap-round" and "optimal" circulant integral operators, proposed by Gohberg *et. al.* in [10], can be derived from this approach.

Using the result in [10], we prove that for sufficiently large τ, if $\{C_\tau\}_\tau$ is uniformly bounded on the real line $\mathbf{R}$ and the convolution product of $\hat{C}_\tau$ with $\hat{k}(t)$ converges to $\hat{k}(t)$ uniformly on $\mathbf{R}$, then the spectra of our circulant preconditioned operators will be clustered around 1. It follows that the conjugate gradient method, when applied to solving the preconditioned operator equation, converges superlinearly. We show that $\{C_\tau(t)\}_\tau$ can be derived easily from the Dirac delta function or from *approximate convolution identities* commonly used in Fourier analysis. Several circulant integral operators possessing the clustering and superlinear convergence properties are constructed explicitly by our scheme.

The outline of the paper is as follows. In §2, we introduce our scheme of constructing circulant integral operators. In §3, we prove that if $\{C_\tau\}_\tau$ is uniformly bounded on $\mathbf{R}$ and the convolution product converges to $\hat{k}(t)$ uniformly on $\mathbf{R}$, then the spectra of the circulant preconditioned operators will be clustered around 1. Finally, we design in §4 some circulant integral operators by using well-known functions C_τ in Fourier analysis. Finally, numerical examples are given in §5 to demonstrate the performance of different circulant integral operators as preconditioners for Wiener-Hopf integral operators.

2. CONSTRUCTION OF CIRCULANT INTEGRAL OPERATORS.

We first recall the relationship between the circulant integral operators and their eigenvalues. Let $c_\tau(t)$ be a τ-periodic and conjugate symmetric function in $L_1[-\tau,\tau]$, i.e.

$$c_\tau(t) = c_\tau(t+\tau) \quad \text{and} \quad c_\tau(t) = \overline{c_\tau(-t)}, \quad \forall t \in [-\tau,\tau].$$

Let $\mathcal{C}_\tau$ be the associated circulant integral operator, see (4). We remark that $\mathcal{C}_\tau$ is a compact, self-adjoint operator on $L_2[-\tau,\tau]$ and its complete set of eigenfunctions is given by

$$\{u_n(t) \ \mid \ u_n(t) = \frac{1}{\sqrt{\tau}} e^{2\pi int/\tau}, \ n \in \mathbf{Z}\},$$

where $\mathbf{Z}$ is the set of all integers. Hence the eigenvalues $\lambda_n(\mathcal{C}_\tau)$ of the operator $\mathcal{C}_\tau$ is given by

$$\lambda_n(\mathcal{C}_\tau) = \int_{-\tau/2}^{\tau/2} c_\tau(t) e^{-2\pi int/\tau} dt, \quad \forall n \in \mathbf{Z}, \tag{6}$$

see [9,p.106]. We note that the eigenvalues of the circulant integral operator are characterized by its corresponding τ-periodic function $c_\tau(t)$. Next we derive a general scheme for constructing circulant integral operators.

LEMMA 1 *Let $\{C_\tau\}_\tau$ be a sequence of conjugate symmetric functions defined on $\mathbf{R}$. Define*

$$c_\tau(t) = \begin{cases} C_\tau(t)k(t) + \ C_\tau(t-\tau)k(t-\tau), & 0 \le t \le \tau, \\ \\ C_\tau(t+\tau)k(t+\tau) + \ C_\tau(t)k(t), & -\tau \le t \le 0, \end{cases} \tag{7}$$

Then c_τ is τ-periodic and conjugate symmetric function. Moreover, if $\mathcal{C}_\tau$ is defined as in (4), then the eigenvalues of $\mathcal{C}_\tau$ are given by

$$\lambda_n(\mathcal{C}_\tau) = \int_{-\tau/2}^{\tau/2} c_\tau(t) e^{-2\pi int/\tau} dt = \int_{-\tau}^{\tau} C_\tau(t)k(t) e^{-2\pi int/\tau} dt, \quad \forall n \in \mathbf{Z}. \tag{8}$$

PROOF: It is easy to check that c_τ is a τ-periodic and conjugate symmetric function. Using the definition of c_τ and (6), (8) can be derived straightforwardly. $\square$

