Preconditioners for Wiener-Hopf Equations with High Order Quadrature Rules

Fu-Rong Lin* Michael K. Ng† Raymond H. Chan‡

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Abstract

We consider solving the Wiener-Hopf equations with high order quadrature rules by preconditioned conjugate gradient (PCG) methods. We propose using convolution operators as preconditioners for these equations. We will show that with proper choice of kernel functions for the preconditioners, the resulting preconditioned equations will have clustered spectra and therefore can be solved by the PCG method with superlinear convergence rate. Moreover, the discretization of these equations by high order quadrature rules leads to matrix systems that involve only Toeplitz or diagonal matrix-vector multiplications and hence can be computed efficiently by FFTs. Numerical results are given to illustrate the fast convergence of the method and the improvement on accuracy by using higher order quadrature rule. We also compare the performance of our preconditioners with the circulant integral operators.

Abbreviated Title: Preconditioners for Wiener-Hopf Equations.

Key Words. Wiener-Hopf equations, projection method, preconditioned conjugate gradient method, Fourier transform, quadrature rules.

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*Department of Mathematics, University of Hong Kong, Pokfulam Road, Hong Kong. Research supported in part by Lee Ka Shing scholarship.

†Computer Sciences Laboratory, Research School of Information Sciences and Engineering, The Australian National University, Canberra, ACT 0200, Australia. Research supported by the Cooperative Research Centre for Advanced Computational Systems.

‡Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong. Research supported in part by HKRGC grant no. CUHK 178/93E and CUHK 316/94E.

1 Introduction

Wiener-Hopf equations arise in a variety of practical applications in mathematics and engineering, especially in the solutions of inverse problems. Typical examples are linear prediction problems for stationary stochastic processes [8, pp.145–146], diffusion problems and scattering problems [8, pp.186–189]. In this paper, we consider the numerical solution of Wiener-Hopf equations defined on the half-line $[0, \infty)$:

$$y(t) + \int_0^\infty a(t-s)y(s)ds = g(t), \quad 0 \le t < \infty. \tag{1}$$

Here $a(t) \in L_1(\mathbb{R})$ and $g(t) \in L_2[0, \infty)$ are given functions. One way of solving (1) is by the projection method [5] where the solution $y(t)$ of (1) is approximated by the solution $y_\tau(t)$ of the finite-section equation

$$y_\tau(t) + \int_0^\tau a(t-s)y_\tau(s)ds = g(t), \quad 0 \le t \le \tau. \tag{2}$$

It is shown in [5, Theorem 3.1] that

$$\lim_{\tau\to\infty} ||y_\tau - y||_{L_p[0,\tau)} = 0, \quad 1 \le p < \infty.$$

The finite-section equation (2) can be solved numerically by either direct or iterative methods. For a fixed τ, the finite-section operator $\mathcal{A}_\tau$ defined by

$$(\mathcal{A}_\tau x)(t) = \begin{cases} \displaystyle\int_0^\tau a(t-s)x(s)ds, & 0 \le t \le \tau, \\ 0, & t > \tau. \end{cases} \tag{3}$$

is a compact operator. Therefore, the spectrum of the Wiener-Hopf operator $\mathcal{I} + \mathcal{A}_\tau$ (where $\mathcal{I}$ is the identity operator) is clustered around 1 and hence solving (2) by iterative methods such as the conjugate gradient (CG) method will be less expensive than direct methods. However, as τ tends to ∞, the spectrum of this finite-section operator $\mathcal{A}_\tau$ becomes dense in the spectrum of the half-line operator defined by (1) and hence the convergence rate of the CG method will deteriorate, see the numerical results in §5.

One standard way of speeding up the convergence rate of the CG method is to apply a preconditioner. Thus instead of solving (2), one solves the preconditioned equation

$$\mathcal{H}(\mathcal{I} + \mathcal{A}_\tau)y_\tau(t) = \mathcal{H}g(t), \quad 0 \le t \le \tau, \tag{4}$$

for some operator $\mathcal{H}$. In [7], Gohberg, Hanke and Koltracht proposed using *circulant integral operators* to precondition (2). Circulant integral operators are operators of the form

$$(\mathcal{C}_\tau x)(t) = \int_0^\tau c_\tau(t-s)x(s)ds, \quad 0 \le t \le \tau, \tag{5}$$

where c_τ is a τ-periodic function in $L_1[-\tau,\tau]$. The preconditioned equation is given by

$$(\mathcal{I}+\mathcal{C}_\tau)^{-1}(\mathcal{I}+\mathcal{A}_\tau)y_\tau(t) = (\mathcal{I}+\mathcal{C}_\tau)^{-1}g(t), \quad 0 \le t \le \tau. \tag{6}$$

It has been shown in [7] that for large τ, the spectra of the circulant preconditioned operators $(\mathcal{I}+\mathcal{C}_\tau)^{-1}(\mathcal{I}+\mathcal{A}_\tau)$ are clustered around 1. Hence the preconditioned conjugate gradient (PCG) method converges superlinearly for large τ.

We note that if the rectangular quadrature rule is used to discretize (6), then we get a matrix system:

$$(\mathbf{I}+\mathbf{C})^{-1}(\mathbf{I}+\mathbf{A})\mathbf{y} = (\mathbf{I}+\mathbf{C})^{-1}\mathbf{g} \tag{7}$$

where the matrices $(\mathbf{I}+\mathbf{C})$ and $(\mathbf{I}+\mathbf{A})$ are n-by-n circulant and Toeplitz matrices respectively. Here n is the number of quadrature points used in the discretization. We see that (7) is basically a circulant preconditioned Toeplitz system which requires only $O(n\log n)$ operations in each iteration by means of Fast Fourier Transforms (FFTs) and the convergence rate of these systems has been analyzed for instances in [3, 11, 15].

