Iterative Methods for Overflow Queueing Models I

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Summary: Markovian queueing networks having overflow capacity are discussed. The Kolmogorov balance equations result in a linear homogeneous system, where the right null-vector is the steady-state probability distribution for the network. Preconditioned conjugate gradient methods are employed to find the null-vector. The preconditioner is a singular matrix which can be handled by separation of variables. The resulting preconditioned system is nonsingular. Numerical results show that the number of iterations required for convergence is roughly constant independent of the queue sizes. Analytic results are given to explain this fast convergence.

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1 Introduction

In Markovian queueing networks, most of the quantities of interest, for example the blocking probability and the waiting time for customers in various queues, can be expressed in terms of the steady-state probability distributions, which are the solutions of the Kolmogorov balance equations. The resulting matrix system has dimension N, where N is the total number of states in the network. This matrix is non-symmetric and known to have a one dimensional null-space. The steady-state probability distribution is the normalized right null-vector of this matrix.

For the models discussed in this paper, the graph of such a matrix derived from a q-queue model is the same as that of a q-dimensional discrete Laplacian. We note that even for systems with relatively small numbers of queues, say 4, and a small number of waiting spaces per queue, say 20, the order N of the matrix can be huge, in this case 16,000. Hence the matrix equations are rarely solved by direct methods such as Gaussian elimination. Kaufman [11] has considered a special direct method which handles one or serveral large submatrices by separation of variables, while the remaining variables are handled by a conventional Gaussian elimination method. Different classical iterative methods, for example, the point and the block SOR methods, are also discussed there. In this paper, preconditioned conjugate gradient methods are empolyed. The preconditioner is a singular matrix of order N which can be handled by separation of variables. Although the orginal matrix is singular, we can reduce the problem to solving a non-singular system by computing the component of the eigenvector which is orthogonal to the null-space of this chosen separable problem. Since the states remaining are precisely those at which interactions between the queues take place, the effective dimension of the problem can usually be reduced by an order n, where n is the individual queue size.

For simplicity, we discuss only one overflow model here. The generalization of the method to other models will be discussed in part II of this paper. In § 2, we first generate the balance equations for the queueing models. We then discuss how to change the problems of finding the null-vectors of these matrices, which are of order $N = n^q$, into linear inhomogeneous systems of order n^{q-1}. Finally we explain how to implement preconditioned conjugate gradient methods to solve the resulting linear systems.

In § 3, we analyze the convergence rate of the method for systems with very large queue size. The convergence rate depends on the spectrum of the iteration matrices. We prove that the eigenvalues of these matrices are clustered with only a few outlying eigenvalues. From these results, we derive the fast convergence of our methods. As a corollary, we also establish the fast convergence of the preconditioned conjugate gradient method when applied to the oblique boundary value problems with the corresponding Neumann problem as preconditioner, see § 3.9.1.

The numerical results are reported in § 4. A comparison is made between this method and the point SOR method. We see that our method has a much better performance. In fact, the number of iterations required to attain a given accuracy is almost constant independent of the queue size.

2 The Equations and The Methods

A Markovian analysis of a queueing network based on solving the Kolmogorov equations (see Neuts [14]) for the steady-state probability distribution involves finding the null-vector of a large, sparse structured matrix. In this section, we will derive these Kolmogorov equations, construct such matrices and describe the iterative methods for finding the null-vectors for such matrices.

Let us first introduce the notations that we will be using. Assume that the network has q queues receiving customers from q independent Poisson sources, see Kleinrock [12]. In the i-th queue there are s_i parallel servers, and $n_i - s_i - 1$ waiting spaces. Customers enter the queue with mean arrival rate $\lambda_i > 0$. The departure distribution is independent and exponential with mean rate $\mu_i > 0$.

Let $p_{i_1,i_2,...,i_q}$ denote the steady-state probability distribution which gives the probability of state $(i_1, ..., i_q)$, i.e., the probability that i_j customers are in the j-th queue, $j = 1, .., q$. Since $0 \le i_j < n_j$, $1 \le j \le q$, the total number of states in the system is $N = \prod_{j=1}^{q} n_j$.

It is important to note that, in general, there is no need to obtain the probabilities of all the states in the system. They may be used however to compute such quantities as the blocking probability, the probability of overflow from one queue to another, or the average waiting time of customers in various queues.

We begin with a simple problem, which gives the idea of how the balance equations and the corresponding matrix are generated. This problem is separable and can be solved easily. It will be used as the preconditioner for more complicated models.

2.1 The Free Model

For the problems which we will discuss in this paper, the balance equations are generated by considering the rate at which a state is left and the rate and state from which that state is entered. For example, let us consider a two-queue network with no interaction between the queues. In particular, customers going into a full queue are lost. If δ_{ij} is the Kronecker delta, then the balance equations are

$$\begin{aligned} &\{\lambda_1(1 - \delta_{in_1-1}) + \lambda_2(1-\delta_{jn_2-1}) + \mu_1 \min(i, s_1) + \mu_2 \min(j, s_2)\} p_{i,j} \\ &\quad = \lambda_1(1-\delta_{i0})p_{i-1,j} + \mu_1(1-\delta_{in_1-1})\min(i+1, s_1)p_{i+1,j} \qquad (2.1.1) \\ &\qquad +\lambda_2(1-\delta_{j0})p_{i,j-1} + \mu_2(1-\delta_{jn_2-1})\min(j+1, s_2)p_{i,j+1}, \end{aligned}$$

for $0 \le i < n_1, 0 \le j < n_2$. The left hand side of (2.1.1) indicates the rate at which state (i, j) is left and the right hand side indicates from which states and the rate at which state (i, j) is entered. If we write the steady-state probability distribution as

$$p_0 = (p_{0,0}, p_{0,1}, ..., p_{0,n_2-1}, p_{1,0}, ..., p_{1,n_2-1}, ..., p_{n_1-1,n_2-1})^*,$$

where * denotes the transposition, then (2.1.1) can be written as $Dp_0 = Cp_0$ where D is diagonal and C has zero diagonal entries but non-positive off-diagonal entries. Let $A_0 \equiv D - C$. Then the steady-state probability distribution is just the right null-vector of A_0, i.e.,

$$A_0 p_0 = 0, \qquad (2.1.2)$$

where A_0 is of order N, $N = n_1 n_2$. Since p_0 is a probability distribution, we require

$$\sum_{j=0}^{n_2-1} \sum_{i=0}^{n_1-1} p_{i,j} = 1, \qquad (2.1.3)$$

$$p_{i,j} \geq 0. \tag{2.1.4}$$

We will see that these constraints will uniquely determine p_0.

From (2.1.1), we see that A_0 is separable, in fact

$$A_0 = G_1 \otimes I_{n_2} + I_{n_1} \otimes G_2, \tag{2.1.5}$$

where

$$G_i = \begin{bmatrix} \lambda_i & -\mu_i & & & & & & 0 \\ -\lambda_i & \lambda_i + \mu_i & -2\mu_i & & & & & \\ & -\lambda_i & \lambda_i + 2\mu_i & -3\mu_i & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & -\lambda_i & \lambda_i + s_i\mu_i & -s_i\mu_i & & \\ & & & & \cdot & \cdot & \cdot & \\ & 0 & & & & -\lambda_i & \lambda_i + s_i\mu_i & -s_i\mu_i \\ & & & & & & -\lambda_i & s_i\mu_i \end{bmatrix} \tag{2.1.6}$$

are matrices of order n_i, and I_k is an identity matrix of order k. Unless ambiguity arises, we will drop the subscript for I. Notice that the graph of A_0 is the same as the graph of the discrete Laplacian on a rectangle with mesh sizes $(n_i - 1)^{-1}$. We claim that the G_i and A_0 have one dimensional null-spaces. In fact we have (see Berman and Plemmons [3])

Lemma 2.1.1 An irreducible matrix A with zero column sums, strictly positive diagonal and non-positive off-diagonal entries, has a one dimensional null-space. The corresponding null-vector can be chosen to have positive entries. □

Thus G_i has a one dimensional null-space with a positive null-vector and, by (2.1.5), so does A_0. Hence the solution p_0 to (2.1.2) - (2.1.4) exists and is unique. In particular, the positivity constraints (2.1.4) can always be satisfied. We remark that by the Gerschgorin theorem (see Varga [19]), except for the zero eigenvalue, all the other eigenvalues of the G_i and hence those of A_0 are positive. In view of (2.1.5), the null-vector p_0 of A_0 can be expressed in terms of the null-vector g_i of G_i as $p_0 = g_1 \otimes g_2$. To find g_i, we first notice that G_i can be symmetrized by a diagonal matrix. More precisely, if we define

$$S_i = \mathrm{diag}({}^i d_1, ..., {}^i d_{n_i}),\ i = 1, 2, \tag{2.1.7}$$

with

$${}^i d_j = a_i \cdot \begin{cases} \displaystyle\prod_{k=1}^{j-1} \left(\frac{\lambda_i}{\min(k, s_i)\mu_i}\right)^{\frac{1}{2}} & 1 < j \leq n_i, \\ 1 & j = 1, \end{cases} \tag{2.1.8}$$

then $S_i^{-1} G_i S_i$ is symmetric. Here a_i is the normalization constant such that

$$\mathbf{1}_i^* S_i^2 \mathbf{1}_i = \sum_{j=1}^{n_i} ({}^i d_j)^2 = 1, \tag{2.1.9}$$

where $\mathbf{1}_i$ denotes the n_i-vector of all ones. Since G_i has zero column sums, i.e., $\mathbf{1}_i^* G_i = 0$, we have,

$$G_i S_i^2 \mathbf{1}_i = S_i^2 G_i^* \mathbf{1}_i = 0. \qquad (2.1.10)$$

Thus $S_i^2 \mathbf{1}_i$ is the null-vector for G_i. Hence,

$$p_0 = S^2 \mathbf{1} = (S_1 \otimes S_2)^2 (\mathbf{1}_1 \otimes \mathbf{1}_2). \qquad (2.1.11)$$

By (2.1.9), p_0 also satisfies the summation constraint (2.1.3).

We remark that the operator A_0 has an analogue in the continuous case. It resembles the finite difference approximation to an elliptic operator with a transport term acting on a rectangular region with Neumann boundary conditions on every side. A simple way to see this is to expand $p_{i,j}$ in (2.1.1) formally in a Taylor series in the mesh sizes $(n_i - 1)^{-1}$ and $(n_j - 1)^{-1}$. We also note that if $\lambda_i = \mu_i = s_i = 1$ in (2.1.6), then the resulting A_0 is just the usual 5-point difference operator for the Laplacian equation with Neumann boundary conditions on every side.

The matrix A_0, though singular, will be used as the preconditioner for more complicated models. It is thus necessary to define an appropriate generalized inverse of A_0. We proceed by first obtaining a spectral decomposition of A_0. Since $S_i^{-1} G_i S_i$ is symmetric, there exist orthogonal matrices Q_i and diagonal matrices Γ_i, $i = 1, 2$, such that

$$Q_i^* S_i^{-1} G_i S_i Q_i = \Gamma_i, \; i = 1, 2. \qquad (2.1.12)$$

Here $\Gamma_i = \mathrm{diag}(\gamma_{i,1}, \gamma_{i,2}, \cdots, \gamma_{i,n_i})$ contains the eigenvalues $\{\gamma_{i,j}\}_{j=1}^{n_i}$ of G_i. By lemma 2.1.1, each G_i has only one zero eigenvalue. Let us set $\gamma_{i,n_i} = 0$. By (2.1.5), A_0 can then be diagonalized by $S_1 Q_1 \otimes S_2 Q_2$. More precisely,

$$(Q_1^* S_1^{-1} \otimes Q_2^* S_2^{-1}) A_0 (S_1 Q_1 \otimes S_2 Q_2) = (\Gamma_1 \otimes I + I \otimes \Gamma_2) \equiv \Sigma, \qquad (2.1.13)$$

where

$$\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2, \cdots, \Sigma_{n_1}),$$

with

$$\Sigma_j = \mathrm{diag}(\gamma_{1,j} + \gamma_{2,1}, \cdots, \gamma_{1,j} + \gamma_{2,n_2}) = \Gamma_2 + \gamma_{1,j} \cdot I, \;\; 1 \le j \le n_1. \qquad (2.1.14)$$

Since only $\gamma_{1,n_1} = \gamma_{2,n_2} = 0$, only the last block $\Sigma_{n_1} = \Gamma_2$ is singular. Because it is diagonal, it is easy to define its generalized inverse, or the $\{1\}$-inverse, $\Sigma_{n_1}^+$; see Ben-Israel and Greville [2]. In fact,

$$\Sigma_{n_1}^+ \equiv \Gamma_2^+ = \mathrm{diag}(\gamma_{2,1}^{-1}, \cdots, \gamma_{2,n_2-1}^{-1}, \gamma), \qquad (2.1.15)$$

with γ defined arbitrarily. Since Σ_j^{-1} is well-defined for $1 \le j < n_2$, the generalized inverse Σ^+ of Σ is given by

$$\Sigma^+ = \mathrm{diag}(\Sigma_1^{-1}, \cdots, \Sigma_{n_1-1}^{-1}, \Sigma_{n_1}^+).$$

By (2.1.13), the generalized inverse A_0^+ of A_0 is thus given by

$$A_0^+ = (S_1 Q_1 \otimes S_2 Q_2)\Sigma^+ (Q_1^* S_1^{-1} \otimes Q_2^* S_2^{-1}). \tag{2.1.16}$$

From the spectral decomposition of A_0, we see that

$$R^N = \text{span} < p_0 > \oplus\, Im(A_0), \tag{2.1.17}$$

where $Im(A_0)$ is the range of A_0. Moreover, A_0^+ is invertible on $Im(A_0)$. In fact we have

$$y = A_0 A_0^+ y = A_0^+ A_0 y, \;\; \forall\, y \in Im(A_0). \tag{2.1.18}$$

This follows easily if we write $y = \sum_{j=1}^{N} y_j e_j$ where $\{e_j\}_{j=1}^{N}$ are the eigenvectors of A_0 with $e_N = p_0$. In this representation, $y \in Im(A_0)$ if and only if $y_N = 0$. Combining (2.1.17) and (2.1.18), we see that for any $p \in R^N$, there exist unique scalar α and $\xi \in Im(A_0)$, such that

$$p = \alpha p_0 + A_0^+ \xi. \tag{2.1.19}$$

2.2 Overflow In One Direction

Let us consider an overflow queueing model which is known to have no closed-form solution as in (2.1.11). Assume that there are two queues in the network. Overflow is permitted only from the first queue into the second. More precisely, customers entering the first queue will be served by the second queue if all the spaces in the first queue are occupied. On the other hand, customers entering the second queue are lost if the second queue is full. This model is discussed in Kaufman [11]. The balance equations for this model are given by:

$$\begin{aligned}
&\{\lambda_1(1-\delta_{in_1-1}\delta_{jn_2-1}) + \lambda_2(1-\delta_{jn_2-1}) + \mu_1 \min(i,s_1) + \mu_2 \min(j,s_2)\} p_{i,j} \\
= \;\; & \lambda_1(1-\delta_{i0}) p_{i-1,j} + \mu_1(1-\delta_{in_1-1}) \min(i+1,s_1) p_{i+1,j} \qquad (2.2.1)\\
& + (\lambda_1 \delta_{in_1-1} + \lambda_2)(1-\delta_{j0}) p_{i,j-1} + \mu_2(1-\delta_{jn_2-1}) \min(j+1,s_2) p_{i,j+1},
\end{aligned}$$

for $0 \le i < n_1$, $0 \le j < n_2$. This differs from (2.1.1) only in the coefficients of $p_{i,j}$ and $p_{i,j-1}$. The coefficient of $p_{i,j-1}$ indicates that customers are gained in the second queue at a rate λ_2, but if the first queue is full, additional customers will also arrive at the second queue at the rate λ_1. Let p be the steady-state probability distribution vector for this problem. Using the notations in § 2.1, we are solving a homogeneous system of order $N = n_1 n_2$, namely,

$$Ap = (A_0 + R_0)p = 0, \tag{2.2.2}$$

$$\sum_{i=1}^{n_1-1} \sum_{j=0}^{n_2-1} p_{i,j} = 1, \tag{2.2.3}$$

$$p_{i,j} \geq 0. \tag{2.2.4}$$

Here

$$R_0 = ({}^1e_{n_1}{}^1e^*_{n_1}) \otimes R_1, \tag{2.2.5}$$

with

$$R_1 = \lambda_i \cdot \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & 0 & \\ & \cdot & \cdot & & & \\ & & \cdot & \cdot & & \\ & 0 & & -1 & 1 & \\ & & & & -1 & 0 \end{bmatrix}, \tag{2.2.6}$$

a square matrix of order n_2. ${}^ie_j$ denotes the j-th unit vector in R^{n_i}.