As examples, we can show that the "wrap-round" and "optimal" circulant integral operators, proposed by Gohberg *et. al.* in [10], can be derived from this approach. Before we begin, let us recall that the Fourier transform $\hat{q}$ of a function q is defined by

$$\hat{q}(s) \equiv \int_{-\infty}^{\infty} q(t) e^{-ist} dt, \quad \forall s \in \mathbf{R},$$

and the convolution product $\hat{C}_\tau * \hat{k}$ is defined by

$$(\hat{C}_\tau * \hat{k})(s) \equiv \int_{-\infty}^{\infty} C_\tau(t)k(t) e^{-ist} dt, \quad \forall s \in \mathbf{R}, \tag{9}$$

where $\hat{C}_\tau$ and $\hat{k}(t)$ are the Fourier transforms of C_τ and $k(t)$ respectively.

(a) "Wrap-round" Circulant Integral Operator $\mathcal{D}_\tau$

Given the operator $\mathcal{K}_\tau$ as in (3), the "wrap-round" circulant integral operator $\mathcal{D}_\tau$ is defined as

$$(\mathcal{D}_\tau y)(t) = \begin{cases} \int_0^\tau d_\tau(t-s)y(s)ds, & 0 \le t \le \tau, \\ 0, & t > \tau. \end{cases}$$

Here the function d_τ is a τ-periodic function defined by

$$d_\tau(t) = k(t), \quad -\tau/2 \le t \le \tau/2,$$

see Gohberg *et. al.* [10]. Such operator can be obtained from Lemma 1 by setting C_τ there to be the function

$$D_\tau(t) = \begin{cases} 1, & |t| \le \tau/2, \\ 0, & |t| > \tau/2. \end{cases}$$

(b) "Optimal" Circulant Integral Operator $\mathcal{F}_\tau$

Given the operator $\mathcal{K}_\tau$, the "optimal" circulant integral operator $\mathcal{F}_\tau$ is defined as

$$(\mathcal{F}_\tau y)(t) = \begin{cases} \int_0^\tau f_\tau(t-s)y(s)ds, & 0 \le t \le \tau, \\ 0, & t > \tau. \end{cases} \tag{10}$$

Here $f_\tau(t)$ is a τ-periodic function defined by

$$f_\tau(t) = \left(\frac{\tau - t}{\tau}\right) k(t) + \left(\frac{t}{\tau}\right) k(t-\tau), \quad 0 \le t \le \tau.$$

This operator can also be obtained from Lemma 1 by setting C_τ there to be the function

$$F_\tau(t) = \begin{cases} \dfrac{\tau - |t|}{\tau}, & |t| \le \tau, \\ 0, & |t| > \tau. \end{cases} \tag{11}$$

By (8) and (9), the eigenvalues of $\mathcal{F}_\tau$ are given by

$$\begin{aligned} \lambda_n(\mathcal{F}_\tau) &= \int_{-\tau}^{\tau} F_\tau(t)k(t)e^{-2\pi int/\tau}dt \\ &= \int_{-\infty}^{\infty} F_\tau(t)k(t)e^{-2\pi int/\tau}dt = (\hat{F}_\tau * \hat{k})(\frac{2\pi n}{\tau}), \quad \forall n \in \mathbf{Z}, \end{aligned} \tag{12}$$

where $\hat{F}_\tau$ is a Fejér convolution kernel given by

$$\hat{F}_\tau(t) = \frac{4\sin^2(\tau t/2)}{\tau t^2}.$$

We remark that Gohberg *et. al.* [10] have derived the same formula (12) by showing that $\mathcal{F}_\tau$ minimizes

$$|||\mathcal{K}_\tau - \mathcal{C}_\tau||| \equiv \int_0^\tau \int_0^\tau [k(s-t) - c_\tau(s-t)]\overline{[k(s-t) - c_\tau(s-t)]}dsdt,$$

over all circulant integral operators $\mathcal{C}_\tau$.

In §4, we will use the method in Lemma 1 to construct other circulant integral operators to be used as preconditioners for Wiener-Hopf equations.

3. SPECTRA OF CIRCULANT PRECONDITIONED OPERATORS.

In this section, we study the superlinear convergence property of the circulant preconditioned operators for those circulant integral operators derived from Lemma 1. In the following, we will assume that the given sequence of functions $\{C_\tau\}_\tau$ satisfies

ASSUMPTION $(\mathbf{A})$:

(i) $\{C_\tau\}_\tau$ is a sequence of conjugate symmetric functions,

(ii) For sufficiently large τ,

$$|C_\tau(t)| \le \beta < \infty, \quad \forall t \in \mathbf{R}, \tag{13}$$

(iii) For all functions $k(t)$ in $L_1(-\infty,\infty)$,

$$\lim_{\tau\to\infty} ||\hat{C}_\tau * \hat{k} - \hat{k}||_\infty = 0, \tag{14}$$

where $||\cdot||_\infty$ denotes the supremum norm.