One main drawback of using the rectangular rule is that the order of accuracy of the discretized solution $\mathbf{y}$ depends only linearly on the number of quadrature points. Thus in order to obtain a reasonable accurate solution for (2), small step-size has to be used and hence the dimension of the resulting matrix system will be large. In order to obtain high order of accuracy, one can use higher order quadrature rules such as the trapezoidal rule or Simpson's rule which have second and fourth order of accuracy respectively. However, the corresponding discretization matrices of the circulant integral operators are $\mathbf{I}+\mathbf{CD}$ where $\mathbf{D}$ is a diagonal matrix. We note that they are in general not circulant and therefore their inversion $(\mathbf{I}+\mathbf{CD})^{-1}$ cannot be computed by using FFT. Hence the cost per iteration of the PCG method will exceed $O(n\log n)$ operations.

The main aim of the paper is to construct preconditioners for (2) such that the preconditioned operators have clustered spectra and that only $O(n\log n)$ operations are required in each iteration of the PCG method even when higher order quadrature rule is employed. Our idea is to precondition (2) as in (4) where only $\mathcal{H}y$ are required for each iteration of the PCG and also its discretization matrix-vector products $\mathbf{Hy}$ can be computed in $O(n\log n)$ operations. In this paper, we use convolution operators for $\mathcal{H}$ to precondition $\mathcal{I}+\mathcal{A}_\tau$. More precisely, given $\mathcal{A}_\tau$, we first construct the convolution operator $\mathcal{B}_\tau$ whose kernel function $b(t)$ satisfies $\hat{b}(t) = \hat{a}(t)/(1+\hat{a}(t))$. (Here and also in the following, we will use $\hat{\cdot}$ to denote the Fourier transform.) Then we precondition (2) as

$$(\mathcal{I}-\mathcal{B}_\tau)(\mathcal{I}+\mathcal{A}_\tau)y_\tau(t) = (\mathcal{I}-\mathcal{B}_\tau)g(t), \quad 0 \le t \le \tau. \tag{8}$$

We will prove that the spectra of the preconditioned operators $(\mathcal{I}-\mathcal{B}_\tau)(\mathcal{I}+\mathcal{A}_\tau)$ are clustered around 1. Hence when we apply the conjugate gradient methods to solve (8), the convergence rate will be superlinear for large τ.

We note that it may be difficult to compute the kernel function $b(t)$ explicitly, and hence B_τ cannot be formed efficiently. In this case, we derive another convolution operator by considering a periodic function from a and using its periodicity to construct the preconditioner. For convergence rate, we will show that the resulting preconditioned Wiener-Hopf operator also has clustered spectrum around 1. Numerical results show that our methods converge faster than those preconditioned by using circulant integral operators. We will see that if Newton-Cotes quadrature rules are employed to discretize the convolution integral operators $\mathcal{A}_\tau$ and $\mathcal{B}_\tau$, the corresponding discretization matrices are just products of Toeplitz matrices and diagonal matrices. Hence FFT can be used to carry out the matrix-vector multiplication in each iteration of the PCG method. Thus the cost per iteration remains $O(n \log n)$.

The outline of the paper is as follows. We introduce the construction of our preconditioners in §2. In §3, we analyze the spectra of the preconditioned Wiener-Hopf operators. The properties of the discretized matrices using higher order quadrature rules are given in §4. In §5, we give numerical examples to illustrate the effectiveness of our preconditioners and the improvement of accuracy by employing higher order quadrature rules. Finally, concluding remarks are given in §6.

2 Construction of the Preconditioners

In the following, we will assume that $a(t) \in L_1(\mathbb{R})$ is conjugate symmetric, i.e. $\overline{a(t)} = a(-t)$ for all t and that $\hat{a}(t) \geq 0$. Thus the operator $\mathcal{A}_\tau$ defined in (3) is a self-adjoint positive operator. In the applications of linear prediction problems, $\hat{a}$ corresponds to the spectral density function of a continuous stationary stochastic process. In these applications. the non-negativeness of $\hat{a}$ is always valid, see Priestley[13, p.127]. The motivation for the construction of our preconditioners is given by the following Lemma.

Lemma 1 (Moiseiwitsch, [12, p.26]) *Let $a(t) \in L_1(\mathbb{R})$ be conjugate symmetric and $g(t) \in L_2(\mathbb{R})$. Then the solution to the whole-line Wiener-Hopf equation*

$$y(t) + \int_{-\infty}^{\infty} a(t-s)y(s)ds = g(t), \quad -\infty < t < \infty, \tag{9}$$

is given by

$$y(t) = g(t) - \int_{-\infty}^{\infty} \frac{\hat{g}(s)\hat{a}(s)}{1+\hat{a}(s)} e^{ist} ds.$$

By Paley-Wiener Theorem [10, Theorem 4.3], if $1 + \hat{a}(t) \neq 0$, then there exists a function $b(t) \in L_1(\mathbb{R})$ such that

$$\hat{b}(t) = \frac{\hat{a}(t)}{1+\hat{a}(t)}, \tag{10}$$

or

$$b(t) + \int_{-\infty}^{\infty} a(t-s)b(s)ds = a(t), \quad -\infty < t < \infty. \tag{11}$$

It follows that the solution $y(t)$ to (9) is given by

$$y(t) = g(t) - \int_{-\infty}^{\infty} b(t-s)g(s)ds.$$

This leads us to consider using the integral operators $\mathcal{I} - \mathcal{B}_\tau$ where

$$(\mathcal{B}_\tau x)(t) = \int_0^\tau b(t-s)x(s)ds, \quad 0 \le t \le \tau, \tag{12}$$

to precondition $\mathcal{I} + \mathcal{A}_\tau$.