By (2.2.5) and the fact that A_0 satisfies the assumptions of lemma 2.1.1, A also satisfies these assumptions. Thus A has a one dimensional null-space with a positive null-vector. Hence the solution p to (2.2.2) - (2.2.4) exists and is unique. Moreover, we have

$$Im(A) = Im(A_0) = \{x \in R^N | \mathbf{1}^*x = 0\}, \tag{2.2.7}$$

where $\mathbf{1} = (1, 1, \cdots, 1)^* \in R^N$. Since $R_0 = A - A_0$, $Im(R_0) \subset Im(A_0)$. By (2.2.5), we see that

$$Im(R_0) = \{x \in R^N | \; x = {}^1e_{n_1} \otimes y \text{ where } y \in R^{n_2} \text{ with } \sum y_j = 0\}. \tag{2.2.8}$$

By (2.1.19), there exist unique α and $\xi_0 \in Im(A_0)$ such that $p = \alpha p_0 + A_0^+\xi_0$. Since $A_0^+\xi_0 \in Im(A_0)$, we have, by (2.2.7), $\mathbf{1}^*p = \mathbf{1}^*\alpha p_0 + \mathbf{1}^*A_0^+\xi_0 = \alpha\mathbf{1}^*p_0$. By constraints (2.1.3) and (2.2.3), $\mathbf{1}^*p_0 = \mathbf{1}^*p = 1$. Thus $\alpha = 1$ and

$$p = p_0 + A_0^+\xi_0. \tag{2.2.9}$$

Substituting (2.2.9) into (2.2.2) and using (2.1.18), we have

$$(I + R_0A_0^+)\xi_0 = -R_0p_0. \tag{2.2.10}$$

Using (2.2.5) and (2.1.11), it is easily checked that $R_0p_0 \neq 0$. Thus the problem of finding a null-vector to (2.2.2) has been transformed into the problem of solving a linear inhomogeneous system (2.2.10).

We remark that $(I + R_0A_0^+)\xi = AA_0^+\xi$ for all $\xi \in Im(A_0)$. Thus we are preconditioning the equation (2.2.2) by A_0 from the right. Although AA_0^+ is singular, we have,

Lemma 2.2.1 The matrix $(I + R_0A_0^+)$ is nonsingular.
Proof: Since $Im(R_0) \subset Im(A_0)$, hence $(I + R_0A_0^+)$ maps $Im(A_0)$ into itself. Moreover, the existence and uniqueness of p implies the existence and uniqueness of a $\xi_0 \in Im(A_0)$ that satisfies (2.2.10). Thus the matrix is invertible in $Im(A_0)$.

Suppose $y \in R^N$ is in the kernel of this matrix. By (2.1.19), there exist unique β and $x \in Im(A_0)$ such that $y = \beta p_0 + x$. Hence $(I + R_0 A_0^+)y = 0$ implies that $-\beta p_0 = (I + R_0 A_0^+)x + \beta R_0 A_0^+ p_0$. Since $Im(R_0) \subset Im(A_0)$, the right hand side is in $Im(A_0)$. Thus by (2.1.17), $\beta = 0$ and hence $(I + R_0 A_0^+)y = (I + R_0 A_0^+)x = 0$. Since $x \in Im(A_0)$, this implies $x = 0$. Hence $y = 0$. □

By this lemma, it is legitimate to solve the inhomogeneous system (2.2.10). By (2.2.10), $\xi_0 = R_0(-p_0 - A_0^+ \xi_0) \in Im(R_0)$. By (2.2.8), $\xi_0 = {}^1e_{n_1} \otimes y_0$ where $y_0 \in R^{n_2}$ and there are only n_2 degrees of freedom in ξ_0. This suggests that the system (2.2.10), which is of order $N = n_1 n_2$, can be reduced to a system of order n_2. To achieve this, we first denote the projection from $Im(R_0)$ onto R^{n_2} by E^*. More precisely, we let

$$E = {}^1e_{n_1} \otimes I_{n_2}. \tag{2.2.11}$$

We note that $E^* \xi_0 = y_0$ and $EE^* \xi_0 = Ey_0 = \xi_0$. Premultiplying (2.2.10) by E^*, and after some simplification, it becomes an n_2 by n_2 system of the form $By_0 = b$, where

$$B \equiv E^*(I + R_0 A_0^+)E = I_{n_2} + R_1 E^* A_0^+ E, \tag{2.2.12}$$

$$b \equiv E^* R_0 p_0 = R_1 E^* p_0 = ({}^1d_{n_1})^2 \cdot R_1 S_2^2 \mathbf{1}_2. \tag{2.2.13}$$

Here ${}^1d_{n_1}$ is given by (2.1.8). By (2.2.9) and (2.1.11), the probability distribution vector p of this model is given by

$$p = p_0 + A_0^+ E y_0 = (S_1 \otimes S_2)^2 \mathbf{1} + A_0^+ E y_0. \tag{2.2.14}$$

We note that the second term in (2.2.12) can be simplified. In fact, using (2.1.16) and some straightforward computations, we have

$$E^* A_0^+ E = S_2 Q_2 E^* (S_1 Q_1 \otimes I) \Sigma^+ (Q_1^* S_1^{-1} \otimes I) E Q_2^* S_2^{-1} = S_2 Q_2 \Phi Q_2^* S_2^{-1}, \tag{2.2.15}$$

where Φ is diagonal and is given by

$$\Phi = \sum_{j=1}^{n_1-1} ({}^1q_{n_1,j})^2 \cdot \Sigma_j^{-1} + ({}^1q_{n_1,n_1})^2 \cdot \Sigma_{n_1}^+. \tag{2.2.16}$$

Here ${}^1q_{n_1,j}$ denotes the (n_1, j) entry of the orthogonal matrix Q_1. Σ_j and $\Sigma_{n_1}^+$ are given by (2.1.14) and (2.1.15). Putting (2.2.15) into (2.2.12), we have

$$B = I + R_1 S_2 Q_2 \Phi Q_2^* S_2^{-1}. \tag{2.2.17}$$

When n_2 is of moderate size, this suggests we can compute and store B, and then solve $By_0 = b$ by a direct method such as Gaussian elimination. However, the numerical results given in § 4 are computed by using conjugate gradient methods because we are also interested in the case when n_2 is very large. For references on conjugate gradient methods, see for instance, Hestenes [9] and Luenberger [13]. Since B is non-symmetric, we find y_0 by solving the normal equation $B^*By_0 = B^*b$. Thus, before the iteration, we generate $\{Q_i, \Gamma_i\}_{i=1}^2$, the eigenpairs of the symmetric matrices $S_i^{-1}G_iS_i$. This may be done by calling a standard eigenvector subroutine, see the EISPACK manual [17]. Since the G_i are tridiagonal and the S_i are diagonal, this requires $O(n_2^2)$ operations. Since Σ_j^{-1}, $1 \le j < n_1$, and $\Sigma_{n_1}^+$ are diagonal, Φ can be generated in $O(n_2^2)$ operations by (2.2.16). In (2.2.17), as R_1 is a bidiagonal matrix, the matrix-vector multiplication By thus requires approximately $2n_2^2 + O(n_2)$ operations. Hence solving the normal equations requires $4n_2^2 + O(n_2)$ operations per iteration. The storage requirement is $n_1^2 + n_2^2 + O(n_i)$, since we need to store the Q_i's. In Chan [4], we mention two alternates that require the storage of only one of the Q_i's.

We note that there is no need to compute the last entry of Φ. In fact,

Lemma 2.2.2 Φ_{n_2}, the last diagonal entry of Φ, can be defined arbitrarily.
<u>Proof</u>: Since $\gamma_{2,n_2} = 0$, it follows from (2.2.16), (2.1.14) and (2.1.15) that

$$\Phi_{n_2} = \sum_{j=1}^{n_1-1} ({}^1q_{n_1,j})^2(\gamma_{1,j})^{-1} + ({}^1q_{n_1,n_1})^2\gamma, \tag{2.2.18}$$

where γ is arbitrary. The lemma is therefore true if ${}^1q_{n_1,n_1} \ne 0$. Since ${}^1q_{\cdot,n_1}$ is the eigenvector corresponding to the zero eigenvalue of $S_1^{-1}G_1S_1$, thus by (2.1.10), ${}^1q_{\cdot,n_1} = S_1\mathbf{1}_1$. In particular, by (2.1.8), ${}^1q_{n_1,n_1} = {}^1d_{n_1} \ne 0$. □

Combining this result and the fact that $Im(R_0)$ is an (n_2-1)-dimensional vector space, we can further reduce the dimension of the system (2.2.10) to $(n_2 - 1)$. We will exploit this fact in § 3.

After obtaining the solution to $By_0 = b$, we can generate the original null-vector p by (2.2.14). We note that this step may require $N(n_1 + n_2)$ operations since $A_0^+Ey_0$ is not sparse and N storage spaces are required for holding p. However, in some particular but interesting cases, we can reduce both the operation count and storage. For example, if only the blocking probability p_{n_1-1,n_2-1} is required, then we only need to calculate an expression of the form $e_N^*A_0^+Ey_0$, where e_N is the N-th unit vector in R^N. By (2.2.11), $e_N^* = e_{n_2}^*E^*$. Hence by (2.2.15), the form can be evaluated in $O(n_2^2)$ operations and requires only $O(n_2)$ storage. As another example, suppose we want the probability of overflow from queue 1 to queue 2, which is given by $\sum_{j=0}^{n_2-2} p_{n_2-1,j}$, or more generally, suppose only a linear functional of p is required. In these cases, there is no need to generate and store p explicitly. The idea is to generate the solution one block at a time, and then accumulate its contribution to the functional before we generate another block. More precisely, suppose we want to calculate l^*p, where l is a vector in R^N. By (2.2.14) and the fact that the entries of S_i are given by (2.1.8), we only need to evaluate an expression of the form

$$s \equiv l^*(A_0^+ E y_0) = l^*(S_1 Q_1 \otimes S_2 Q_2)\Sigma^+ (Q_1^* S_1^{-1} \otimes Q_2^* S_2^{-1}) E y_0.$$

Let us first partition l into n_1 block, $\hat{l}_1, \cdots, \hat{l}_{n_1}$, each with n_2 entries, and define

$$w_j = \begin{cases} {}^1q_{n_1,j}({}^1d_{n_1})^{-1} \cdot S_2 Q_2 \Sigma_j^{-1}(Q_2^* S_2^{-1} y_0), & 1 \le j < n_1, \\ {}^1q_{n_1,n_1}({}^1d_{n_1})^{-1} \cdot S_2 Q_2 \Sigma_{n_1}^{+}(Q_2^* S_2^{-1} y_0), & j = n_1, \end{cases} \tag{2.2.19}$$

then it is easy to check that

$$s = \sum_{j=1}^{n_1} \sum_{k=1}^{n_1} ({}^1d_k {}^1q_{k,j}) \hat{l}_k^* w_j.$$

Thus it is clear that we can generate w_j one at a time and accumulate its contribution to l^*p before we generate another w_j. Hence no extra storage is required for p. Notice that this way of accumulating the result blockwise does not increase the work. Similar techniques are discussed in Banegas [1].

Let us consider an important special case, the single server case, where $s_l = 1$. In this case, we can derive explicit formulas for Q_l, Γ_l and Φ. We will show that $Q_l x$ can be computed by using the Fast Fourier Transform. Hence the operations count and storage required for each iteration can be further reduced by almost a factor of n_i. For references on Fast Fourier Transform, see Cooley and Tukey [5].

We first give the formula for Q_l and Γ_l, $l = 1, 2$. By (2.1.6), with $s_l = 1$, we have

$$S_l^{-1} G_l S_l = \sqrt{\lambda_l \mu_l} \cdot \begin{bmatrix} \rho_l & -1 & & & & \\ -1 & \rho_l + \frac{1}{\rho_l} & -1 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & -1 & \rho_l + \frac{1}{\rho_l} & -1 \\ & & & & -1 & \frac{1}{\rho_l} \end{bmatrix} \tag{2.2.20}$$

for $l = 1, 2$, where $\rho_l = (\frac{\lambda_l}{\mu_l})^{\frac{1}{2}}$. Let $\theta_{l,j} = \frac{j\pi}{n_l}$, and define $\psi_{l,j}$ by

$$\sin(\psi_{l,j} - \theta_{l,j}) = \frac{1}{\rho_l} \sin \psi_{l,j}, \tag{2.2.21}$$

for $1 \le j < n_l$, or equivalently,

$$\frac{1}{\rho_l^2} \sin^2 \psi_{l,j} = \frac{\sin^2 \theta_{l,j}}{1 - 2\rho_l \cos \theta_{l,j} + \rho_l^2}. \tag{2.2.22}$$

It is easy to check that

$${}^lq_j = \begin{cases} (\frac{2}{n_l})^{\frac{1}{2}} (\sin \psi_{l,j}, \sin(\theta_{l,j} + \psi_{l,j}), \cdots, \sin((n_l - 1)\theta_{l,j} + \psi_{l,j}))^* & 1 \le j < n_l, \\ (\frac{1-\rho_l^2}{1-\rho_l^{2n_l}})^{\frac{1}{2}} (1, \rho_l, \cdots, \rho_l^{n_l - 1})^* & j = n_l, \end{cases} \tag{2.2.23}$$

are the normalized eigenvectors of $S_l^{-1} G_l S_l$ with eigenvalues

$$\gamma_{l,j} = \begin{cases} \sqrt{\lambda_l \mu_l}(\rho_l + \frac{1}{\rho_l} - 2\cos\theta_{l,j}) & 1 \le j < n_l \\ 0 & j = n_l \end{cases} \tag{2.2.24}$$

Thus $Q_l = ({}^l q_1, {}^l q_2, \cdots, {}^l q_{n_l})$ can be generated without calling any EISPACK subroutine.