As an example, we show that

LEMMA 2 *The sequence of functions $\{F_\tau\}_\tau$ given by (11) satisfies Assumption* $(\mathbf{A})$.

PROOF: Clearly F_τ satisfies $(\mathbf{A})$ **(i)** and **(ii)**. To prove $(\mathbf{A})$ **(iii)**, we first note that the Fourier transform of any function in $L_1(-\infty,\infty)$ is bounded and uniformly continuous on $(-\infty,\infty)$, (see for instance Champeney [2,Theorem 8.1]), hence by Theorem 8.10 (vi) in [2], the result follows. $\square$

Now we state our main lemma on clustering.

LEMMA 3 *Let $k(t) \in L_1(-\infty,\infty)$ and $\mathcal{K}_\tau$ be the operator given by (3). Let $\{C_\tau\}_\tau$ be a sequence of functions satisfying Assumption* $(\mathbf{A})$. *If $\mathcal{C}_\tau$ is the circulant integral operator with c_τ defined by (7), then for any given $\epsilon > 0$, there exist a positive integer N and a $\tau^* > 0$ such that for all $\tau > \tau^*$, there exists a decomposition*

$$\mathcal{K}_\tau - \mathcal{C}_\tau = \mathcal{R}_\tau + \mathcal{E}_\tau$$

with self-adjoint operators $\mathcal{R}_\tau$ and $\mathcal{E}_\tau$ satisfying

$$\text{rank } \mathcal{R}_\tau \le N$$

and

$$||\mathcal{E}_\tau||_2 \leq \epsilon.$$

Here $||\cdot||_2$ *is the operator norm on the Hilbert space* $L_2[0,\infty)$.

We emphasize that Lemma 3 basically states that a sequence of conjugate symmetric functions satisfying Assumption (**A**) will satisfy the conditions of Theorems 2.1 and 3.1 in Gohberg *et. al.* [10]. In fact, by Lemma 2, the conclusion of Lemma 3 should hold for the "optimal" circulant integral operator $\mathcal{F}_\tau$ defined by (10). But this result for $\mathcal{F}_\tau$ was already proved in Gohberg *et. al.* [10] as a corollary of Theorem 2.1 there. We restate this result here as the following Lemma.

LEMMA 4 *Let* $k(t) \in L_1(-\infty,\infty)$ *and* $\mathcal{K}_\tau$ *be the operator given by (3). Let* $\mathcal{F}_\tau$ *be the "optimal" circulant integral operator defined by (10). Then for any given* $\epsilon > 0$, *there exist a positive integer* N *and a* $\tau^* > 0$ *such that for all* $\tau > \tau^*$, *there exists a decomposition*

$$\mathcal{K}_\tau - \mathcal{F}_\tau = \mathcal{R}_\tau + \mathcal{E}_\tau$$

with self-adjoint operators $\mathcal{R}_\tau$ *and* $\mathcal{E}_\tau$ *satisfying*

$$\text{rank } \mathcal{R}_\tau \leq N$$

and

$$||\mathcal{E}_\tau||_2 \leq \epsilon.$$

Using this Lemma, we can easily prove Lemma 3.

PROOF OF LEMMA 3: Given that $\{C_\tau\}_\tau$ satisfies Assumption (**A**), we first rewrite $\mathcal{K}_\tau - \mathcal{C}_\tau$ as

$$\mathcal{K}_\tau - \mathcal{F}_\tau + \mathcal{F}_\tau - \mathcal{C}_\tau,$$

where $\mathcal{F}_\tau$ is the "optimal" circulant integral operator given by (10). In view of Lemma 4, it suffices to show that

$$\lim_{\tau\to\infty} ||\mathcal{F}_\tau - \mathcal{C}_\tau||_2 = 0.$$

We note that

$$||\mathcal{F}_\tau - \mathcal{C}_\tau||_2 \leq \sup_{n\in\mathbf{Z}} |\lambda_n(\mathcal{F}_\tau) - \lambda_n(\mathcal{C}_\tau)|,$$