We note that the construction of $\mathcal{B}_\tau$ involves the Fourier transform of a and the inverse Fourier transform of $\hat{a}/(1+\hat{a})$. The kernel function $b(t)$ cannot be obtained explicitly in some cases. Therefore, we construct another convolution operator as preconditioner such that its corresponding kernel function can be computed more efficiently.

To construct such operator, we first define the functions

$$D_\tau(t) = \begin{cases} 1, & |t| \le \tau, \\ 0, & |t| > \tau, \end{cases} \tag{13}$$

and $a_\tau(t) = D_\tau(t)a(t)$. We note that

$$\mathcal{A}_\tau x(t) = \int_0^\tau a(t-s)x(s)ds = \int_0^\tau a_\tau(t-s)x(s)ds, \quad 0 \le t \le \tau.$$

Thus as far as the projected operator $\mathcal{A}_\tau$ is concerned, $a(\cdot)$ and $a_\tau(\cdot)$ are equal.

Similar to (11), we consider the solution $p_{2\tau}$ of the following equation to construct the kernel function of the preconditioner,

$$p_{2\tau}(t) + \int_{-\tau}^{\tau} c_{2\tau}(t-s)p_{2\tau}(s)ds = c_{2\tau}(t), \quad -\tau \le t \le \tau, \tag{14}$$

where $c_{2\tau}(t)$ is a 2τ-periodic function defined by

$$c_{2\tau}(t) = a_\tau(t), \quad -\tau \le t \le \tau.$$

We remark that as $c_{2\tau}$ is 2τ-periodic and in $L_1([-\tau,\tau])$, also is $p_{2\tau}$, see [5, p.6, Theorem 0.3]. Let $\mathcal{P}_{2\tau}$ be the convolution operator with kernel function given by $p_{2\tau}$,

$$(\mathcal{P}_{2\tau}x)(t) = \int_{-\tau}^{\tau} p_{2\tau}(t-s)x(s)ds, \quad 0 \le t \le 2\tau.$$

Here, we employ the periodicity of $c_{2\tau}$ instead of using the Fourier transform of a as in (10). Therefore, by using the lemma in [6, p.119, Theorem 8.1], the eigenvalues of $\mathcal{P}_{2\tau}$ are given by

$$\lambda_n(\mathcal{P}_{2\tau}) = \int_{-\tau}^{\tau} p_{2\tau}(t)e^{-i\pi nt/\tau}dt = \frac{\hat{a}_\tau(\frac{n\pi}{\tau})}{1+\hat{a}_\tau(\frac{n\pi}{\tau})}, \quad n \in \mathbb{Z}, \tag{15}$$

where

$$\hat{a}_\tau(t) = \int_{-\infty}^{\infty} D_\tau(s)a(s)e^{-ist}ds = \int_{-\tau}^{\tau} a(s)e^{-ist}ds = \hat{D}_\tau * \hat{a}(t). \tag{16}$$

Similar to (12), we define $w_\tau(t)$ by

$$w_\tau(t) = D_\tau(t)p_{2\tau}(t), \tag{17}$$

and consider using the integral operators $\mathcal{I} - \mathcal{W}_\tau$ where

$$(\mathcal{W}_\tau x)(t) = \int_0^\tau w(t-s)x(s)ds, \quad 0 \le t \le \tau, \tag{18}$$

to precondition $\mathcal{I} + \mathcal{A}_\tau$.

In §3, we will analyze the spectra of these preconditioned Wiener-Hopf equations (8) with the operators $\mathcal{I} - \mathcal{B}_\tau$ and $\mathcal{I} - \mathcal{W}_\tau$ as preconditioners. In §4, we will see that the discretization matrix of $\mathcal{I} - \mathcal{W}_\tau$ can be obtained efficiently by using FFTs.

3 Convergence Analysis

In this section, we analyze the spectra of the preconditioned Wiener-Hopf equations (8). We first show that the operators $\mathcal{I} + \mathcal{A}_\tau$ and $\mathcal{I} - \mathcal{B}_\tau$ are positive.

Lemma 2 (Grenander and Szegö [9, p.139]) *Let $f(t) \in L_1(\mathbb{R})$ be conjugate symmetric and $\mathcal{F}_\tau$ be the convolution operator defined on $[0,\tau]$ with kernel function $f(t)$. Then for all $\tau > 0$, the spectrum $\sigma(\mathcal{F}_\tau)$ of $\mathcal{F}_\tau$ satisfies $\sigma(\mathcal{F}_\tau) \subseteq [m, M]$, where m and M are the infimum and supremum of $\hat{f}(t)$ respectively.*

By applying Lemma 2 to the operators $\mathcal{A}_\tau$ and $\mathcal{B}_\tau$ we have

Lemma 3 *Let $a(t) \in L_1(\mathbb{R})$ be conjugate symmetric and $\hat{a}(t) \ge 0$. Then $\sigma(\mathcal{I} + \mathcal{A}_\tau) \subseteq [1, \gamma]$ and , $\sigma(\mathcal{I} - \mathcal{B}_\tau) \subseteq [\gamma^{-1}, 1]$, where*

$$\gamma = 1 + \sup_{t \in \mathbb{R}} \hat{a}(t) < \infty.$$

In particular, the operator $(I + \mathcal{A}_\tau)^{1/2}$ is a well-defined, invertible operator from $L_2[0,\tau)$ to $L_2[0,\tau)$ with

$$||(\mathcal{I} + \mathcal{A}_\tau)^{1/2}||_2 \le \gamma^{1/2} < \infty. \tag{19}$$