Next we claim that, for any real vector x, $Q_l x$, $l = 1, 2$ can be computed by using the Fast Fourier Transform. In fact, by (2.2.23), the k-th entry of this vector is given by

$$\begin{aligned} (Q_l x)_k &= (\frac{2}{n_l})^{\frac{1}{2}} \sum_{j=1}^{n_l - 1} \sin((k-1)\theta_{l,j} + \psi_{l,j}) x_j + (\frac{1-\rho_l^2}{1-\rho_l^{2n_l}})^{\frac{1}{2}} \rho_l^{k-1} x_{n_l} \\ &= (\frac{2}{n_l})^{\frac{1}{2}} \mathrm{IMAG} \left\{ \sum_{j=1}^{n_l-1} e^{i(k-1)\theta_{l,j}} \cdot (e^{i\psi_{l,j}} x_j) \right\} + (\frac{1-\rho_l^2}{1-\rho_l^{2n_l}})^{\frac{1}{2}} \rho_l^{k-1} x_{n_l}, \end{aligned}$$

where IMAG means taking the imaginary part. The numbers $z_j \equiv e^{i\psi_{l,j}} x_j$, $j = 1, \cdots, n_l$ can be computed by using n_l complex multiplications. The expression $\sum_{j=1}^{n_l} e^{i(k-1)\theta_{l,j}} z_j$ can be evaluated by the Fast Fourier Transform. This requires only $O(n_l \log n_l)$ operations for arbitrary n_l, see Chan [4]. From (2.2.17), we see that the work and storage required for computing the matrix-vector multiplication By are thus reduced to $O(n_2 \log n_2)$ and $O(n_2)$ respectively. The work of generating the whole null-vector p can also be reduced to $O(n_2^2 \log n_2)$. We remark that when using the Fast Fourier Transform, there is no need to store Q_l. Thus the storage requirement is reduced from $O(n_l^2)$ to $O(n_l)$. If only one of the $s_l = 1$, we can still reduce the work and storage by the same amount, see Chan [4].

Finally we give a formula for Φ. First we recall from lemma 2.2.2 that Φ_{n_2} can be set arbitrarily. For $1 \le j < n_2$, by (2.2.16), and using the formulas for Q_1, $\gamma_{i,j}$ and $\psi_{i,j}$ in (2.2.22) - (2.2.24), the j-th diagonal entry of Φ is given by

$$\begin{aligned} \Phi_j &= \frac{2}{n_1} \sum_{k=1}^{n_1-1} \frac{\sin^2((n_1 - 1)\theta_{1,k} + \psi_{1,k})}{(\gamma_{1,k} + \gamma_{2,j})} + \frac{1-\rho_1^2}{1-\rho_1^{2n_1}} \cdot \frac{\rho_1^{2n_1-2}}{\gamma_{2,j}} \\ &= \frac{2}{n_1} \sum_{k=1}^{n_1-1} \frac{\sin^2(\psi_{1,k} - \theta_{1,k})}{(\gamma_{1,k} + \gamma_{2,j})} + \frac{1-\rho_1^2}{1-\rho_1^{2n_1}} \cdot \frac{\rho_1^{2n_1-2}}{\gamma_{2,j}} \\ &= \frac{2}{n_1} \sum_{k=1}^{n_1-1} \frac{\sin^2 \theta_{1,k}}{(1 - 2\rho_1 \cos\theta_{1,k} + \rho_1^2)(\mu_1(1 - 2\rho_1\cos\theta_{1,k} + \rho_1^2) + \gamma_{2,j})} \\ &\qquad + \frac{1-\rho_1^2}{1-\rho_1^{2n_1}} \cdot \frac{\rho_1^{2n_1-2}}{\gamma_{2,j}} \end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{2}{n_1}\frac{1}{\gamma_{2,j}}\sum_{k=1}^{n_1-1}\frac{\sin^2\theta_{1,k}}{(1-2\rho_1\cos\theta_{1,k}+\rho_1^2)} \\
& -\frac{2}{n_1}\frac{\mu_1}{\gamma_{2,j}}\sum_{k=1}^{n_1-1}\frac{\sin^2\theta_{1,k}}{\mu_1(1-2\rho_1\cos\theta_{1,k}+\rho_1^2)+\gamma_{2,j}}+\frac{1-\rho_1^2}{1-\rho_1^{2n_1}}\cdot\frac{\rho_1^{2n_1-2}}{\gamma_{2,j}}.
\end{aligned}
$$

By the Poisson summation formula, see [16], the first term is equal to $\frac{(1-\rho_1^{2n_1-2})}{\gamma_{2,j}(1-\rho_1^{2n_1})}$, which combines with the third term and gives

$$
\Phi_j=\frac{1}{\gamma_{2,j}}-\frac{2}{n_1}\frac{\mu_1}{\gamma_{2,j}}\sum_{k=1}^{n_1-1}\frac{\sin^2\theta_{1,k}}{\mu_1(1-2\rho_1\cos\theta_{1,k}+\rho_1^2)+\gamma_{2,j}}. \tag{2.2.25}
$$

The second term in (2.2.25) can also be computed by using the Poisson summation formula. To apply the formula, we notice that

$$
(1-2\rho_1\cos\theta_{1,k}+\rho_1^2)+\frac{\gamma_{2,j}}{\mu_1}=\frac{\rho_1}{a_j}\left\{1-2a_j\cos^2\theta_{1,k}+a_j^2\right\},
$$

where a_j is the smallest root of

$$
a_j^2-(\rho_1+\frac{1}{\rho_1}+\frac{\gamma_{2,j}}{\sqrt{\lambda_1\mu_1}})a_j+1=0. \tag{2.2.26}
$$

Thus,

$$
\begin{aligned}
\frac{2}{n_1}\sum_{k=1}^{n_1-1}\frac{\sin^2\theta_{1,k}}{(1-2\rho_1\cos\theta_{1,k}+\rho_1^2)+\frac{\gamma_{2,j}}{\mu_1}} &= \frac{2}{n_1}\frac{a_j}{\rho_1}\sum_{k=1}^{n_1-1}\frac{\sin^2\theta_{1,k}}{1-2a_j\cos\theta_{1,k}+a_j^2} \\
&= \frac{a_j}{\rho_1}\left(\frac{1-a_j^{2n_1-2}}{1-a_j^{2n_1}}\right).
\end{aligned}
$$

Putting this into (2.2.25) we have,

Lemma 2.2.3 In the single server case,

$$
\Phi_j=\begin{cases}\dfrac{1}{\gamma_{2,j}}\left\{1-\dfrac{a_j}{\rho_1}\left(\dfrac{1-a_j^{2n_1-2}}{1-a_j^{2n_1}}\right)\right\} & 1\le j<n_2, \\ \text{arbitrary} & j=n_2,\end{cases} \tag{2.2.27}
$$

where a_j are given by (2.2.26). □

Thus Φ can be generated in $O(n_2)$ operations.

Let us remark that the matrix A also has an analogy in the continuous case. Notice that R_1 in (2.2.6) is the forward difference operator on a line. Thus R_0 in (2.2.5) resembles an operator which is zero in a rectangular region, but with a tangential derivative along one of the sides. This particular side corresponds to those states where the first queue is full. Recall that A_0 resembles a finite difference approximation of an elliptic operator with a transport term on the same region and with Neumann boundary conditions on every sides. Thus the continuous analogy of $A = A_0 + R_0$ is the finite difference approximation of the same operator as A_0, but with an oblique derivative on the particular side. Thus an intuitive explanation for the fast convergence of the method is that the preconditioner A_0 is a good approximation to the operator A, in the sense that it changes its oblique boundary condition into a Neumann boundary condition.

In the single server case, the underlying elliptic operator has constant coefficients and is given by

$$\begin{aligned}(\lambda_1+\mu_1)p_{xx} \quad &+ \quad (\lambda_2+\mu_2)p_{yy} \;+\; 2(n_1-1)(\mu_1-\lambda_1)p_x \\ &+ \quad 2(n_2-1)(\mu_2-\lambda_2)p_y = 0. \qquad (2.2.28)\end{aligned}$$

Thus for n_i large, one reasonable limit to consider is $\mu_i - \lambda_i = O((n_i-1)^\alpha)$ for a certain α. In § 3, we will analyse the method under this limit.

3 Analysis of a Model Problem

In this section, we establish the fast convergence for the model we discussed in § 2.2 in the single-server case, i.e. $s_1 = s_2 = 1$. We will show that the number of iterations required to attain a given accuracy increases no faster than $O(\log^2 n_2)$ when n_i increase. For simplicity, we consider only the case where $\lambda_i < \mu_i$ here. The case where $\lambda_i \geq \mu_i$ can be proved similarly and is given in Chan [4]. First, let us define clearly the limit we are taking.

3.1 Defining the Parameters

Recall that when we expand $p_{i,j}$ in (2.2.1) with $s_i = 1$ and

$$h_i = (n_i - 1)^{-1}, \quad i = 1, 2, \qquad (3.1.1)$$

then the underlying continuous equation is of the form (2.2.28). Thus to obtain a reasonable limit when the queue sizes n_i, $i = 1, 2$, are large, we assume that the traffic density and the queue size satisfy the relation

$$\frac{\lambda_i}{\mu_i} = 1 - \beta_i h_i^\alpha, \quad i = 1, 2, \qquad (3.1.2)$$

where $\mu_i, \beta_i, i = 1, 2$ and α are constants independent of h_i. We remark that

Lemma 3.1.1 Given arbitrary $\{\lambda_i, \mu_i, h_i\}_{i=1}^2$, such that $0 < \lambda_i < \mu_i$ and $0 < h_i \leq 1$, $i = 1, 2$, if we define α by

$$\alpha \equiv \min_{i=1,2} \left\{ \frac{\log(1 - \frac{\lambda_i}{\mu_i})}{\log h_i} \right\}, \qquad (3.1.3)$$

and β by (3.1.2), then $\alpha > 0$ and $0 < \beta_i \leq 1$, for $i = 1, 2$. $\square$

The proof is easy. In view of this lemma, we assume in the following that

$$\alpha \geq 0 \text{ and } 0 < \beta_i \leq 1, i = 1, 2. \tag{3.1.4}$$

Moreover, in order to avoid taking two limits simultaneously, we assume that when h_i, i=1,2, tend to zero, the compatibility condition

$$\frac{n_2}{n_1} = C_0 \tag{3.1.5}$$

holds, where C_0 is a constant independent of h_i. We remark that by choosing a suitable time scale, we can assume that $\mu_1 = 1$. This implies $\lambda_1 < 1$. In the following, we use C to denote any generic positive constant that depends only on α, μ_i and β_i, $i = 1, 2$, and is independent of h_i. We note that by (3.1.2) and (3.1.5)

$$(\frac{\lambda_1}{\mu_1})/(\frac{\lambda_2}{\mu_2}) = 1 \pm O(h_i^\alpha). \tag{3.1.6}$$

Thus, there exists a C such that

$$\frac{1}{C} < \frac{\lambda_1}{\lambda_2} < C. \tag{3.1.7}$$

To begin the analysis, we first note that with $s_i = 1$, (2.1.7) becomes

$$S_i = a_i \cdot \text{diag}(1, \rho_i, ..., \rho_i^{n_i - 1}), \qquad i = 1, 2, \tag{3.1.8}$$

where

$$\rho_i \equiv (\frac{\lambda_i}{\mu_i})^{\frac{1}{2}} = (1 - \beta_i h_i^\alpha)^{\frac{1}{2}} < 1, \tag{3.1.9}$$

and

$$a_i = \left(\frac{1 - \rho_i^2}{1 - \rho_i^{2n_i}} \right)^{\frac{1}{2}}. \tag{3.1.10}$$

Notice that for $\alpha \geq 0$ and h_i sufficiently small,

$$\rho_i^{2(n_i-1)} = (1 - \beta_i h_i^\alpha)^{\frac{1}{h_i}} = e^{\frac{1}{hi} log(1-\beta_i h_i^\alpha)} = e^{-\beta_i h_i^{\alpha-1} - O(h_i^{2\alpha-1})}.$$

Thus we have,

Lemma 3.1.2 For all $\alpha \geq 0$, there exists $H_i = H_i(\alpha, \beta_i)$, $i = 1, 2$, such that for all $h_i < H_i$, we have

$$\rho_i^{2(n_i-1)} \leq e^{-\beta_i h_i^{\alpha-1}} < c_i < 0.5, \text{ if } 0 \leq \alpha < 1, \tag{3.1.11}$$

and

$$\rho_i > \rho_i^{2(n_i-1)} \geq e^{-2\beta_i h_i^{\alpha-1}} > c_i^2 > 0, \text{ if } \alpha \geq 1, \tag{3.1.12}$$

where

$$c_i = c_i(\alpha, \beta_i) \equiv e^{-\beta_i H_i^{\alpha-1}}. \quad \Box \tag{3.1.13}$$

From this lemma, we see that there are two different cases to be considered, namely, the cases where $\alpha < 1$ and $\alpha \geq 1$.

3.2 The Case when $\alpha < 1$

For $\alpha < 1$, the problem of solving p in $Ap = 0$ approaches the separable problem $A_0 p_0 = 0$. More precisely, we have

Theorem 3.2.1 If $0 \leq \alpha < 1$ and $0 < h_i < H_i$, for $i = 1, 2$, then

$$||Ap_0||_2 \leq 8e^{-\beta_1 h_1^{\alpha-1}}.$$

Proof: By (2.2.2), (2.1.2), (2.1.11) and (3.1.8), we have

$$\begin{aligned} Ap_0 = (A_0 + R_0)p_0 = R_0 p_0 &= ({}^1e_{n_1}{}^1e_{n_1}^* \otimes R_1)(S_1^2 \mathbf{1}_1 \otimes S_2^2 \mathbf{1}_2) \\ &= (a_1)^2 \rho_1^{2(n_1-1)}({}^1e_{n_1}{}^1e_{n_1}^* \otimes R_1 S_2^2 \mathbf{1}_2). \end{aligned}$$

Thus by (2.2.6), (3.1.10) and (3.1.11),

$$\begin{aligned} ||Ap_0||_2 &= (a_1)^2 \rho_1^{2(n_1-1)} ||R_1 S_2^2 \mathbf{1}_2||_2 \\ &= \lambda_1 \rho_1^{2(n_1-1)} \left(\frac{1-\rho_1^2}{1-\rho_1^{2n_1}} \right) \left(\frac{1-\rho_2^2}{1-\rho_2^{2n_2}} \right) \left[2 \cdot \frac{1+\rho_2^{4n_2-6}}{(1+\rho_2^2)} \right]^{\frac{1}{2}} \\ &< 2 \cdot \frac{1}{1-\rho_1^{2n_1}} \cdot \frac{1}{1-\rho_2^{2n_2}} \rho_1^{2(n_1-1)} \\ &< 8e^{-\beta_1 h_1^{\alpha-1}}. \ \Box \end{aligned}$$

This lemma shows that when $\alpha < 1$ and n_i is sufficiently large, p_0 is already a good approximation to p. In fact, when $\alpha < 1$, $(p_0)_{ij}$ is exponentially small for i close to $n_1 - 1$. Hence the direction of the derivative along the boundary $i = n_1 - 1$, oblique or Neumann, does not have much effect on the solution. As an example, we note that when $\alpha = 0$ and $\rho_i^2 = \beta_i = \frac{1}{2}$, $i = 1, 2$, then $||Ap_0||_2 \leq 10^{-10}$ when $n_i \geq 32$.