see [9,p.112]. Since the eigenvalues of $\mathcal{F}_\tau$ and $\mathcal{C}_\tau$ are given by (12) and (8) respectively, we have

$$\begin{aligned} ||\mathcal{F}_\tau - \mathcal{C}_\tau||_2 &\leq \sup_{n\in\mathbf{Z}} |\lambda_n(\mathcal{F}_\tau) - \lambda_n(\mathcal{C}_\tau)| \\ &\leq \sup_{n\in\mathbf{Z}} |(\hat{F}_\tau * \hat{k})(\frac{2\pi n}{\tau}) - \int_{-\tau}^{\tau} C_\tau(t)k(t)e^{-2\pi int/\tau}dt| \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{n\in\mathbf{Z}}|(\hat{F}_\tau * \hat{k})(\frac{2\pi n}{\tau}) - \hat{k}(\frac{2\pi n}{\tau})| + \sup_{n\in\mathbf{Z}}|\hat{k}(\frac{2\pi n}{\tau}) - (\hat{C}_\tau * \hat{k})(\frac{2\pi n}{\tau})| + \\
&\quad \sup_{n\in\mathbf{Z}}|(\hat{C}_\tau * \hat{k})(\frac{2\pi n}{\tau}) - \int_{-\tau}^{\tau} C_\tau(t)k(t)e^{-2\pi int/\tau}dt| \\
&\leq ||\hat{F}_\tau * \hat{k} - \hat{k}||_\infty + ||\hat{k} - \hat{C}_\tau * \hat{k}||_\infty + \\
&\quad \sup_{n\in\mathbf{Z}}|\int_{|t|\geq\tau} C_\tau(t)k(t)e^{-2\pi int/\tau}dt|. \qquad (15)
\end{aligned}$$

Here the last term of (15) is obtained by using definition (9). In view of (14) and Lemma 2, it suffices to estimate the last term of (15). However, we note that

$$|\int_{|t|\geq\tau} C_\tau(t)k(t)e^{-2\pi int/\tau}dt| \leq \int_{|t|\geq\tau}|C_\tau(t)||k(t)|dt \leq \beta\int_{|t|\geq\tau}|k(t)|dt,$$

where β is the constant given by (13). Since $k(t)$ is in $L_1(-\infty,\infty)$, the result follows. $\square$

Next we show that the circulant integral operator $\sigma\mathcal{I} + \mathcal{C}_\tau$ is positive definite and uniformly invertible for large τ.

LEMMA 5 *Let $k(t) \in L_1(-\infty,\infty)$ and $\{C_\tau\}_\tau$ be a sequence of functions satisfying Assumption* (**A**). *If $\mathcal{C}_\tau$ is the circulant integral operator with c_τ defined by (7), then for any given $0 < \epsilon < \sigma$, there exists a $\tau^* > 0$ such that for all $\tau > \tau^*$, we have*

$$||(\sigma\mathcal{I} + \mathcal{C}_\tau)^{-1}||_2 \leq \frac{1}{\sigma - \epsilon}.$$

PROOF: We first note that all the eigenvalues of the "optimal" circulant integral operator $\mathcal{F}_\tau$ are positive, see Gohberg *et. al.* [10,Corollary 4.3]. From (15), we see that for τ sufficiently large, we have

$$|\lambda_n(\mathcal{C}_\tau) - \lambda_n(\mathcal{F}_\tau)| \leq \epsilon, \quad \forall n \in \mathbf{Z}.$$

Hence the lemma follows. $\square$

Combining Lemmas 3 and 5, we have our main theorem.

THEOREM *Let $k(t) \in L_1(-\infty,\infty)$ and $\mathcal{K}_\tau$ be given by (3). Let $\{C_\tau\}_\tau$ be a sequence of functions satisfying Assumption* (**A**). *If $\mathcal{C}_\tau$ is the circulant integral operator with c_τ defined by (7), then for any given $0 < \epsilon < \sigma$, there exist a positive integer N and a $\tau^* > 0$ such that for all $\tau > \tau^*$, at most N eigenvalues of $(\sigma\mathcal{I}+\mathcal{C}_\tau)^{-1/2}(\sigma\mathcal{I}+\mathcal{K}_\tau)(\sigma\mathcal{I}+\mathcal{C}_\tau)^{-1/2}$ are at distance greater than ϵ from 1.*