Proof: We first note that since $\lim_{t\to\infty} \hat{a}(t) = 0$ and $\hat{a}(t)$ is uniformly continuous on $\mathbb{R}$ (see Champeney [2, Theorem 8.1]), $\gamma < \infty$. Next we note that by (10),

$$0 \le \hat{b}(t) = \frac{\hat{a}(t)}{1+\hat{a}(t)} = 1 - \frac{1}{1+\hat{a}(t)} \le 1 - \gamma^{-1}, \quad \forall t \in \mathbb{R}. \quad \square$$

In order to prove that the preconditioned operators have clustered spectra, we first introduce the circulant integral operator proposed by Chan *et al.* [4]. For any given kernel function $f(t)$ in $L_1(\mathbb{R})$, the circulant integral operator $\mathcal{G}_\tau$ is defined as

$$(\mathcal{G}_\tau y)(t) = \begin{cases} \int_0^\tau g_\tau(t-s)y(s)ds, & 0 \le t \le \tau, \\ 0, & t > \tau \end{cases} \tag{20}$$

where the function g_τ is a τ-periodic function defined by

$$g_\tau(t) = f(t) + f(t-\tau), \quad 0 \le t \le \tau.$$

The eigenvalues of $\mathcal{G}_\tau$ are given by the convolution of $\hat{D}_\tau$ and $\hat{f}$ at the point $2\pi n/\tau$, i.e.

$$\lambda_n(\mathcal{G}_\tau) = (\hat{D}_\tau * \hat{f})(\frac{2\pi n}{\tau}), \quad \forall n \in \mathbb{Z}, \tag{21}$$

where $D_\tau(t)$ is defined in (13). The following lemma shows that $\mathcal{G}_\tau$ is a good preconditioner for $\mathcal{A}_\tau$.

Lemma 4 (Gohberg, Hanke and Koltracht [7], Chan, Jin and Ng [4]) *Let $a(t) \in L_1(\mathbb{R})$ be conjugate symmetric and $\hat{a}(t) \ge 0$. Then for any given $\epsilon > 0$, there exist positive integers N and τ^* such that for all $\tau > \tau^*$, there exists a decomposition $\mathcal{A}_\tau - \mathcal{G}_\tau = \mathcal{R}_\tau + \mathcal{E}_\tau$, with self-adjoint operators $\mathcal{R}_\tau$ and $\mathcal{E}_\tau$ satisfying* rank $\mathcal{R}_\tau \le N$ *and $||\mathcal{E}_\tau||_2 \le \epsilon$. Moreover, the preconditioned operator $(\mathcal{I}+\mathcal{G}_\tau)^{-1}(\mathcal{I}+\mathcal{A}_\tau)$ has at most N eigenvalues outside the interval $(1-\epsilon, 1+\epsilon)$.*

We now show that the preconditioner differs from ours by a low rank and small norm perturbation.

Lemma 5 *Let $a(t) \in L_1(\mathbb{R})$ be conjugate symmetric and $\hat{a}(t) \ge 0$. Then for any given $\epsilon > 0$, there exist positive integers N and τ^* such that for all $\tau > \tau^*$,*

$$(\mathcal{I}+\mathcal{G}_\tau)^{-1} - (\mathcal{I}-\mathcal{B}_\tau) = \mathcal{R}_\tau + \mathcal{E}_\tau \tag{22}$$

with self-adjoint operators $\mathcal{R}_\tau$ and $\mathcal{E}_\tau$ satisfying rank $\mathcal{R}_\tau \le N$ *and $||\mathcal{E}_\tau||_2 \le \epsilon$.*

Proof: For the convolution operator $\mathcal{B}_\tau$, we let $\mathcal{K}_\tau$ be its preconditioner as defined in (20). In view of Lemma 4, it suffices to show that

$$\lim_{\tau\to\infty} ||(\mathcal{I}+\mathcal{G}_\tau)^{-1} - (\mathcal{I}-\mathcal{K}_\tau)||_2 = 0.$$

However, since $\mathcal{G}_\tau$ and $\mathcal{K}_\tau$ are circulant operators, we have

$$||(\mathcal{I}+\mathcal{G}_\tau)^{-1} - (\mathcal{I}-\mathcal{K}_\tau)||_2 \le \sup_{n\in\mathbb{Z}} |\lambda_n((\mathcal{I}+\mathcal{G}_\tau)^{-1}) - \lambda_n(\mathcal{I}-\mathcal{K}_\tau)|,$$

see [9,p.112]. By (21)

$$\lambda_n(\mathcal{K}_\tau) = (\hat{D}_\tau * \hat{b})(\frac{2\pi n}{\tau}), \quad \forall n \in \mathbb{Z}$$

and

$$\lambda_n((\mathcal{I}+\mathcal{G}_\tau)^{-1}) = \left[1 + (\hat{D}_\tau * \hat{a})(\frac{2\pi n}{\tau})\right]^{-1}, \quad \forall n \in \mathbb{Z}.$$

Therefore we have

$$\begin{aligned}
& \sup_{n\in\mathbb{Z}} |\lambda_n((\mathcal{I}+\mathcal{G}_\tau)^{-1}) - \lambda_n(\mathcal{I}-\mathcal{K}_\tau)| \\
= & \sup_{n\in\mathbb{Z}} \left| \frac{1}{1+(\hat{D}_\tau * \hat{a})(\frac{2\pi n}{\tau})} - 1 + (\hat{D}_\tau * \hat{b})(\frac{2\pi n}{\tau}) \right| \\
= & \sup_{n\in\mathbb{Z}} \left| \frac{\hat{a}(\frac{2\pi n}{\tau}) - (\hat{D}_\tau * \hat{a})(\frac{2\pi n}{\tau})}{[1+\hat{a}(\frac{2\pi n}{\tau})][1+(\hat{D}_\tau * \hat{a})(\frac{2\pi n}{\tau})]} + (\hat{D}_\tau * \hat{b})(\frac{2\pi n}{\tau}) - \hat{b}(\frac{2\pi n}{\tau}) \right|.
\end{aligned}$$

As $a(t)$ and $b(t)$ are in $L_1(\mathbb{R})$, $\hat{a}$ and $\hat{b}$ are uniformly continuous on $\mathbb{R}$ and hence $\hat{D}_\tau * \hat{a}$ and $\hat{D}_\tau * \hat{b}$ converge uniformly to $\hat{a}$ and $\hat{b}$ respectively on $\mathbb{R}$. Hence the result follows. □

Now we are ready to prove the fast convergence of our method for the preconditioner $(\mathcal{I}-\mathcal{B}_\tau)$.