3.3 An Equivalent Problem

In the next few sections, we will analyse the spectrum of the iteration matrix B^*B when $\alpha \geq 1$. Here B is given by (2.2.17). However, we find that it is easier to work with the following similar transformation of B. We define

$$B_s \equiv Q_2^* S_2^{-1} B S_2 Q_2 = I + Q_2^* S_2^{-1} R_1 S_2 Q_2 \Phi. \tag{3.3.1}$$

We note that the transformation matrix is bounded in the l_2 norm. More precisely,

Lemma 3.3.1 For $\alpha \geq 1$ and $h_2 < H_2$,

$$||a_2^{-1} S_2 Q_2||_2 = a_2^{-1} ||S_2||_2 = 1, \tag{3.3.2}$$

$$||a_2 Q_2^* S_2^{-1}||_2 = a_2 ||S_2^{-1}||_2 = \rho_2^{-(n_2-1)} \leq \frac{1}{c_2}. \tag{3.3.3}$$

Here a_2 and c_2 are given by (3.1.10) and (3.1.13) respectively. □

The proof follows immediately from (3.1.8) and (3.1.12). Using this lemma, we claim that the spectrum of B^*B and $B_s^*B_s$ are equivalent.

Lemma 3.3.2 Let the singular values of B and B_s be $0 \leq \sigma_1 \leq \cdots \leq \sigma_{n_2}$ and $0 \leq \tilde{\sigma}_1 \leq \cdots \leq \tilde{\sigma}_{n_2}$ respectively. If $\alpha \geq 1$ and $h_2 < H_2$ then

$$c_2 \sigma_j \leq \tilde{\sigma}_j \leq \frac{1}{c_2} \sigma_j, \quad 1 \leq j \leq n_2. \tag{3.3.4}$$

<u>Proof</u>: Given an arbitrary subspace R^j of dimension j in R^{n_2}, by (3.3.1), we have, for all $u \in Q_2^* S_2^{-1} \cdot R^j$,

$$\frac{u^* B_s^* B_s u}{u^* u} = \frac{z^* a_2^2 S_2^{-2} z}{z^* z} \frac{y^* B^* B y}{y^* y} \frac{x^* a_2^{-2} S_2^2 x}{x^* x},$$

where $x = Q_2 u$, $y = S_2 x$ and $z = By$. By the Courant-Fischer theorem, see Parlett [15], and lemma 3.3.1,

$$\begin{aligned}
\tilde{\sigma}_j^2 &\leq \max_{u \in Q_2^* S_2^{-1} R^j} \frac{u^* B_s^* B_s u}{u^* u} \\
&\leq \max_{z \in R^{n_2}} \frac{z^* a_2^2 S_2^{-2} z}{z^* z} \cdot \max_{y \in R^j} \frac{y^* B^* B y}{y^* y} \cdot \max_{x \in R^{n_2}} \frac{x^* a_2^{-2} S_2^2 x}{x^* x} \\
&\leq \frac{1}{c_2^2} \cdot \max_{y \in R^j} \frac{y^* B^* B y}{y^* y}.
\end{aligned}$$

Hence using the Courant-Fischer theorem again, we have $c_2^2 \tilde{\sigma}_j^2 \leq \sigma_j^2$. The other inequality in (3.3.4) can be established similarly by using the maximin characterization of the j-th singular value. □

3.4 Matrix Identities and the Norm of Φ

In this section, we will prove three lemmas that are the tools for deriving the bounds for $\tilde{\sigma}_1$ and $\tilde{\sigma}_{n_2}$ in the next subsection. Let us begin by defining

$$R_s \equiv \frac{\rho_2}{\lambda_1} S_2^{-1} R_1 S_2 = \begin{bmatrix} \rho_2 & & & & & \\ -1 & \rho_2 & & & 0 & \\ & \cdot & \cdot & & & \\ & & \cdot & \cdot & & \\ & 0 & & -1 & \rho_2 & \\ & & & & -1 & 0 \end{bmatrix}, \tag{3.4.1}$$

$$G_s \equiv (\lambda_2\mu_2)^{-\frac{1}{2}} S_2^{-1} G_2 S_2 = \begin{bmatrix} \rho_2 & -1 & & & & \\ -1 & \rho_2 + \frac{1}{\rho_2} & -1 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & -1 & \rho_2 + \frac{1}{\rho_2} & -1 \\ & & & & -1 & \frac{1}{\rho_2} \end{bmatrix}, \tag{3.4.2}$$

$$Z_s \equiv \begin{bmatrix} \frac{1}{\rho_2} & 1 & & & & \\ -1 & 0 & 1 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & -1 & 0 & 1 \\ & & & & -1 & -\rho_2 \end{bmatrix} \tag{3.4.3}$$

and

$$\Gamma_s \equiv (\lambda_2\mu_2)^{-\frac{1}{2}} \cdot \Gamma_2 \equiv \operatorname{diag}(\gamma_1, ..., \gamma_{n_2}); \tag{3.4.4}$$

where by (2.2.24),

$$\gamma_j = \frac{\gamma_{2,j}}{(\lambda_2\mu_2)^{\frac{1}{2}}} = \begin{cases} \rho_2 + \frac{1}{\rho_2} - 2\cos\theta_j & 1 \le j < n_2 \\ 0 & j = n_2, \end{cases} \tag{3.4.5}$$

with

$$\theta_j = \frac{j\pi}{n_2}, \quad 1 \le j < n_2. \tag{3.4.6}$$

By (2.1.12), we have

$$Q_2^* G_s Q_2 = \Gamma_s. \tag{3.4.7}$$

It is straightforward to verify

Lemma 3.4.1 The following matrix identities hold for all $\rho_2 > 0$.

$$(1) \qquad R_s R_s^* = \rho_2 G_s, \tag{3.4.8}$$

$$(2) \qquad Z_s G_s + G_s Z_s^* = 0, \tag{3.4.9}$$

$$(3) \qquad Q_2^* Z_s Q_2 \Gamma_s + \Gamma_s Q_2^* Z_s^* Q_2 = 0, \tag{3.4.10}$$

$$(4) \qquad 2R_s = G_s + (\rho_2 - \frac{1}{\rho_2}) I_{n_2} + Z_s. \ \Box \tag{3.4.11}$$

We remark that the variables λ_i and μ_i have dimension (1 / time). In contrast, the variables ρ_2 and γ_j, $1 \le j < n_2$, are all dimensionless. Thus the matrices R_s, G_s, Z_s and Γ_s are dimensionless. Accordingly, we define the dimensionless

$$\Phi_s \equiv (\lambda_2 \mu_2)^{\frac{1}{2}} \cdot \Phi, \tag{3.4.12}$$

where Φ is given by (2.2.27). We note that by (3.3.1), (3.4.1) and (3.4.12),

$$B_s = I + \frac{\lambda_1}{\lambda_2} Q_2^* R_s Q_2 \Phi_s. \tag{3.4.13}$$

Moreover, the diagonal entries ϕ_j of Φ_s are given by

$$\phi_j = \frac{1}{\gamma_j} \left(1 - \frac{a_j}{\rho_1} \frac{1 - a_j^{2n_1 - 2}}{1 - a_j^{2n_1}} \right), \quad 1 \le j < n_2, \tag{3.4.14}$$

where a_j are given by (2.2.26). In view of (3.4.5), a_j is the smallest root of

$$a_j^2 - (\rho_1 + \frac{1}{\rho_1} + \zeta \gamma_j) a_j + 1 = 0, \quad 1 \le j < n_2, \tag{3.4.15}$$

where

$$\zeta \equiv \left(\frac{\lambda_2 \mu_2}{\lambda_1 \mu_1} \right)^{\frac{1}{2}} > 0. \tag{3.4.16}$$

We note that by (3.1.7), there exists a $C > 0$ such that,

$$C^{-1} < \zeta < C. \tag{3.4.17}$$

Since the constant term in (3.4.15) is 1, thus

$$a_j = \frac{\omega_j - (\omega_j^2 - 4)^{\frac{1}{2}}}{2} = \frac{2}{\omega_j + (\omega_j^2 - 4)^{\frac{1}{2}}}, \quad 1 \le j < n_2, \tag{3.4.18}$$

where

$$\omega_j \equiv \rho_1 + \frac{1}{\rho_1} + \zeta \gamma_j, \quad 1 \le j < n_2. \tag{3.4.19}$$

Since

$$x + \frac{1}{x} \ge 2, \quad \forall x \ge 0, \tag{3.4.20}$$

$\gamma_j > 0$ by (3.4.5). Hence $\omega_j > 2$ by (3.4.19) and $a_j < 1$ by (3.4.18).

Lemma 3.4.2 ϕ_j is a decreasing function of j for $1 \le j < n_2$.

Proof: By (3.4.14),

$$\phi_j = \frac{1}{\rho_1}\left(\frac{\rho_1 - a_j}{\gamma_j}\right) + \frac{1}{\rho_1}\left(\frac{1-a_j}{\gamma_j}\right)\cdot\left(a_j^{2n_1-1}\cdot\frac{1+a_j}{1-a_j^{2n_1}}\right), \quad 1 \le j < n_2. \tag{3.4.21}$$

We claim that the terms in the right hand side are decreasing functions of j. By (3.4.5), we see that γ_j is an increasing function of j for $1 \le j < n_2$. Thus by (3.4.19), ω_j increases w.r.t. j. Hence by (3.4.18), a_j decreases w.r.t. j and so is $a_j^{2n_1-1}\dfrac{1+a_j}{1-a_j^{2n_1}}$. Next we observe that by (3.4.15),

$$(a_j - \rho_1)(a_j - \frac{1}{\rho_1}) = \zeta\gamma_j a_j.$$

Thus the first term in (3.4.21) can be rewritten as

$$\frac{1}{\rho_1}\frac{\rho_1 - a_j}{\gamma_j} = \frac{\zeta a_j}{1-\rho_1 a_j}. \tag{3.4.22}$$

Since $a_j, \rho_1 < 1$, the right hand side is a decreasing function of j. Finally, since $\dfrac{1-a_j}{\gamma_j} = \dfrac{1-\rho_1}{\gamma_j} + \dfrac{\rho_1 - a_j}{\gamma_j}$, is a sum of two decreasing functions of j, $\dfrac{1}{\rho_1}\left(\dfrac{1-a_j}{\gamma_j}\right)$ is also a decreasing fuction of j. $\Box$

We note that the right hand side in (3.4.22) is positive, thus

$$1 > \rho_1 \ge a_j > 0. \tag{3.4.23}$$

Lemma 3.4.3 For $\alpha \ge 1$ and $h_i < H_i$, $i = 1, 2$, we have,

$$(1) \quad \phi_j \ge C > 0, \;\; 1 \le j < n_2, \tag{3.4.24}$$

$$(2) \quad \phi_j \le C \cdot \frac{n_2}{j}, \;\; 1 \le j < n_2, \tag{3.4.25}$$

Proof: To prove (1), we first note that by (3.4.5),

$$\gamma_j = \rho_2 + \frac{1}{\rho_2} - 2\cos\theta_j < \rho_2 + \frac{1}{\rho_2} + 2, \quad 1 \le j < n_2. \tag{3.4.26}$$

Hence by (3.4.18) and (3.4.19)

$$\frac{1}{a_j} < \omega_j < \rho_1 + \frac{1}{\rho_1} + \zeta(\rho_2 + \frac{1}{\rho_2} + 2) \quad 1 \le j < n_2. \tag{3.4.27}$$

Hence,

$$\frac{1-\rho_1 a_j}{a_j} = \frac{1}{a_j} - \rho_1 < \frac{1}{\rho_1} + \zeta(\rho_2 + \frac{1}{\rho_2} + 2), \ \ 1 \le j < n_2.$$

Since a_j, $\rho_1 < 1$, the first term is positive, hence

$$\frac{a_j}{1-\rho_1 a_j} > [\frac{1}{\rho_1} + \zeta(\rho_2 + \frac{1}{\rho_2} + 2)]^{-1} > 0. \tag{3.4.28}$$

Notice that the second term in (3.4.21) is nonnegative. Thus by (3.4.22),

$$\phi_j \ge \frac{\rho_1 - a_j}{\rho_1 \gamma_j} = \frac{\zeta a_j}{1-\rho_1 a_j} > \zeta[\frac{1}{\rho_1} + \zeta(\rho_2 + \frac{1}{\rho_2} + 2)]^{-1}, \ 1 \le j < n_2.$$

By (3.1.12) and (3.4.17), (3.4.24) follows.

Next we prove (2). By (3.4.14),

$$\phi_j = \frac{1}{\gamma_j}\left(1 - \frac{a_j}{\rho_1}\cdot\frac{1-a_j^{2(n_1-1)}}{1-a_j^{2n_1}}\right), \quad 1 \le j < n_2.$$

We note that $\frac{1-a_j^{2(n_1-1)}}{1-a_j^{2n_1}} \ge \frac{n_1-1}{n_1}$. This is because $a_j < 1$ and $f(t) = \frac{1}{t}(1-a^t)$ is a decreasing function of t when $0 < a < 1$ and $t > 0$. Thus

$$\phi_j \le \frac{1}{\gamma_j}\left(1 - \frac{a_j}{\rho_1}\frac{n_1-1}{n_1}\right) = \frac{\rho_1 - a_j}{\gamma_j \rho_1} + \frac{a_j}{\gamma_j \rho_1 n_1} < \frac{\zeta a_j}{1-a_j} + \frac{1}{\gamma_j n_1}, \tag{3.4.29}$$

where the last inequality follows from (3.4.22) and (3.4.23). Notice that by (3.4.15) and (3.4.20), we have

$$(a_j - 1)^2 = (\rho_1 + \frac{1}{\rho_1} - 2 + \zeta\gamma_j)a_j \ge \zeta\gamma_j a_j, \quad 1 \le j < n_2.$$

Since $a_j < 1$, this implies

$$\frac{\zeta a_j}{1-a_j} \le (\frac{\zeta a_j}{\gamma_j})^{\frac{1}{2}}, \quad 1 \le j < n_2. \tag{3.4.30}$$

Hence (3.4.29) becomes

$$\phi_j \le (\frac{\zeta a_j}{\gamma_j})^{\frac{1}{2}} + \frac{1}{\gamma_j n_1}, \quad 1 \le j < n_2. \tag{3.4.31}$$

Notice that by (3.4.20) and the inequality $\sin\frac{j\pi}{2n_2} \ge \frac{j}{n_2}$, we have

$$\gamma_j = \rho_2 + \frac{1}{\rho_2} - 2 + 4\sin^2\frac{\theta_j}{2} \ge 4\sin^2\frac{\theta_j}{2} \ge \frac{4j^2}{n_2^2}, \quad 1 \le j < n_2. \tag{3.4.32}$$

Thus (3.4.31) becomes

$$\phi_j \le \frac{\sqrt{\zeta} n_2}{2j} + \frac{n_2^2}{4j^2 n_1} \le \frac{1}{2}(\sqrt{\zeta} + \frac{C_0}{2j})\frac{n_2}{j} \le C \cdot \frac{n_2}{j},\ 1 \le j < n_2. \ \Box \tag{3.4.33}$$

Recall that by lemmas 2.2.2, ϕ_{n_2} can be defined arbitarily. Thus for simplicity, we assume in the following that

$$\phi_{n_2} = 0. \tag{3.4.34}$$

With this, lemmas 3.4.2 and 3.4.3 give

$$||\Phi_s||_2 \le C n_2. \tag{3.4.35}$$

3.5 Condition Number of B

We are now able to derive an upper bound on the condition number of B_s and B. We first obtain an upper bound for the largest singular value of B_s.