PROOF: We just note that

$$(\sigma\mathcal{I} + \mathcal{C}_\tau)^{-1/2}(\sigma\mathcal{I} + \mathcal{K}_\tau)(\sigma\mathcal{I} + \mathcal{C}_\tau)^{-1/2} = \mathcal{I} + (\sigma\mathcal{I} + \mathcal{C}_\tau)^{-1/2}(\mathcal{K}_\tau - \mathcal{C}_\tau)(\sigma\mathcal{I} + \mathcal{C}_\tau)^{-1/2}. \quad \square$$

It follows easily from the theorem that the conjugate gradient method, when applied to solving preconditioned operator equation, converges superlinearly, see Chan and Strang [5] or Gohberg *et. al.* [10].

4. APPROXIMATE CONVOLUTION IDENTITY.

In this section, we give a useful method for constructing sequences of functions $\{C_\tau\}_\tau$ that satisfy Assumption ($\mathbf{A}$). The idea is to use approximate convolution identity, see Champeney [2,p.33] for its definition.

LEMMA 6 *Let $h(t)$ be a conjugate symmetric function in $L_1(-\infty,\infty)$. If we define for each τ,*

$$\hat{C}_\tau(t) = \frac{\tau h(\tau t)}{||h||_{L_1(-\infty,\infty)}} \qquad \text{a.e. on } \mathbf{R}, \tag{16}$$

then the sequence of functions $\{C_\tau\}_\tau$ satisfies Assumption ($\mathbf{A}$).

PROOF: Clearly ($\mathbf{A}$) **(i)** is satisfied. For ($\mathbf{A}$) **(ii)**, we just note that

$$|C_\tau(t)| = \frac{1}{||h||_{L_1(-\infty,\infty)}}\left|\int_{-\infty}^{\infty} h(s)e^{ist/\tau}ds\right| \le \frac{1}{||h||_{L_1(-\infty,\infty)}}\int_{-\infty}^{\infty}|h(s)|ds \le 1, \quad \forall t \in \mathbf{R}.$$

To prove ($\mathbf{A}$) **(iii)**, we first note that the sequence $\{\hat{C}_\tau\}_\tau$ forms an approximate convolution identity. Recall that the Fourier transform of any function in $L_1(-\infty,\infty)$ is bounded and uniformly continuous on $(-\infty,\infty)$, (see for instance Champeney [2,Theorem 8.1]), ($\mathbf{A}$) **(iii)** then follows from Theorem 5.2 (iii) of [2]. $\square$

It follows from Lemma 3 that circulant integral operators constructed from approximate convolution identity as in (16) will be good preconditioners for Wiener-Hopf equations. Table 1 lists some functions $h(t)$ together with their corresponding $\hat{C}_\tau$ and C_τ. We note that the $\hat{C}_\tau$ listed are summation functions frequently used in Fourier analysis, see [15,p.85] and we have here an approach of constructing circulant integral operators from any given summation functions.

Let us illustrate our construction by using the Poisson function given in Table 1. By (7), the τ-periodic function c_τ is equal to

$$c_\tau(t) = \frac{1}{2}e^{-|t/\tau|}k(t) + \frac{1}{2}e^{-|(t-\tau)/\tau|}k(t-\tau), \quad 0 \le t \le \tau.$$

The associated circulant integral operator is then given by (4).

We emphasize that there are sequences of functions $\{C_\tau\}_\tau$ that satisfy Assumption ($\mathbf{A}$) but not approximate convolution identities. For instance, by the convolution identity, the Dirac delta function δ satisfies

$$(\delta * \hat{k})(t) = \hat{k}(t), \quad \forall t \in \mathbf{R},$$

$h(t)$	$\hat{C}_\tau(t)$	$C_\tau(t)$
$\frac{1}{1+t^2}$ (Poisson function [15,p.85])	$\frac{\tau}{1+\tau^2 t^2}$	$\frac{1}{2}e^{-\lvert t/\tau\rvert}$
e^{-t^2} (Gauss-Weierstrass function [15,p.86])	$\tau e^{-(\tau t)^2}$	$e^{-(t/\tau)^2}$
$\frac{12\sin^4 t}{t^4}$ (Jackson function [1, p.119])	$\frac{12\sin^4(\tau t/2)}{\tau^3 t^4}$	$\begin{cases} 1-\frac{3t^2}{2\tau^2}+\frac{3\lvert t\rvert^3}{4\tau^3}, & \lvert t\rvert \le \tau, \\ \frac{1}{4}(2-\frac{\lvert t\rvert}{\tau})^3, & \tau \le \lvert t\rvert \le 2\tau, \\ 0, & \lvert t\rvert \ge 2\tau \end{cases}$
$\frac{e^{-\lvert t\rvert}}{2}$ (Abel-Poisson function [1,p.119])	$\frac{\tau}{2}e^{-\lvert\tau t\rvert}$	$\frac{\tau^2}{\tau^2+t^2}$