Theorem 1 *Let $a(t) \in L_1(\mathbb{R})$ be conjugate symmetric and $\hat{a}(t) \ge 0$. Then for any given $\epsilon > 0$, there exist positive integers N and τ^* such that for all $\tau > \tau^*$, the spectrum of $(\mathcal{I}-\mathcal{B}_\tau)(\mathcal{I}+\mathcal{A}_\tau)$ has at most N eigenvalues outside the interval $(1-\epsilon, 1+\epsilon)$.*

Proof: By Lemma 3, $(\mathcal{I}+\mathcal{A}_\tau)^{1/2}$ is an invertible operator. Hence the spectra of

$$(\mathcal{I}-\mathcal{B}_\tau)(\mathcal{I}+\mathcal{A}_\tau) \quad \text{and} \quad (\mathcal{I}+\mathcal{A}_\tau)^{1/2}(\mathcal{I}-\mathcal{B}_\tau)(\mathcal{I}+\mathcal{A}_\tau)^{1/2}$$

are the same. By (22), we have

$$\begin{aligned}
(\mathcal{I}+\mathcal{A}_\tau)^{1/2}(\mathcal{I}-\mathcal{B}_\tau)(\mathcal{I}+\mathcal{A}_\tau)^{1/2} &= (\mathcal{I}+\mathcal{A}_\tau)^{1/2}[(\mathcal{I}+\mathcal{G}_\tau)^{-1} - \mathcal{R}_\tau - \mathcal{E}_\tau](\mathcal{I}+\mathcal{A}_\tau)^{1/2} \\
&= (\mathcal{I}+\mathcal{A}_\tau)^{1/2}(\mathcal{I}+\mathcal{G}_\tau)^{-1}(\mathcal{I}+\mathcal{A}_\tau)^{1/2} - \tilde{\mathcal{R}}_\tau - \tilde{\mathcal{E}}_\tau,
\end{aligned}$$

where by Lemma 5 and (19),

$$\text{rank } \tilde{\mathcal{R}}_\tau = \text{rank } \left\{(\mathcal{I}+\mathcal{A}_\tau)^{1/2}\mathcal{R}_\tau(\mathcal{I}+\mathcal{A}_\tau)^{1/2}\right\} \le N$$

and

$$||\tilde{\mathcal{E}}_\tau||_2 = ||(\mathcal{I}+\mathcal{A}_\tau)^{1/2}\mathcal{E}_\tau(\mathcal{I}+\mathcal{A}_\tau)^{1/2}||_2 \le \gamma\epsilon.$$

Since by Lemma 4, the operator $(\mathcal{I}+\mathcal{A}_\tau)^{1/2}(\mathcal{I}+\mathcal{G}_\tau)^{-1}(\mathcal{I}+\mathcal{A}_\tau)^{1/2}$ has clustered spectrum, our theorem follows by applying the minimax theorem [6, p.123]. □

We end this section by considering the operator $\mathcal{I}-\mathcal{W}_\tau$ as preconditioner. We first let $\tilde{\mathcal{K}}_\tau$ be the preconditioner of $\mathcal{W}_\tau$ as defined in (20).

Lemma 6 *Let $\mathcal{G}_\tau$ be the circulant integral operator as defined in Lemma 5. Then*

$$(\mathcal{I}+\mathcal{G}_\tau)^{-1} = \mathcal{I}-\tilde{\mathcal{K}}_\tau.$$

Proof: We just note from (15) and (17) that

$$\hat{w}_\tau(\frac{n\pi}{\tau}) = \lambda_n(\mathcal{P}_{2\tau}) = \frac{\hat{a}_\tau(\frac{n\pi}{\tau})}{1+\hat{a}_\tau(\frac{n\pi}{\tau})}, \quad n \in \mathbb{Z}. \tag{23}$$

Hence, it is easily seen that

$$\lambda_n(\mathcal{I}-\tilde{\mathcal{K}}_\tau) = \frac{1}{1+\hat{a}_\tau(\frac{2\pi n}{\tau})} = \lambda_n((\mathcal{I}+\mathcal{G}_\tau)^{-1}), \quad n \in \mathbb{Z}.$$

□

Next we prove that the operators $\mathcal{I}-\mathcal{W}_\tau$ is positive.