Lemma 3.5.1 For $\alpha \ge 1$ and $h_i < H_i$, $i = 1, 2$,

$$||B_s||_2 \le C \cdot n_2.$$

Proof: By (3.4.13), (3.4.35) and the fact that Q_2 is orthogonal,

$$||B_s||_2 \le 1 + \frac{\lambda_1}{\lambda_2}||R_s||_2||\Phi_s||_2 \le 1 + C\frac{\lambda_1}{\lambda_2}n_2||R_s||_2.$$

By (3.4.8), (3.4.7) and (3.4.26),

$$||R_s||_2^2 = ||R_s R_s^*||_2 = \rho_2||G_s||_2 = \rho_2||\Gamma_s||_2 \le (\rho_2 + 1)^2 \le 4.$$

Thus $||B_s||_2 \le 1 + C\frac{\lambda_1}{\lambda_2}n_2$. By (3.1.7), the lemma follows. $\Box$

To derive a lower bound for the smallest singular value of $B_s^* B_s$, we need

Lemma 3.5.2 Let $B_w \equiv B_s W$. If $\lambda_{\min}(B_w + B_w^*) \ge \delta > 0$ and W is nonsingular, then

$$||B_s^{-1}||_2 \le \frac{2}{\delta}||W||_2.$$

Proof: For arbitrary x, using the Cauchy-Schwartz inequality

$$\delta||x||_2^2 \le \lambda_{\min}(B_w + B_w^*)||x||_2^2 \le x^*(B_w + B_w^*)x = 2x^* B_w x \le 2||x||_2||B_w x||_2.$$

Since $B_w x$ is arbitrary, this implies $||B_w^{-1}||_2 \le 2/\delta$ and

$$||B_s^{-1}||_2 = ||WW^{-1}B_s^{-1}||_2 \le ||W||_2||B_w^{-1}||_2 \le \frac{2}{\delta}||W||_2. \ \Box$$

An appropriate W is given by

$$W \equiv \mathrm{diag}(\gamma_1 \phi_1^{-1}, ..., \gamma_{n_2-1}\phi_{n_2-1}^{-1}, 1), \tag{3.5.1}$$

where γ_j and ϕ_j^{-1} are diagonal entries of Φ_s and Γ_s respectively. We note that

$$\Phi_s W = \Gamma_s. \tag{3.5.2}$$

We establish

Lemma 3.5.3 For all $\alpha > 1$, there exists an $0 < H_3 \leq \min\{H_1, H_2\}$, such that for all $h_1, h_2 < H_3$, we have

$$\lambda_{\min}(B_w + B_w^*) \geq C \cdot h_2^3. \tag{3.5.3}$$

Proof: By (3.4.13), (3.5.2), (3.4.11) and (3.4.7),

$$B_w = B_s W = W + \frac{1}{2}\frac{\lambda_1}{\lambda_2}[\Gamma_s + (\rho_2 - \frac{1}{\rho_2}) \cdot I + Q_2^* Z_s Q_2]\Gamma_s.$$

Thus by (3.4.10),

$$B_w^* + B_w = 2W + \frac{\lambda_1}{\lambda_2}[\Gamma_s + (\rho_2 - \frac{1}{\rho_2})I]\Gamma_s,$$

which is a diagonal matrix. Hence by (3.4.5)

$$\lambda_{\min}(B_w^* + B_w) = 2 \cdot \min_{1 \leq j < n_2} \{\gamma_j \cdot [\phi_j^{-1} + \frac{\lambda_1}{\lambda_2}(\rho_2 - \cos\theta_j)], 1\}.$$

By lemma 3.4.2, ϕ_j^{-1} is an increasing function of j for $1 \leq j < n_2$, and by (3.4.5) and (3.4.6), so are γ_j and $-\cos\theta_j$. Thus

$$\lambda_{\min}(B_w^* + B_w) \geq 2 \cdot \min\{\gamma_1[\phi_1^{-1} + \frac{\lambda_1}{\lambda_2}(\rho_2 - \cos\theta_1)], 1\}. \tag{3.5.4}$$

Notice that by lemma 3.4.3, $\phi_1^{-1} \geq C \cdot h_2$, while by (3.1.2) and (3.4.6),

$$\rho_2 - \cos\theta_1 = \pi^2 h_2^2 - \frac{1}{2}\beta_2 h_2^\alpha + \text{higher order terms}. \tag{3.5.5}$$

Thus when $\alpha > 1$ and h_2 sufficiently small, $\phi_1^{-1} + \frac{\lambda_1}{\lambda_2}(\rho_2 - \cos\theta_1) > C \cdot h_2$. Hence by (3.5.4), $\lambda_{\min}(B_w^* + B_w) \geq C \cdot \gamma_1 h_2$. By (3.4.32), (3.5.3) follows. $\Box$

For $\alpha = 1$, by comparing (3.4.33) and (3.5.5), we see that for all sufficiently small h_2, the right hand side in (3.5.4) is positive if

$$2(\sqrt{\zeta} + \frac{C_0}{2})^{-1} > \frac{1}{2}\beta_2 \frac{\lambda_1}{\lambda_2}. \tag{3.5.6}$$

Thus we have

Corollary 3.5.4 If $\alpha = 1$ and (3.5.6) holds, then there exists an $H_3 > 0$, such that for all $h_1, h_2 < H_3$, (3.5.3) holds. $\Box$

We remark that in the case where $\lambda_1 = \lambda_2, \mu_1 = \mu_2$ and $n_1 = n_2$, then (3.5.6) is always satisfied.

Since ϕ_j^{-1} and γ_j are increasing functions of j, $||W||_2 \leq \max\{\gamma_{n_2-1}\phi_{n_2-1}^{-1}, 1\}$. By (3.4.26) and (3.4.24), we thus have

$$||W||_2 < C.$$

Hence by lemmas 3.5.2 and 3.5.3, we get

Lemma 3.5.5 Assume that either $\alpha = 1$ and (3.5.6) holds or $\alpha > 1$, then for $h_1, h_2 < H_3$, we have

$$||B_s^{-1}||_2 \leq C \cdot n_2^3. \Box$$

Combining this result with lemmas 3.5.1 and 3.3.2, we obtain

Theorem 3.5.6 Let $\kappa(B)$ be the condition number of B. Assume that either $\alpha = 1$ and (3.5.6) holds or $\alpha > 1$, then for $h_1, h_2 < H_3$, $\kappa(B) \leq Cn_2^4$. $\Box$

This theorem suggests that the convergence rate of the ordinary conjugate gradient method, when applied to the normal equations corresponding to B or B_s, may be extremely slow. However, in § 3.8, we will show that the method converges quickly as a consequence of a clustering of the singular values. To do so, we need more information about the matrix Φ_s and Z_s. For simplicity, we will also reduce the dimension of the problem further to $n_2 - 1$.

3.6 More Matrix Identities and the Approximation of Φ

Define the projection Q_p of Q_2 as the n_2 by $n_2 - 1$ matrix

$$Q_p \equiv [q_1, ..., q_{n_2-1}], \tag{3.6.1}$$

where q_i are the i-th column of Q_2. Define also the $n_2 - 1$ by $n_2 - 1$ matrices

$$\begin{aligned}
\Phi_p &\equiv \text{diag}(\phi_1, ..., \phi_{n_2-1}), && (3.6.2)\\
\Gamma_p &\equiv \text{diag}(\gamma_1, ..., \gamma_{n_2-1}), && (3.6.3)\\
L_1 &\equiv \frac{1}{\rho_2} Q_p^*({}^2e_1)({}^2e_1^*)Q_p, && (3.6.4)\\
L_2 &\equiv \rho_2 Q_p^*({}^2e_{n_2})({}^2e_{n_2}^*)Q_p, && (3.6.5)\\
L_0 &\equiv L_1 - L_2, && (3.6.6)
\end{aligned}$$

$$L_3 \equiv \begin{bmatrix} 1 & 0 & & & & \\ 0 & 0 & 0 & & 0 & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & 0 & 0 & \rho_2 \\ & & & & \rho_2 & -1 \end{bmatrix}, \tag{3.6.7}$$

$$L_4 \equiv \Phi_p L_0 + L_0 \Phi_p. \tag{3.6.8}$$

We remark that L_1 and L_2 are matrices of rank 1, while L_0, L_3 and L_4 are matrices of rank 2, 3 and 4 respectively. Moreover, these matrices are symmetric. We note that by (2.2.23) and (2.2.21), the (k,j)-entry of the L_i satisfy

$$(L_1)_{k,j} = \frac{1}{\rho_2}\frac{2}{n_2}\sin\psi_{2,k}\sin\psi_{2,j} = (-1)^{k+j}(L_2)_{k,j}, \quad 1 \le k,j < n_2. \tag{3.6.9}$$

Lemma 3.6.1 The following identities hold for all $\rho_2 > 0$.

(1) $$R_s^* R_s = \rho_2 G_s + L_3, \tag{3.6.10}$$

(2) $$L_0 = \frac{1}{2} Q_p^*(Zs^* + Zs)Q_p, \tag{3.6.11}$$

(3) $$R_s^* Q_2 \cdot ({}^2e_{n_2}) = \mathbf{0}_{n_2}, \tag{3.6.12}$$

(4) $$Q_2^* Z_s Q_2 \Phi_s = \begin{bmatrix} Q_p^* Z_s Q_p \Phi_p & \mathbf{0}_{n_2-1} \\ \mathbf{0}_{n_2-1}^* & 0 \end{bmatrix}, \tag{3.6.13}$$

(5) $$Q_2^* Z_s Q_2 \Gamma_s = \begin{bmatrix} Q_p^* Z_s Q_p \Gamma_p & \mathbf{0}_{n_2-1} \\ \mathbf{0}_{n_2-1}^* & 0 \end{bmatrix}, \tag{3.6.14}$$

(6) $$\frac{1}{2}\{Q_p^* Z_s Q_p \Phi_p + \Phi_p Q_p^* Z_s^* Q_p\} = L_4 - \frac{1}{2}\{\Phi_p Q_p^* Z_s Q_p + Q_p^* Z_s^* Q_p \Phi_p\}. \tag{3.6.15}$$

Proof: (1) and (2) can be proved by straightforward computations. (3) follows directly from (2.2.23) and (3.4.1). To prove (4), observe that by (3.4.34),

$$Q_2^* Z_s Q_2 \Phi_s {}^2e_{n_2} = Q_2^* Z_s Q_2 \mathbf{0}_{n_2} = \mathbf{0}_{n_2}$$

On the other hand, by (3.4.11), (3.4.7), (3.6.12) and (3.4.34),

$${}^2e_{n_2}^* Q_2^* Z_s Q_2 \Phi_s = \{2 \cdot {}^2e_{n_2}^* Q_2^* R_s Q_2 - {}^2e_{n_2}^*(\rho_2 - \frac{1}{\rho_2})I - {}^2e_{n_2}^* \Gamma_s\}\Phi_s = \mathbf{0}_{n_2}.$$

The proof of (5) is similar to that of (4). (6) follows from (2) and (3.6.8). □

For i = 1,2, we also define the $(n_2 - 1)$ by $(n_2 - 1)$ matrices

$$W_i \equiv \sum_{l=1}^{n_1-1} \tilde{\gamma}_l ({}^1q_{n_1,l})^2 (\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1} L_i (\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1}, \tag{3.6.16}$$

where $({}^1q_{n_1,l})$, given by (2.2.23), is the (n_1,l) entry of Q_1 and

$$\tilde{\gamma}_l = \frac{\gamma_{1,l}}{(\lambda_2\mu_2)^{\frac{1}{2}}} = \begin{cases} \frac{1}{\zeta}(\rho_1 + \frac{1}{\rho_1} - 2\cos\frac{l\pi}{n_1}) & 1 \le l < n_1, \\ 0 & l = n_1. \end{cases} \tag{3.6.17}$$

Here $\gamma_{1,l}$ and ζ given by (2.2.24) and (3.4.16) respectively. We have,

Lemma 3.6.2 W_1 and W_2 are positive semi-definite matrices and satisfy

$$(1) \qquad (W_1)_{k,j} = (-1)^{k+j}(W_2)_{k,j}, \;\; 1 \le k,j < n_2, \tag{3.6.18}$$

$$(2) \qquad W_1 - W_2 = \frac{1}{2}\{\Phi_p Q_p^* Z_s Q_p + Q_p^* Z_s^* Q_p \Phi_p^*\}. \tag{3.6.19}$$

Proof: To prove that the W_i are positive semi-definite, we first note that, the L_i are symmetric, and the $(\Gamma_p + \tilde{\gamma}_l \cdot I)$, $1 \le l < n_1$, are diagonal. Thus the W_i are symmetric. Next we observe that by (3.6.17) and (3.4.20),

$$\tilde{\gamma}_l \ge \frac{1}{\zeta}(2 - 2\cos\frac{l\pi}{n_1}) = \frac{4}{\zeta}\sin^2\frac{l\pi}{n_1} > 0, \;\; 1 \le l < n_1. \tag{3.6.20}$$

Hence for all $x \in R^{n_2-1}$, by the definition of W_i and L_i,

$$x^* W_1 x = \frac{1}{\rho_2}\sum_{l=1}^{n_1-1} \tilde{\gamma}_l({}^1q_{n_1,l})^2[x^*(\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1} Q_p^*({}^2e_1)]^2 \ge 0, \tag{3.6.21}$$

$$x^* W_2 x = \rho_2\sum_{l=1}^{n_1-1} \tilde{\gamma}_l({}^1q_{n_1,l})^2[x^*(\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1} Q_p^*({}^2e_{n_2})]^2 \ge 0. \tag{3.6.22}$$

To prove (1), we observe that by (3.6.16),

$$(W_i)_{k,j} = (L_i)_{k,j}\sum_{l=1}^{n_1-1}\frac{\tilde{\gamma}_l({}^1q_{n_1,l})^2}{(\gamma_k + \tilde{\gamma}_l)(\gamma_j + \tilde{\gamma}_l)}, \;\; 1 \le k,j < n_2, \; i = 1,2. \tag{3.6.23}$$

Thus by (3.6.9), (1) follows.