Table 1: The definitions of $h(t)$ and their corresponding $\hat{C}_\tau$ and C_τ.

see Walker [15,p.87]. Thus (14) is satisfied by letting $\hat{C}_\tau = \delta$. Since the Dirac delta function is just the Fourier transform of

$$C_\tau(t) = 1, \quad \forall t \in \mathbf{R},$$

see [2,p.118] and $C_\tau(t)$ is uniformly bounded and conjugate symmetric, the assumptions (**A**) **(i)** and **(ii)** are also satisfied. The τ-periodic function c_τ is given by

$$c_\tau(t) = k(t) + k(t-\tau), \quad 0 \le t \le \tau,$$

see (7). By our main theorem, this circulant integral operator is also a good choice of preconditioner for the finite-section Wiener-Hopf equations.

5. NUMERICAL RESULTS.

In this section, we present numerical results on the performance of different circulant integral operators as preconditioners for Wiener-Hopf operators. The kernel function we tested is

$$k(t) = \begin{cases} (1+e^{-1})/2, & \lvert t\rvert \le 1, \\ e^{-\lvert t\rvert}, & \lvert t\rvert > 1. \end{cases}$$

The right hand side function $g(t)$ in (1) is selected such that the corresponding solution of the Wiener-Hopf equation is

$$x(t) = \begin{cases} 1, & 0 \le t \le \gamma, \\ 0, & t > \gamma. \end{cases} \tag{17}$$

We used the rectangular rule to discretize the finite-section Wiener-Hopf integral operator and all circulant integral operators $\mathcal{C}_\tau$. We remark that the kernel function has a jump at $t = 1$. The mesh-size we used in the discretization scheme is of 1 unit. Therefore the size of the matrix is the same as τ for integral-valued τ.

In the test, zeros vector is used as our initial guess and the stopping criterion is $||\mathbf{r}_k||_2/||\mathbf{r}_0||_2 < 10^{-12}$, where $\mathbf{r}_k$ is the residual vector of the preconditioned conjugate gradient method after k iterations. We have chosen σ (given in (1)) to be $(1 - e^{-1})/2$ so that the main diagonal entry of the discretization matrix is 1. The parameter γ in (17) is arbitrarily chosen to be 8. All computations are done by Matlab on an HP-715 workstation. Table 2 shows the numbers of iterations required for convergence for different choices of preconditioners. In the table, $\mathcal{I}$ denotes no preconditioner is used, $\mathcal{D}$ and $\mathcal{F}$ are the "wrap-round" and "optimal" circulant integral operators respectively. For $i = 1, 2, \cdots, 5$, $\mathcal{C}_i$ are respectively the circulant integral operators constructed by using Poisson, Gauss-Weierstrass, Jackson, Abel-Poisson and Dirac delta functions as discussed in §4.

We see from the table that the numbers of iterations of the preconditioned systems are significantly less than that of the non-preconditioned system. The circulant integral operator $\mathcal{C}_5$ constructed by using Dirac delta functions is the overall best amongst the preconditioners tested. Moreover, we see that the convergence performance of the "optimal" circulant operators is not good compared with that of the other circulant integral operators constructed by approximate convolution identities.

	τ							
Preconditioners	16	32	64	128	256	512	1024	2048
$\mathcal{I}$	18	44	128	180	667	**	**	**
$\mathcal{D}$	9	6	6	5	6	5	5	6
$\mathcal{F}$	14	16	19	17	19	20	20	18
$\mathcal{C}_1$	14	16	19	17	19	20	20	18
$\mathcal{C}_2$	10	9	9	8	8	9	7	8
$\mathcal{C}_3$	11	9	10	8	9	10	7	8
$\mathcal{C}_4$	10	9	9	8	8	9	7	8
$\mathcal{C}_5$	7	5	6	5	6	5	5	6

Table 2. Number of Iterations for Different Preconditioners.
(** means > 1000 iterations)

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