Lemma 7 *Let $a(t) \in L_1(\mathbb{R})$ be conjugate symmetric and $\hat{a}(t) \ge 0$. Then for sufficiently large τ, the operator $\mathcal{I}-\mathcal{W}_\tau$ is positive definite.*

Proof: We first define the following functions $q_\tau^{(k)}$ for $k = 0, 1$:

$$q_\tau^{(k)} = \begin{cases} \frac{1}{2}[w_\tau(t) + e^{\pi ik}w_\tau(t-\tau)], & 0 \le t \le \tau, \\ \frac{1}{2}[e^{-\pi ik}w_\tau(t+\tau) + w_\tau(t)], & -\tau \le t \le 0. \end{cases} \tag{24}$$

We note that $w_\tau = q_\tau^{(0)} + q_\tau^{(1)}$. Therefore, we can write

$$\mathcal{I}-\mathcal{W}_\tau = \frac{1}{2}[\mathcal{I}-\mathcal{Q}_\tau^{(0)}] + \frac{1}{2}[\mathcal{I}-\mathcal{Q}_\tau^{(1)}],$$

where

$$\mathcal{Q}_\tau^{(k)} = \int_0^\tau q_\tau^{(k)}(t-s)x(s)ds, \quad k = 0, 1.$$

By using the spectral decomposition of the operators $\mathcal{Q}_\tau^{(k)}$ [6, p.119, Theorem 8.1] and (23), it follows that the eigenvalues of $\mathcal{I} - \mathcal{Q}_\tau^{(k)}$ for $k = 0, 1$ are given by

$$\lambda_n(\mathcal{I} - \mathcal{Q}_\tau^{(k)}) = \frac{1}{1 + \hat{a}_\tau(\frac{2\pi n}{\tau} + \frac{\pi k}{\tau})}, \quad n \in \mathbb{Z}.$$

Noting that $-1/2 < \hat{a}_\tau(t) < \max_{t\in\mathbb{R}}\{\hat{a}(t)\} + 1/2$ for sufficiently large τ, the result follows. □

Combining Lemmas 3, 4, 6 and 7, we have the following theorem which shows that $\mathcal{I} - \mathcal{W}_\tau$ is also a good preconditioner for $\mathcal{I} + \mathcal{A}_\tau$.

Theorem 2 *Let $a(t) \in L_1(\mathbb{R})$ be conjugate symmetric and $\hat{a}(t) \geq 0$. For any given $\epsilon > 0$, there exist positive integers N and τ^* such that for all $\tau > \tau^*$, the spectrum of $(\mathcal{I} - \mathcal{W}_\tau)(\mathcal{I} + \mathcal{A}_\tau)$ has at most N eigenvalues outside the interval $(1-\epsilon, 1+\epsilon)$.*

Thus the CG method, when applied to solving the preconditioned operator equation (8) with preconditioners $\mathcal{I} - \mathcal{B}_\tau$ or $\mathcal{I} - \mathcal{W}_\tau$, will converge superlinearly, see Axelsson and Barker [1]. Finally, we remark that Theorems 1 and 2 also holds if the condition $\hat{a}(t) \geq 0$ is relaxed to $1 + \hat{a}(t) > \delta > 0$ for some constant δ.

4 Discretization of Preconditioned Operators

Let us consider the linear systems generated by discretizing (8) by Netwon-Cotes type quadrature rules. Let the interval $[0, \tau]$ be divided into n subintervals of equal length h, i.e. $hn = \tau$. Given any convolution operator $\mathcal{F}_\tau$ defined on $[0, \tau]$ with kernel function $f(t)$, it is straightforward to show that its discretization matrix will be of the form $\mathbf{TD}$ where $\mathbf{T}$ is a Toeplitz matrix whose first column is given by

$$[\mathbf{T}]_{j0} = hf(jh), \quad j = 0, 1, \cdots, n, \tag{25}$$

and $\mathbf{D}$ is a diagonal matrix that depends only on the quadrature formula used. For examples, for the rectangular rule, the trapezoidal rule and Simpson's rule, the diagonals of $\mathbf{D}$ are given respectively by

$$(1, 1, \cdots, 1, 1), \quad (\frac{1}{2}, 1, 1, \cdots, 1, 1, \frac{1}{2}) \quad \text{and} \quad (\frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \cdots, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{3})$$

Thus we see that after discretization, (8) becomes

$$(\mathbf{I}-\mathbf{BD})(\mathbf{I}+\mathbf{AD})\mathbf{y}=(\mathbf{I}-\mathbf{BD})\mathbf{g} \tag{26}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{A}$ and $\mathbf{B}$ are Toeplitz matrices. This discretized system can then be solved by the PCG method. In each iteration, we only need Toeplitz matrix-vector multiplications and diagonal matrix-vector multiplications. Thus the cost per iteration is of order $O(n\log n)$ by using FFTs. We note that by using the transformation

$$\tilde{\mathbf{y}}=\mathbf{D}^{1/2}\mathbf{y} \quad \text{and} \quad \tilde{\mathbf{g}}=\mathbf{D}^{1/2}\mathbf{g}, \tag{27}$$

(26) can be symmetrized as

$$(\mathbf{I}-\mathbf{D}^{1/2}\mathbf{B}\mathbf{D}^{1/2})(\mathbf{I}+\mathbf{D}^{1/2}\mathbf{A}\mathbf{D}^{1/2})\tilde{\mathbf{y}}=(\mathbf{I}-\mathbf{D}^{1/2}\mathbf{B}\mathbf{D}^{1/2})\tilde{\mathbf{g}}. \tag{28}$$

If the operator $\mathcal{I}-\mathcal{W}_\tau$ is used as preconditioner, then from (25), we know that we only need the values of $w_\tau(jh)$ for $j=0,1,\cdots,n$. Here the values of $w_\tau(\cdot)$ are defined in (17), i.e. we need to compute the values of $p_{2\tau}(jh)$ for $j=0,1,\cdots,n$. In order to compute $\hat{a}_\tau(t)$ in the integrand of (15), we partition the interval $[-\tau,\tau]$ into $2n$ equal subintervals of step-size h and approximate (16) as

$$\hat{a}_\tau(t)\approx h\sum_{k=-n}^{n-1}a(kh)e^{-ikht}. \tag{29}$$

We note that the approximate values of $\hat{a}_\tau(\pi j/\tau)$ for $j=0,\pm1,\cdots,\pm n$ can be computed by (29) by one FFT of length $2n$. From these values, we can then calculate the approximate values of $\lambda_j(\mathcal{P}_{2\tau})$ for $j=0,\pm1,\cdots,\pm n$ according to (15). Then the approximate values of $p_{2\tau}(kh)$ for $k=0,\pm1,\cdots,\pm n$ can be obtained by approximating the integral (15) by a formula of the form (29) again. This just requires one inverse FFT.