Next we prove (2). By (3.6.13), (3.4.10) and (3.6.11), we have, for $1 \le l \le n_1$,

$$\begin{aligned} Q_p^* Z_s Q_p(\Gamma_p + \tilde{\gamma}_l \cdot I) + (\Gamma_p + \tilde{\gamma}_l \cdot I) Q_p^* Z_s^* Q_p &= \tilde{\gamma}_l\{Q_p^*(Z_s + Z_s^*)Q_p\} \\ &= 2\tilde{\gamma}_l L_0. \end{aligned} \tag{3.6.24}$$

Since $\gamma_j > 0$ for $1 \le j < n_2$ and $\tilde{\gamma}_l \ge 0$ for $1 \le l \le n_1$, hence $(\Gamma_p + \tilde{\gamma}_l \cdot I)$, $1 \le l \le n_1$, are positive definite and thus invertible. By (3.6.24),

$$\begin{aligned} &(\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1} Q_p^* Z_s Q_p + Q_p^* Z_s^* Q_p (\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1} \\ &\quad = 2\tilde{\gamma}_l(\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1} L_0 (\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1}. \end{aligned} \tag{3.6.25}$$

Recall that by (2.2.16), (2.1.14) and (2.1.15),

$$\Phi = \sum_{l=1}^{n_1-1} ({}^1q_{n_1,l})^2 \cdot (\Gamma_2 + \gamma_{1,l} \cdot I)^{-1} + ({}^1q_{n_1,n_1})^2 \Gamma_2^+.$$

Restricting to the (n_2-1)-dimensional space and noting that $\tilde{\gamma}_{n_1} = 0$, it is easy to check that

$$\Phi_p = \sum_{l=1}^{n_1} ({}^1q_{n_1,l})^2 \cdot (\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1}. \tag{3.6.26}$$

Thus multiplying (3.6.25) by $({}^1q_{n_1,l})^2$ and taking the sum from $l = 1$ to n_1, we get

$$\begin{aligned}\Phi_p Q_p^* Z_s Q_p + Q_p^* Z_s^* Q_p \Phi_p^* &= 2 \sum_{l=1}^{n_1-1} \tilde{\gamma}_l ({}^1q_{n_1,l})^2 \cdot (\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1} L_0 (\Gamma_p + \tilde{\gamma}_l \cdot I)^{-1} \\ &= 2\{W_1 - W_2\},\end{aligned}$$

where the last equality follows from (3.6.6) and (3.6.16). □

Finally we give an approximation $\tilde{\Phi}_p$ of Φ_p. According to (3.4.21), we define

$$\tilde{\Phi}_p \equiv \operatorname{diag}(\tilde{\phi}_1, ..., \tilde{\phi}_{n_2-1}), \tag{3.6.27}$$

where $\tilde{\phi}_j$ is the first term of (3.4.21), i.e.,

$$\tilde{\phi}_j = \frac{\rho_1 - a_j}{\rho_1 \gamma_j}, \quad 1 \le j < n_2. \tag{3.6.28}$$

Define C_1 to be a constant such that

$$C_1 > \frac{C_0}{\sqrt{\zeta}} = \frac{n_2}{n_1} \left(\frac{\lambda_1 \mu_1}{\lambda_2 \mu_2} \right)^{\frac{1}{4}}. \tag{3.6.29}$$

It exists by (3.4.17). We claim that $\tilde{\Phi}_p$ is an approximation of Φ_p.

Lemma 3.6.3 For all $\alpha \ge 1$, there exists an $0 < H_4 < H_3$ and $C > 0$ such that for h_1 and $h_2 < H_4$ and $C_1 \log n_2 \le j < n_2$, we have

$$(1) \qquad a_j^{2n_1} < C/n_2^3, \tag{3.6.30}$$

$$(2) \qquad 0 < \phi_j - \tilde{\phi}_j < \frac{C}{j^2 n_2}. \tag{3.6.31}$$

Proof: We prove (1) first. Observe that by (3.4.19), (3.4.20) and (3.4.32),

$$(\omega_j^2 - 4)^{\frac{1}{2}} \ge [(2 + \zeta\gamma_j)^2 - 4]^{\frac{1}{2}} \ge [(2 + 4\zeta \frac{j^2}{n_2^2})^2 - 4]^{\frac{1}{2}} \ge 4\sqrt{\zeta} j/n_2, \ \ 1 \le j < n_2.$$

Thus (3.4.18) gives

$$a_j < 2(2 + 4\zeta\frac{j^2}{n_2^2} + 4\sqrt{\zeta}\frac{j}{n_2})^{-1} < (1 + \sqrt{\zeta}\frac{j}{n_2})^{-2}, \quad 1 \le j < n_2. \tag{3.6.32}$$

If $j \ge n_2/\sqrt{\zeta}$, this and (3.1.5) give

$$a_j^{2n_1} \le (1+1)^{-4n_1} = e^{-4n_1 \log 2} \le e^{-2n_1} \le \frac{1}{n_1^3} \le \frac{C_0^3}{n_2^3}. \tag{3.6.33}$$

If $j < n_2/\sqrt{\zeta}$, then (3.6.32) and (3.6.29) give

$$a_j^{2n_1} \le (1 + \sqrt{\zeta}\frac{j}{n_2})^{-4n_1} = e^{-4\sqrt{\zeta}j/C_0 + O(h_2)} \le e^{-3j/C_1},$$

for h_1 and h_2 sufficiently small. Hence if $C_1 \log n_2 \le j < n_2/\sqrt{\zeta}$,

$$a_j^{2n_1} \le e^{-3\log n_2} \le \frac{1}{n_2^3}.$$

Combining this with (3.6.33), we get (1).

To prove (2), we observe that by (3.4.21) and the definition of $\tilde{\phi}_j$, we have

$$\phi_j - \tilde{\phi}_j = \frac{1 - a_j^2}{\rho_1 a_j (1 - a_j^{2n_1})} \frac{a_j^{2n_1}}{\gamma_j}, \quad 1 \le j < n_2. \tag{3.6.34}$$

Since $a_j < 1$, this implies $\phi_j - \tilde{\phi}_j > 0$. By (3.4.27) and (3.6.30), we see that for $h_i < H_4$ and $C_1 \log n_2 \le j < n_2$,

$$\frac{1 - a_j^2}{\rho_1 a_j (1 - a_j^{2n_1})} \le \frac{1 + Ch_2^3}{\rho_1 a_j} \le C, \quad \forall\, C_1 \log n_2 \le j < n_2.$$

Hence by (3.6.30) and (3.4.32), (3.6.34) becomes

$$\phi_j - \tilde{\phi}_j \le \frac{C a_j^{2n_1}}{\gamma_j} \le \frac{C}{\gamma_j n_2^3} \le \frac{C}{j^2 n_2}, \quad \forall\, C_1 \log n_2 \le j < n_2. \Box$$

3.7 Clustering of Singular Values of B

Using the lemmas in the previous section, we are able to prove that the singular values of B are clustered. More precisely, we will show that $B^*B = (1 + \frac{\lambda_1}{\lambda_2}) \cdot I + L + U$, where L is a matrix of low rank and U is a matrix of small l_2 norm.

Recalling (3.4.13) and using (3.6.10), we have

$$\begin{aligned} B_s^* B_s &= I + \frac{\lambda_1}{\lambda_2}\{Q_2^* R_s Q_2 \Phi_s + \Phi_s Q_2^* R_s^* \Phi_s\} \\ &\quad + \rho_2 (\frac{\lambda_1}{\lambda_2})^2 \Phi_s Q_2^* G_s Q_2 \Phi_s + (\frac{\lambda_1}{\lambda_2})^2 \Phi_s Q_2^* L_3 Q_2 \Phi_s. \end{aligned} \tag{3.7.1}$$

For simplicity, we will use $\tilde{L}_j$ to denote matrices of rank less than or equal to j. Using (3.4.11) and (3.4.7), (3.7.1) becomes

$$B_s^* B_s = D + W + \tilde{L}_3, \tag{3.7.2}$$

where

$$D \equiv I + \frac{\lambda_1}{\lambda_2}\{\Gamma_s + (\rho_2 - \frac{1}{\rho_2})I + \rho_2(\frac{\lambda_1}{\lambda_2})\Phi_s\Gamma_s\}\Phi_s \tag{3.7.3}$$

is a diagonal matrix, and

$$W \equiv \frac{1}{2}(\frac{\lambda_1}{\lambda_2})\{Q_2^* Z_s Q_2 \Phi_s + \Phi_s Q_2^* Z_s^* \Phi_s\}. \tag{3.7.4}$$

By (3.6.13), $W = \begin{bmatrix} W_p & \mathbf{0}_{n_2-1} \\ \mathbf{0}_{n_2-1}^* & 0 \end{bmatrix}$, where

$$W_p \equiv \frac{1}{2}(\frac{\lambda_1}{\lambda_2})\{Q_p^* Z_s Q_p \Phi_p + \Phi_p Q_p^* Z_s^* \Phi_p\}$$

is a matrix of order $n_2 - 1$. By (3.6.15) and (3.6.19),

$$W_p = \frac{\lambda_1}{\lambda_2}\{L_4 + W_2 - W_1\}. \tag{3.7.5}$$

By (3.4.34), $D = \begin{bmatrix} D_p & \mathbf{0}_{n_2-1} \\ \mathbf{0}_{n_2-1}^* & 1 \end{bmatrix}$, where D_p is given by

$$\begin{aligned} D_p &\equiv \operatorname{diag}(d_1, ..., d_{n_2-1}) \\ &= I + \frac{\lambda_1}{\lambda_2}\{\Gamma_p + (\rho_2 - \frac{1}{\rho_2})I + \rho_2(\frac{\lambda_1}{\lambda_2})\Phi_p\Gamma_p\}\Phi_p. \end{aligned} \tag{3.7.6}$$

Combining this with (3.7.5), (3.7.2) gives

$$B_s^* B_s = \begin{bmatrix} D_p + \frac{\lambda_1}{\lambda_2}(W_2 - W_1) & \mathbf{0}_{n_2-1} \\ \mathbf{0}_{n_2-1}^* & 1 + \frac{\lambda_1}{\lambda_2} \end{bmatrix} + \tilde{L}_8. \tag{3.7.7}$$

Corresponding to (3.6.28), we define the approximation $\tilde{D}_p$ of D_p as

$$\begin{aligned} \tilde{D}_p &\equiv \operatorname{diag}(\tilde{d}_1, ..., \tilde{d}_{n_2-1}) \\ &\equiv I + \frac{\lambda_1}{\lambda_2}\{\Gamma_p + (\rho_2 - \frac{1}{\rho_2})I + \rho_2(\frac{\lambda_1}{\lambda_2})\tilde{\Phi}_p\Gamma_p\}\tilde{\Phi}_p. \end{aligned} \tag{3.7.8}$$

Lemma 3.7.1 For $\alpha \geq 1$, there exists a $C_2 > 0$ such that for $h_i < H_4$, $i = 1,2$,

$$|\tilde{d}_j - d_j| < \frac{C_2 h_2}{j^2}, \ \ \forall\, C_1 \log n_2 \leq j < n_2. \tag{3.7.9}$$

Here C_1 and H_4 are given by (3.6.29) and lemma 3.6.3 respectively.
Proof: By (3.7.6) and (3.7.8),

$$\begin{aligned} d_j - \tilde{d}_j &= (\frac{\lambda_1}{\lambda_2})\{\gamma_j + (\rho_2 - \frac{1}{\rho_2})I + \rho_2(\frac{\lambda_1}{\lambda_2})\gamma_j\phi_j\}(\phi_j - \tilde{\phi}_j) \\ &\quad +(\frac{\lambda_1}{\lambda_2})^2\rho_2\gamma_j\tilde{\phi}_j(\phi_j - \tilde{\phi}_j), \quad 1 \le j < n_2. \end{aligned} \tag{3.7.10}$$

Notice that by (3.4.31), (3.4.23) and (3.4.26), we have

$$\gamma_j\phi_j \le (\zeta\gamma_j)^{\frac{1}{2}} + n_1^{-1} \le C, \quad 1 \le j < n_2.$$

Hence by (3.6.31),

$$\gamma_j\tilde{\phi}_j \le \gamma_j\phi_j \le C, \quad C_1 \log n_2 \le j < n_2. \tag{3.7.11}$$

Thus by (3.1.7), (3.4.26) and (3.6.31), (3.7.10) gives

$$|d_j - \tilde{d}_j| < C|\phi_j - \tilde{\phi}_j| < \frac{C_2 h_2}{j^2}, \quad C_1 \log n_2 \le j < n_2. \;\Box$$

Lemma 3.7.2 For $\alpha \ge 1$, there exists a constant $C_3 > 0$ such that when $h_i < H_4$, $i = 1,2$,

$$|\tilde{d}_j - (1 + \frac{\lambda_1}{\lambda_2})| < C_3 h_2^{\alpha-1}/j, \quad \forall\, 1 \le j < n_2. \tag{3.7.12}$$

Proof: By (3.7.8) and (3.6.28), we have

$$\tilde{d}_j = 1 + \frac{\lambda_1}{\lambda_2}\frac{\rho_1 - a_j}{\rho_1\gamma_j}\{\gamma_j + (\rho_2 - \frac{1}{\rho_2}) + \rho_2(\frac{\lambda_1}{\lambda_2})\frac{\rho_1 - a_j}{\rho_1}\}, \quad 1 \le j < n_2.$$

By (3.4.16), this can be rewritten as

$$\tilde{d}_j = 1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_2}\frac{(\rho_1 - a_j)}{\rho_1\gamma_j\zeta}\{-\frac{\zeta a_j\gamma_j}{\rho_1 - a_j} + \zeta(\rho_2 - \frac{1}{\rho_2}) + (\rho_1 - a_j)\}.$$

By (3.4.22) and after some simplification, this becomes

$$(1 + \frac{\lambda_1}{\lambda_2}) - \tilde{d}_j = \frac{\lambda_1}{\lambda_2}\frac{a_j}{1 - \rho_1 a_j}\{(\frac{1}{\rho_1} - \rho_1) + \zeta(\frac{1}{\rho_2} - \rho_2)\}.$$

Since $a_j, \rho_1 < 1$, we have

$$|\tilde{d}_j - (1 + \frac{\lambda_1}{\lambda_2})| \le \frac{\lambda_1}{\lambda_2}\frac{a_j}{1 - a_j}|(\frac{1}{\rho_1} - \rho_1) + \zeta(\frac{1}{\rho_2} - \rho_2)|, \quad 1 \le j < n_2. \tag{3.7.13}$$

Notice that by (3.4.30) and (3.4.32),

$$\frac{a_j}{1-a_j} \le (\frac{a_j}{\zeta\gamma_j})^{\frac{1}{2}} \le \frac{1}{\sqrt{\zeta}\gamma_j} \le \frac{n_2}{2\sqrt{\zeta}j}, \;\; 1 \le j < n_2. \qquad (3.7.14)$$

On the other hand, by the definition of ζ and ρ_i in (3.4.16) and (3.1.9),

$$\frac{\lambda_1}{\lambda_2}|(\frac{1}{\rho_1} - \rho_1) + \zeta(\frac{1}{\rho_2} - \rho_2)| = \frac{\lambda_1}{\lambda_2}\frac{1}{\rho_1}|\beta_1 h_1^\alpha + \frac{\mu_2}{\mu_1}\beta_2 h_2^\alpha| \le C \cdot h_2^\alpha. \qquad (3.7.15)$$

Hence the lemma follows by putting this and (3.7.14) into (3.7.13). □

Combining lemmas 3.7.1 and 3.7.2, we have, for $h_i < H_4$, $i = 1$,2,

$$\begin{aligned} |d_j - (1+\frac{\lambda_1}{\lambda_2})| &\le \frac{C_2 h_2}{(C_1 \log n_2)^2} + \frac{C_3 h_2^{\alpha-1}}{C_1 \log n_2} \\ &= C_4 h_2^{\alpha_0} / \log n_2, \; \forall\, C_1 \log n_2 \le j < n_2. \end{aligned} \qquad (3.7.16)$$

Here

$$\alpha_0 \equiv \min\{\alpha - 1, 1\}, \qquad (3.7.17)$$

and C_4 is any constant such that

$$C_4 \ge \frac{1}{C_1}(\frac{C_2}{C_1 \log n_2} + C_3). \qquad (3.7.18)$$

Hence we have

Corollary 3.7.3 For $\alpha \ge 1$ and $h_i < H_4$, $i = 1$,2,

$$D_p = (1 + \frac{\lambda_1}{\lambda_2}) \cdot I + \tilde{L}_{C_1 \log n_2} + U_p, \qquad (3.7.19)$$

where U_p is a diagonal matrix with

$$||U_p||_2 < C_4 h_2^{\alpha_0} / \log n_2. \square \qquad (3.7.20)$$

Next we claim that the eigenvalues of W_i, $i = 1$,2, and hence of W_p, are clustered around zero. From lemma 3.6.2, the W_i are positive semi-definite matrices. Thus

$$\mathrm{tr}(W_i) \equiv \sum_{j=1}^{n_2-1} (W_i)_{jj} = \sum_{j=1}^{n_2-1} \lambda_j(W_i) \ge 0, \;\; i = 1, 2, \qquad (3.7.21)$$

where $\lambda_j(W_i) \ge 0$ are the eigenvalues of W_i.