The main feature of our proposed preconditioner is that it is already inverted. Hence only Toeplitz matrix-vector products (plus some inner products) are required in each step of PCG algorithm. In contrast, if circulant integral operators (see [7]) are used with high order quadrature rules, then one has to invert matrix of the form $\mathbf{I}+\mathbf{CD}$ which in general has no fast inversion formula. However, one can still remedy the drawback of circulant integral operators by using our approach and thus avoids the inversion of $\mathbf{I}+\mathbf{CD}$.

The construction of such circulant preconditioners is as follows. For circulant integral operator $\mathcal{C}_\tau$ given in (5), we note that if $\mathcal{I}+\mathcal{C}_\tau$ is invertible, then its inverse can be expressed in the form:

$$(\mathcal{I}+\mathcal{C}_\tau)^{-1}=\mathcal{I}-\mathcal{M}_\tau$$

where the kernel function of $\mathcal{M}_\tau$ is given by

$$m_\tau(t) = \frac{1}{\tau} \sum_{k \in \mathbb{Z}} \frac{\lambda_k}{1+\lambda_k} e^{2\pi i k t/\tau}.$$

Here λ_k are the eigenvalues of $\mathcal{C}_\tau$ given by

$$\lambda_k = \int_{-\tau/2}^{\tau/2} c_\tau(t) e^{-2\pi i k t/\tau} dt, \quad \forall k \in \mathbb{Z}.$$

We remark that our preconditioned operator equation will become

$$(\mathcal{I} - \mathcal{M}_\tau)(\mathcal{I} + \mathcal{A}_\tau) y_\tau(t) = (\mathcal{I} - \mathcal{M}_\tau) g(t), \quad 0 \le t \le \tau. \tag{30}$$

We know again from (25) that we only need the values of $m_\tau(jh)$ for $j = 0, 1, \cdots, n$ in order to construct matrix preconditioner. To approximate these values, we partition the interval $[-\tau/2, \tau/2]$ into n equal subintervals of step-size h and approximate λ_k as

$$\tilde{\lambda}_k = h \sum_{k=-n/2}^{n/2-1} c(kh) e^{-2\pi i j k/n}.$$

Then the approximate values of $m_\tau(jh)$ for $j = 0, \pm 1, \cdots, \pm n$ can be calculated from

$$m_\tau(jh) \approx \frac{1}{nh} \sum_{k=-n/2}^{n/2-1} \frac{\tilde{\lambda}_k}{1+\tilde{\lambda}_k} e^{2\pi i j k/n}.$$

Hence the matrix preconditioner can be constructed by using the inverse of circulant integral operator. However, we remark that numerical results in §5 show that our proposed convolution preconditioners are better than these *inverted-circulant* type preconditioners.

To conclude this section, we compare the computational costs of using our proposed preconditioners and inverted-circulant preconditioners. We recall that the multiplication of an n-by-n circulant matrix to an n-vector requires only two n-dimensional FFTs. For Toeplitz matrix $\mathbf{T}$, products of the form $\mathbf{Tv}$ can be obtained by first embedding the matrices into $2n$-by-$2n$ circulant matrices and using $2n$-dimensional FFTs, see Strang [14]. Thus the cost per iteration when $\mathcal{I} - \mathcal{B}_\tau$ or $\mathcal{I} - \mathcal{W}_\tau$ is used as preconditioner is about the same as the cost of applying four $2n$-dimensional FFTs. If $\mathcal{I} - \mathcal{M}_\tau$ is used, then one of the Toeplitz matrix-vector product will be replaced by circulant matrix-vector product. Hence the cost per iteration of our method is roughly $4/3$ times higher than that required by inverted-circulant preconditioned systems. In all cases, the cost is $O(n \log n)$ operations per iteration.

	τ=16				τ=32				τ=64				τ=128			
n	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$
512	8	7	17	24	12	6	14	37	21	5	10	55	70	4	7	79
1024	8	9	17	26	9	6	14	37	14	4	10	58	32	4	7	86
2048	7	9	17	26	8	6	14	40	10	4	10	60	16	4	7	87
4096	7	9	17	26	8	6	14	40	9	4	10	60	12	4	7	89
8192	6	10	17	26	7	6	14	40	8	4	10	62	9	4	7	93

Table 1: The numbers of iterations for the rectangular rule.

5 Numerical Examples

In this section, we test the effectiveness of our proposed preconditioners by two test kernel functions. The first one is

$$a_1(t) = \frac{1}{\mu} e^{-\alpha|t|}.$$

We note that its $b(t)$ (as defined in (10)) is given by

$$b(t) = \frac{\alpha}{\mu\beta} e^{-\beta|t|},$$

where

$$\beta = \sqrt{\frac{2\alpha}{\mu} + \alpha^2}.$$

In practical applications, the parameter μ is the regularization parameter and is usually a small positive number. In the test, we set $\mu = 0.05$ and $\alpha = 0.1$. We choose our right hand side function $g(t)$ such that the corresponding solution for the Wiener-Hopf equation (1) is

$$s(t) = \begin{cases} (16-t)^2, & 0 \le t \le 16, \\ 0, & t > 16. \end{cases} \tag{31}$$

The preconditioned equation (8) is first discretized by quadrature rules to obtain the linear system (26). Then the transformation (27) is used to symmetrize the linear systems, see (28). We use the same random vector as our initial guess for all methods. The stopping criterion is

$$||\tilde{\mathbf{g}} - (\mathbf{I} + \mathbf{D}^{1/2}\mathbf{A}\mathbf{D}^{1/2})\tilde{\mathbf{y}}^{(k)}||_2 \le 10^{-6}$$

where $\tilde{\mathbf{y}}^{(k)}$ is the kth iterant and the vector 2-norm is defined as $||\mathbf{x}||_2^2 \equiv \mathbf{x}^T\mathbf{x}$. All computations are done by Matlab.