Lemma 3.7.4 For $\alpha \ge 1$, there exists a $C_5 > 0$ and $0 < H_5 \le H_4$, such that for all h_1 and $h_2 < H_5$,

$$\mathrm{tr}(W_i) < C_5 \log n_2, \;\; i = 1, 2. \qquad (3.7.22)$$

Proof: By (3.6.23) and (3.6.9)

$$(W_1)_{jj} = (W_2)_{jj} = \frac{1}{\rho_2}\frac{2}{n_2}\sin^2\psi_{2,j}\sum_{l=1}^{n_1-1}\frac{\tilde{\gamma}_l({}^1q_{n_1,l})^2}{(\tilde{\gamma}_l+\gamma_j)^2},\quad 1 \le j < n_2.$$

Using (2.2.21) - (2.2.24) and (3.6.17), this becomes

$$(W_i)_{jj} = \frac{4}{\zeta\rho_1\rho_2 n_1 n_2}\sin^2\psi_{2,j}\sum_{l=1}^{n_1-1}\frac{\sin^2\theta_{1,j}}{(\tilde{\gamma}_l+\gamma_j)^2},\quad 1 \le j < n_2. \tag{3.7.23}$$

By (3.4.32), $\gamma_j \ge 4j^2/n_2^2$. Similarly, $\tilde{\gamma}_l \ge 4l^2/(\zeta n_1^2)$. Using these and the fact that $\sin\theta_{1,l} = \sin\dfrac{l\pi}{n_1} \le \dfrac{l\pi}{n_1}$ and $\sin^2\psi_{2,j} \le 1$, (3.7.23) becomes

$$(W_i)_{jj} \le \frac{\zeta\pi^2}{4\rho_1\rho_2 n_1 n_2}\sum_{l=1}^{n_1-1}\frac{(\frac{l}{n_1})^2}{[\zeta(\frac{j}{n_2})^2+(\frac{l}{n_1})^2]^2},\quad 1 \le j < n_2. \tag{3.7.24}$$

Let $y_j^2 = \zeta j^2/n_2^2$. Consider the function $f_j(x) \equiv x^2/(y_j^2+x^2)^2$. The maximum of $f_j(x)$ for $x \ge 0$ is at $x = y_j$ where $f_j(y_j) = (2y_j)^{-2}$. Hence for $1 \le j < n_2$,

$$\begin{aligned}
(W_i)_{jj} &\le \frac{\zeta\pi^2}{4\rho_1\rho_2 n_2}\left\{\int_0^1 \frac{x^2}{(y_j^2+x^2)^2}dx + \frac{2}{n_1}\max_{(0,1)}\frac{x^2}{(y_j^2+x^2)^2}\right\} \\
&\le \frac{\zeta\pi^2}{4\rho_1\rho_2 n_2}\left\{\frac{\pi}{4y_j} + \frac{1}{2n_1 y_j^2}\right\} = \frac{\sqrt{\zeta}\pi^3}{16\rho_1\rho_2 j} + \frac{C_0\pi^2}{8\rho_1\rho_2 j^2},
\end{aligned}$$

where C_0 is given in (3.1.6). Hence for i = 1,2,

$$\begin{aligned}
\mathrm{tr}(W_i) &\le \frac{\sqrt{\zeta}\pi^3}{16\rho_1\rho_2}\sum_{j=1}^{n_2-1} j^{-1} + \frac{C_0\pi^2}{8\rho_1\rho_2}\sum_{j=1}^{n_2-1} j^{-2}, \\
&\le \frac{\sqrt{\zeta}\pi^3}{16\rho_1\rho_2}(\log n_2 + \gamma) + \frac{C_0\pi^2}{8\rho_1\rho_2}\sum_{j=1}^{\infty} j^{-2},
\end{aligned}$$

where γ is the Euler constant. Using the Euler formula $\sum_{j=1}^{\infty} j^{-2} = \pi^2/6$, we have

$$\mathrm{tr}(W_i) \le \frac{\pi^3}{16\rho_1\rho_2}\left\{\sqrt{\zeta}\log n_2 + \gamma + \frac{\pi C_0}{3}\right\},\ i = 1, 2. \tag{3.7.25}$$

Thus there exists a C_5 such that (3.7.22) holds for all sufficiently large n_2. $\square$

From this lemma, we see that the number of eigenvalues of W_i that are larger than 1 cannot exceed $C_5 \log n_2$. Since the W_i are symmetric, they can be diagonalized. By separating those eigenvalues that are greater than 1 from those that are smaller, we have

Collorary 3.7.5 If $\alpha \geq 1$ and h_1, $h_2 < H_5$, then

$$W_i = V_i + \tilde{L}^i_{C_5 \log n_2}, \; i = 1, 2. \tag{3.7.26}$$

Here the V_i are positive semi-definite matrices with $||V_i||_2 < 1$ and the $\tilde{L}^i_{C_5 \log n_2}$ are positive definite matrices with

$$\lambda_{\min}(\tilde{L}^i_{C_5 \log n_2}) \geq 1. \; \Box \tag{3.7.27}$$

Define the constant

$$C_6 \equiv 2C_5 + C_1. \tag{3.7.28}$$

By (3.7.19) and (3.7.27), we get

$$D_p + \frac{\lambda_1}{\lambda_2}(W_2 - W_1) = I + \frac{\lambda_1}{\lambda_2}(I + V_2 - V_1) + \tilde{L}_{C_6 \log n_2} + U_p, \tag{3.7.29}$$

where

$$\tilde{L}_{C_6 \log n_2} \equiv \tilde{L}^2_{C_5 \log n_2} - \tilde{L}^1_{C_5 \log n_2} + \tilde{L}_{C_1 \log n_2}. \tag{3.7.30}$$

Clearly, the eigenvalues of $I + \frac{\lambda_1}{\lambda_2}(I + V_2 - V_1)$ lie in the interval $[1, 1 + 2\frac{\lambda_1}{\lambda_2}]$. Thus by (3.7.7) and (3.7.20), we have

Corollary 3.7.6 If $\alpha \geq 1$ and $h_i < H_5$, i = 1,2, then

$$B_s^* B_s = V_s + U_s + \tilde{L}_{C_6 \log n_2 + 8}. \tag{3.7.31}$$

Here

$$V_s \equiv \begin{bmatrix} I + \frac{\lambda_1}{\lambda_2}(I + V_2 - V_1) & \mathbf{0}_{n_2-1} \\ \mathbf{0}^*_{n_2-1} & 1 + \frac{\lambda_1}{\lambda_2} \end{bmatrix} \tag{3.7.32}$$

is symmetric with

$$\lambda_j(V_s) \in [1, 1 + 2\frac{\lambda_1}{\lambda_2}], \;\; 1 \leq j \leq n_2, \tag{3.7.33}$$

$$U_s \equiv \begin{bmatrix} U_p & \mathbf{0}_{n_2-1} \\ \mathbf{0}^*_{n_2-1} & 0 \end{bmatrix} \tag{3.7.34}$$

is diagonal with

$$||U_s||_2 < C_4 h_2^{\alpha_0} / \log n_2, \tag{3.7.35}$$

and

$$\tilde{L}_{C_6 \log n_2 + 8} \equiv \tilde{L}_{C_6 \log n_2} + \tilde{L}_8. \; \Box \tag{3.7.36}$$

From (3.1.7), (3.7.33) and (3.7.35), we see that there exists d_1 and $d_2 > 0$, independent of h_i, such that for all h_i sufficiently small, all the eigenvalues of $V_s + U_s$ lie in the interval $[d_1, d_2]$. Notice that the matrices V_s and U_s are symmetric, hence $\tilde{L}_{C_6 \log n_2 + 8}$ is also symmetric. By writing $\tilde{L}_{C_6 \log n_2 + 8}$ as a sum of $(C_6 \log n_2 + 8)$'s symmetric rank one matrices and using the Cauchy interlace theorem (see Parlett [15]) repeatedly for $(C_6 \log n_2 + 8)$ times, we see that all except $(C_6 \log n_2 + 8)$ eigenvalues of $B_s^* B_s$ lie in the interval $[d_1, d_2]$. Using lemma 3.3.2, we immediate get

Corollary 3.7.7 For $\alpha \geq 1$, there exists b_1 and $b_2 > 0$, independent of h_i, such that for h_1, $h_2 < H_6$, all except $(C_6 \log n_2 + 8)$ eigenvalues of $B^* B$ lie in the interval $[b_1, b_2]$. $\square$

Using this corollary, we are able to derive an upper bound for the number of iterations required to attain a given accuracy.

3.8 The Rate of Convergence

In this section, we will show that the number of iterations required to attain a given accuracy grows no faster than $O(\log^2 n_2)$. We begin with

Lemma 3.8.1 Let x be the solution to $B^* B x = B^* b$ and x_j be the j-th iterant of the ordinary conjugate gradient method applied to this normal equation. If the eigenvalues $\{\delta_j\}$ of $B^* B$ are such that

$$0 < \delta_1 \leq \ldots \leq \delta_{p-1} < \delta_p \equiv b_1 \leq \ldots \leq \delta_{n_2 - q} \equiv b_2 < \delta_{n_2 - q + 1} \leq \ldots \leq \delta_{n_2},$$

then

$$\frac{||B(x - x_j)||_2}{||B(x - x_0)||_2} \leq 2 \left(\frac{b-1}{b+1} \right)^{j-p-q} \cdot \max_{\delta \in [b_1, b_2]} \left\{ \prod_{j=1}^{p} \left(\frac{\delta - \delta_j}{\delta_j} \right) \prod_{j=n_2-q+1}^{n_2} \left(\frac{\delta_j - \delta}{\delta_j} \right) \right\}. \tag{3.8.1}$$

Here

$$b \equiv \left(\frac{b_2}{b_1} \right)^{\frac{1}{2}} \geq 1. \ \square$$

The proof can be found in Van der Vorst [18], see also Daniel [6]. Notice that for $n_2 - q + 1 \leq j \leq n_2$ and $\delta \in [b_1, b_2]$, we have, $0 \leq \dfrac{\delta_j - \delta}{\delta_j} \leq 1$. Thus (3.8.1) can be rewritten as

$$\frac{||B(x - x_j)||_2}{||B(x - x_0)||_2} \leq 2 \left(\frac{b-1}{b+1} \right)^{j-p-q} \cdot \max_{\delta \in [b_1, b_2]} \prod_{j=1}^{p} \left(\frac{\delta - \delta_j}{\delta_j} \right). \tag{3.8.2}$$

From lemmas 3.5.5, we see that if $\alpha > 1$ or if $\alpha = 1$ such that (3.5.6) holds, then $\delta_j \geq C \cdot h_2^6$, for $1 \leq j \leq n_2$. Thus for $1 \leq j \leq p$ and $\delta \in [b_1, b_2]$, we have, $0 \leq \dfrac{\delta - \delta_j}{\delta_j} \leq C n_2^6$. Hence (3.8.2) becomes

$$\frac{||B(x - x_j)||_2}{||B(x - x_0)||_2} \leq C^p n_2^{6p} \cdot \left(1 - \frac{2}{b+1} \right)^{j-p-q} = C^p n_2^{6p} \cdot e^{(j-p-q) \log(1 - \frac{2}{b+1})}.$$

Since $\log(1-x) \le -x$ for $0 \le x \le 1$, we have

$$\frac{||B(x-x_j)||_2}{||B(x-x_0)||_2} \le e^{p\log C + 6p\log n_2 + (p+q-j)\frac{2}{b+1}}.$$

Thus given arbitrary $\epsilon > 0$, an upper bound for the number of iterations required to make

$$\frac{||B(x-x_j)||_2}{||B(x-x_0)||_2} \le \epsilon$$

is given by

$$j_0 \equiv \frac{b+1}{2}\{p\log C + 6p\log n_2 - \log\epsilon\} + p + q.$$

Notice that by corollary 3.7.7, $b = b_2/b_1$ is independent of n_2. Moreover, we have

$$p, q < p + q \le C_6 \log n_2 + 8.$$

Thus j_0 is bounded above by $C \cdot \log^2 n_2$. Hence we have,

Theorem 3.8.2 Assume that either $\alpha = 1$ and (3.5.6) holds or $\alpha > 1$, then the number of iterations required to reduce $||B(x-x_j)||_2/||B(x-x_0)||_2$ by a given accuracy can grow no faster than $O(\log^2 n_2)$ as h_i tends to zero. □

3.9 Concluding Remarks

3.9.1 The Case when $\lambda_i \ge \mu_i$ and $\alpha \ge 1$

We remark that we can also establish the fast convergence in these cases. The idea is to expand ρ_i around $h_i = 0$. Let us assume

$$\frac{\lambda_i}{\mu_i} = 1 + \beta_i h_i^\alpha > 1, \tag{3.9.1}$$

where

$$\alpha \ge 1 \text{ and } 0 \le \beta_i \le 1. \tag{3.9.2}$$

Here α, μ_i and β_i are assumed to be constant independent of h_i. The proof for the following theorem is given in Chan [4].

Theorem 3.9.1 If (3.9.1) and (3.9.2) hold, then the number of iterations required to reduce $||B(x-x_j)||_2/||B(x-x_0)||_2$ by a given accuracy can grow no faster than $O(\log^2 n_2)$ as h_i tends to zero. □

We note that the case where $\lambda_i = \mu_i, i = 1{,}2$, can be treated as a particular case of (3.1.2) with $\alpha = \infty$. We remark that in this case, we are in fact preconditioning an oblique BVP by a Neumann BVP. To see this, we first observe that by (2.1.5) and the fact that $\lambda_i = \mu_i$,

$$A_0 = \lambda_1(\tilde{G}_1 \otimes I_{n_2}) + \lambda_2(I_{n_1} \otimes \tilde{G}_2), \tag{3.9.3}$$

where the $\tilde{G}_i$ are of order n_i and are given by

$$\tilde{G}_i \equiv \text{tridiag}(-1,2,-1) - e_1 e_1^* - e_{n_i} e_{n_i}^*, \quad i = 1,2. \tag{3.9.4}$$

Thus A_0 is the 5-point formula for the Neumann problem

$$\begin{cases} \lambda_1 \dfrac{\partial^2 p}{\partial x^2} + \lambda_2 \dfrac{\partial^2 p}{\partial y^2} = 0 & \text{in } [0,1]^2, \\ \qquad\qquad\qquad \dfrac{\partial p}{\partial \eta} = 0 & \text{on } \partial[0,1]^2, \end{cases} \tag{3.9.5}$$

with mesh-size $h_i = (n_i - 1)^{-1}, i = 1,2$. Here η denotes the unit outward normal of the square $[0,1]^2$. On the other hand, from (2.2.6) and (3.4.1), we see that

$$R_0 = ({}^1e_{n_1} {}^1e_{n_1}^*) \otimes R_1 = ({}^1e_{n_1} {}^1e_{n_1}^*) \otimes (\lambda_1 \cdot R_s),$$

where R_s is given by (3.4.1) with $\rho_2 = 1$ there. Thus R_s is a forward difference approximation of ∂_x. Hence it is easy to check that $A = A_0 + R_0$ is a 5-point formula for the oblique BVP

$$\begin{cases} \lambda_1 \dfrac{\partial^2 p}{\partial x^2} + \lambda_2 \dfrac{\partial^2 p}{\partial y^2} = 0 & \text{in } [0,1]^2, \\ \qquad\qquad\qquad \dfrac{\partial p}{\partial \gamma} = 0 & \text{on } \partial[0,1]^2, \end{cases} \tag{3.9.6}$$

Here γ is a directional vector defined on $\partial[0,1]^2$ and is given by

$$\gamma = \begin{cases} \eta & \text{if } x \neq 1, \\ \eta + \tau & \text{if } x = 1, \end{cases}$$

where τ is the unit tangential vector.

Thus the preconditioning of A by A_0 is the discrete version of preconditioning the oblique problem (3.9.6) by the Neumann problem (3.9.5). By theorem 3.9.1, the matrix A_0 is a very good preconditioner for A. In fact, by the results in § 3.7, the singular values of the preconditioned matrix $AA_0^+ = I + R_0 A_0^+$ are clustered around $(1 + \frac{\lambda_1}{\lambda_2})^{\frac{1}{2}}$.

3.9.2 The Multi-Server Case

For $s_i > 1$, then instead of (2.2.28), the underlying continous equation is of the form

$$\begin{aligned} &(\lambda_1 + s_1\mu_1)p_{xx} + (\lambda_2 + s_2\mu_2)p_{yy} \\ &\qquad + 2h_1(s_1\mu_1 - \lambda_1)p_x + 2h_2(s_2\mu_2 - \lambda_2)p_y \approx 0, \end{aligned} \tag{3.9.7}$$

in the region where $s_1 h_1 \leq x = ih_1 \leq 1$ and $s_2 h_2 \leq y = jh_2 \leq 1$. In other part of the square $[0,1]^2$, the equation has variable coefficients. Thus if s_i are constant independent of h_i, then for sufficiently small h_i, a reasonable limit to consider is

$$s_i\mu_i = \lambda_i \pm \eta_i h_i^\alpha, \quad i = 1, 2, \tag{3.9.8}$$

for some constants η_i and α. We remark that when $s_i = 1$ and $\eta_i \leq \mu_i$, $i = 1, 2$, this reduces to the limit we discussed previously.

When s_i increases proportionally to n_i, (3.9.8) may not be the right limit to consider. We remark that in this case, the diagonal matrix S_i that is used in the transformation (3.3.1) is no longer well-conditioned. In fact, by (2.1.8), the last diagonal entry of S_i is given by

$$^i d_{n_i} = a_i \cdot \left[\frac{1}{s_i!} (\frac{\lambda_i}{\mu_i})^{s_i} (\frac{\lambda_i}{s_i\mu_i})^{n_i - s_i - 1} \right]^{\frac{1}{2}}, \quad i = 1, 2.$$

Thus by (3.9.8) and Stirling's formula, we have, for $\alpha \geq 1$,

$$^i d_{n_i} \approx a_i \cdot \left[\frac{(s_i)^{s_i}}{s_i!} \right]^{\frac{1}{2}} \approx C s_i^{\frac{1}{4}} e^{-\frac{1}{2}s_i}, \quad i = 1, 2.$$

Thus the condition number of S_i increases exponentially for all $\alpha \geq 1$.

We remark that the ordinary conjugate gradient method indeed converges within a few steps when $s_i =$ constant, while it diverges when $s_i = n_i - 1$, see § 4.

This concludes our discussion of the model problems.

4 Numerical Results

In this section, we report on the numerical results for the models discussed in § 2.2. The computations were carried out on the Cyber-760 at the Mathematics and Computing Laboratory of the Courant Institute and Cyber-730 at the University of Massachusetts at Amherst. Single precision, between fourteen and fifteen decimal digits, was used throughout. Craig's method, used in these computations, is a version of the ordinary conjugate gradient method applied to the normal equations; see Elman [7]. The initial iterant x_0 is chosen to be identically zero and the tolerance is set to 10^{-10}. We note that a larger tolerance will be very adequate in most situations. In the tables β_i, η_i and α are parameters defined by (3.1.3) or (3.9.8).

Let us first consider the case where $\lambda_i < \mu_i$ and $s_i = 1$. Tables 1 and 2 give the number of iterations required to converge for two different choices of β_i. We see that the convergence rate is almost independent of β_i. The results also show that the case where $\alpha = 1$ is critical for Craig's method. More precisely, when $\alpha > 1$, the number of iterations seems to be constant as $n_i \to \infty$. When $\alpha = 1$, it increases like $O(\log n_i)$. This is consistent with (3.7.17). When $\alpha < 1$, Craig's method does not converge for sufficiently large n_i. However, from theorem 3.2.1, we see that for sufficiently large n_i, there is no need to solve the matrix equation numerically.

Next we consider the case where $\lambda_i \geq \mu_i$ and $s_i = 1$. Table 3 gives the results of our method for different choices of α. In the table, $\alpha = \infty$ represents the problems where $\lambda_i = \mu_i$. We see that the convergence rate in this case is almost the same as in the case where $\lambda_i < \mu_i$.

Table 4 gives the time in seconds required for the different stages of the algorithms. 'Initialization' refers to the generation of Φ, S_i, Q_i and the right hand side b. 'Iteration' refers to the solving of $By_0 = b$ by the iterative methods. We have used the Fast Fourier Transform in computing the matrix-vector multiplication in each iteration. 'Generating p' refers to the computation of p in (2.2.14). We note that the timings are consistent with the theoretical estimates. We remark that substantial saving would result if only a few entries of p were needed.

Table 5 gives the results for a family of multi-server problems that satisfy (3.9.8). As remarked there, when $s_i = O(n_i)$, (3.9.8) may not be a good limit. Indeed, Craig's method diverges when $s_i = n_i - 1$. We note that in the multi-server case, $s_i > 1$, we cannot use the Fast Fourier Transform. Hence the work and storage requirement per iteration are $2n_i^2 + O(n_i)$ and $n_i^2 + O(n_i)$ respectively.

Let us compare our method with other conventional methods. First we assume that the system (2.2.2) is solved by a classical iterative method such as the point SOR method, see Kaufman [11]. Since the graph of the generating matrix A is the same as the graph of a discrete Laplacian, it is clear that the point SOR method requires $7n_i^2 + O(n_i)$ work and $n_i^2 + O(n_i)$ storage spaces per iteration. We note that this method converges very slowly. Tables 6 and 7 give the numerical evidence. Table 6 lists the number of iterations required by the two methods and table 7 compares the time required in seconds. In the tables, ω^* denotes the optimal relaxation factor up to three decimal points obtained experimentally. We see that the point SOR method has a very slow convergence rate especially when s_i is small. In the three cases we considered, our method converges 3 to 29 times faster than the point SOR method.

Let us now consider the approach of solving the system by a direct method. Since the band-width of the generating matrix A is n_i, the band Gaussian elimination will require $O(n_i^4)$ work and $O(n_i^3)$ storage space. A direct method that takes advantage of the separablity of the problem will reduce these counts to $O(n_i^3)$ and $O(n_i^2)$ respectively, see Kaufman [11]. Since the graph of A is the same as the graph of a discrete Laplacian, nested dissection method can also be used, see George and Liu [8]. The counts for this method are $O(n_i^3)$ and $O(n_i^2 \log n_i)$ respectively. Let us remark that our preconditioned system $By_0 = b$ can also be solved by direct methods. In fact, we can compute and store B by using (2.2.17). This would require $O(n_i^3)$ operations and $O(n_i^2)$ storage spaces. Thus we see that solving the preconditioned system $By_0 = b$ by conjugate gradient type methods requires the least amount of work and storage.

Table 1: Craig's Method, $\dfrac{\lambda_i}{\mu_i} = 1 - \beta_i h_i^{\alpha}, \mu_i = s_i = 1, \beta_i = \frac{1}{2}, i = 1,2$

(n_1, n_2)	α				(n_1, n_2)	α				(n_1, n_2)	α			
	0.	1.	2.	3.		0.	1.	2.	3.		0.	1.	2.	3.
(5,8)	8	8	8	7	(8,8)	8	8	7	7	(8,5)	5	5	5	5
(10,16)	13	11	9	9	(16,16)	13	10	9	9	(16,10)	10	9	8	7
(20,32)	14	12	10	10	(32,32)	14	11	10	10	(32,20)	13	11	9	9
(40,64)	19	13	12	12	(64,64)	19	13	12	12	(64,40)	15	12	11	11
(80,128)	**	14	12	12	(128,128)	**	14	12	12	(128,80)	23	13	12	12

** more than 30 iterations.

Table 2: Craig's Method, $\dfrac{\lambda_i}{\mu_i} = 1 - \beta_i h_i^{\alpha}, \mu_i = s_i = \beta_i = 1, i = 1,2$

(n_1, n_2)	α			(n_1, n_2)	α			(n_1, n_2)	α		
	1.	2.	3.		1.	2.	3.		1.	2.	3.
(5,8)	8	8	7	(8,8)	8	8	7	(8,5)	5	5	5
(10,16)	11	10	9	(16,16)	11	9	9	(16,10)	9	8	7
(20,32)	12	10	10	(32,32)	12	10	10	(32,20)	11	9	9
(40,64)	14	12	12	(64,64)	14	12	12	(64,40)	13	10	11
(80,128)	15	12	12	(128,128)	16	12	12	(128,80)	14	12	12

Table 3: Craig's Method, $\dfrac{\lambda_i}{\mu_i} = 1 + \beta_i h_i^{\alpha}, \mu_i = s_i = 1, \beta_i = 1, i = 1,2$

(n_1, n_2)	α				(n_1, n_2)	α				(n_1, n_2)	α			
	1.	2.	3.	∞		1.	2.	3.	∞		1.	2.	3.	∞
(5,8)	8	8	7	6	(8,8)	8	8	7	6	(8,5)	5	5	5	4
(10,16)	12	10	8	7	(16,16)	11	9	8	7	(16,10)	9	8	7	6
(20,32)	14	10	10	9	(32,32)	13	10	10	9	(32,20)	11	9	9	8
(40,64)	15	12	12	11	(64,64)	14	12	12	11	(64,40)	13	11	11	9
(80,128)	16	12	12	11	(128,128)	16	12	12	11	(128,80)	14	12	12	11

Table 4: Craig's Method, $\dfrac{\lambda_i}{\mu_i} = 1 - \beta_i h_i^2,\ \mu_i = s_i = 1, \beta_i = 1, i = 1,2$

$n_1 = n_2$	8	16	32	64	128
Initialization	0.001	0.002	0.006	0.010	0.023
Iteration	0.044	0.097	0.215	0.519	1.144
No. of iterations	8	9	10	12	12
Time per iteration	0.0055	0.0108	0.0215	0.0433	0.0953
Generating p	0.014	0.053	0.192	0.794	3.381
Total time	0.059	0.152	0.413	1.323	4.548

.tex Table 5: Craig's Method, $s_i\mu_i = \lambda_i + \eta_i h_i^{\alpha}, \mu_i = 2\eta_i = 1, i = 1,2$

s_i	1				5				$n_i - 1$			
α	0.	1.	2.	3.	0.	1.	2.	3.	0.	1.	2.	3.
(8,8)	8	8	7	7	8	8	8	8	8	8	8	8
(16,16)	13	10	9	9	12	11	11	11	14	14	14	14
(32,32)	14	11	10	10	16	13	13	13	15	16	16	16
(64,64)	19	13	12	12	18	15	14	14	**	**	**	**
(128,128)	**	14	12	12	**	15	15	15	**	**	**	**

** more than 30 iterations

Table 6: Comparison with the Point SOR method

Parameters			$s_i\mu_i = \lambda_i + \eta_i h_i^\alpha, \mu_i = \eta_i = 1, i = 1, 2, \alpha = 2$									
Method			point SOR: Initial guess $p = p_0$									Craig's
n_i	s_i	N	ω^*	Relaxation factor ω								Iterations
				1.0	1.3	1.5	1.6	1.7	1.8	1.9	ω^*	
4	1	16	1.610	361	191	108	63	78	141	520	54	4
4	3	16	1.435	133	64	37	51	73	123	311	31	4
8	1	64	1.794	**	974	600	444	300	158	981	155	8
8	7	64	1.606	350	183	103	58	72	122	312	50	8

** more than 1000 iterations

Table 7: Time in Seconds Required by Craig's and Point SOR

Parameters	$s_i\mu_i = \lambda_i + \eta_i h_i^\alpha, \mu_i = \eta_i = 1, i = 1, 2, \alpha = 2$					
Problem	$n_i = 16, s_i = 1$		$n_i = 16, s_i = 15$		$n_i = 40, s_i = 39$	
Dimension N	256		256		1600	
Method	Craig's	pt SOR	Craig's	pt SOR	Craig's	pt SOR
ω^*	...	1.889	...	1.725	...	1.832
No. of iterations	9	406	14	79	16	132
Time for iteration	0.097	4.421	0.158	0.859	0.976	9.075
Time per iteration	0.0108	0.0109	0.0113	0.0109	0.0610	0.0688
Total time	0.152	4.439	0.263	0.878	2.196	9.106

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