	τ=16				τ=32				τ=64				τ=128			
n	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$
512	8	7	16	24	11	6	13	37	19	5	9	56	74	4	7	83
1024	7	9	16	26	9	6	13	40	13	5	9	60	31	4	7	86
2048	7	9	17	26	7	6	14	40	10	4	10	60	16	4	7	88
4096	7	9	17	26	8	6	14	40	9	4	10	60	12	4	7	89
8192	6	10	17	27	7	6	14	40	8	4	10	62	9	4	7	93

Table 2: The numbers of iterations for the trapezoidal rule.

	τ=16				τ=32				τ=64				τ=128			
n	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$
512	8	10	16	24	12	7	13	36	23	13	13	56	76	34	38	84
1024	8	9	16	26	9	6	13	40	14	9	11	60	38	16	17	87
2048	7	9	17	26	7	6	14	40	10	8	10	60	17	10	11	87
4096	7	9	17	26	8	6	14	40	9	7	10	60	11	8	9	90
8192	6	10	17	27	7	6	14	40	8	6	10	61	9	7	8	95

Table 3: The numbers of iterations for Simpson's rule.

Tables 1-3 give the numbers of iterations required for convergence when different preconditioners are used. The preconditioners are discretized according to the discussion in §4. The symbol $\mathcal{I}$ represents that (2) is solved without using any preconditioner. The discretization rule used is listed in the caption.

From the tables, we see that without using preconditioner, the method will converge very slowly especially for large τ and n. However, our preconditioner $\mathcal{B}_\tau$ and $\mathcal{W}_\tau$ work well provided that the step-size $h = \tau/n$ is sufficiently small. For moderate size h, $\mathcal{W}_\tau$ is already good enough to give us fast convergence. We see from numerical results that our methods converges faster than those preconditioned by using circulant integral operators.

We note that $a_1(t)$ is not differentiable at $t = 0$, therefore the higher order quadratures, for instance, Simpson's rule, are not useful in this case. In order to preserve high order of accuracy, we approximate the finite-section Wiener-Hopf integral operator on $[0, \tau]$ by using Simpson's rule except that the section of integral

$$\int_{(k-1)h}^{(k+1)h} a(kh - s)y(s)ds, \qquad \text{for odd integers } k, \tag{32}$$

is discretized by using trapezoidal rule. More precisely, we approximate the integral (32) by

$$\frac{1}{2}a(h)y((k-1)h) + a(0)y(kh) + \frac{1}{2}a(-h)y((k+1)h).$$

This combined Simpson-trapezoidal scheme gives higher order of accuracy as compared with those of trapezoidal and Simpson's rules alone. We note that the resulting coefficient matrix will be a sum of an identity matrix, Toeplitz times diagonal matrix and a band matrix with odd rows being zeros. Since these discretization matrices cannot be symmetrized as in (28), we solve its normal equations by preconditioned conjugate gradient methods. Table 4 gives the number of iterations of solving the preconditioned normal equations when this combined rule is used. We see from the table that our proposed preconditioner performs very well and our methods converges faster than those non-preconditioned or preconditioned by using inverted-circulant integral operators.

To illustrate the usefulness of using higher order quadrature rules, we give in Table 5, the error of the numerical solutions. The error is computed as

$$\left\{ h * \sum_{j=0}^{n} |y(jh) - s(jh)|^2 \right\}^{1/2} \approx \left\{ \int_0^\tau |y(t) - s(t)|^2 dt \right\}^{1/2}$$

where $\{y(jh)\}_{j=0}^n$ is the computed solution and $s(t)$ is the true solution given by (31). Since a_1 is not differentiable at $t = 0$, the error of the numerical solution by using Simpson's rule is larger than that of using trapezoidal rule. However, for the combined Simpson-trapezoidal rule, the error decreases like $O(h^3)$ which is faster than the $O(h^2)$ rate of using trapezoidal rule alone.

	τ=16				τ=32				τ=64				τ=128			
n	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{B}_\tau$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$
512	20	20	28	60	25	25	31	136	47	51	56	180	95	84	85	880
1024	15	17	29	55	19	20	26	121	26	28	29	294	53	65	68	1108
2048	14	16	28	56	17	17	25	112	20	21	22	267	26	31	32	678
4096	13	16	29	56	16	16	25	110	17	17	20	243	20	22	23	576
8192	12	15	28	60	14	14	27	114	16	16	21	255	16	18	21	580

Table 4: The number of iterations for the combined Simpson-trapezoidal rules.

	τ = 64				τ = 128			
n	Rect.	Trap.	Simp.	Comb.	Rect.	Trap.	Simp.	Comb.
512	15.320	0.0894	2.496	0.214	30.610	0.391	9.694	0.547
1024	7.641	0.0212	0.627	0.00855	15.320	0.0894	2.496	0.214
2048	3.813	0.00515	0.157	7.952e-4	7.641	0.0212	0.627	0.00855
4096	1.904	0.00127	0.0391	8.804e-5	3.813	0.00515	0.157	7.952e-4
8192	0.951	3.147e-4	0.00978	1.041e-5	1.904	0.00127	0.0391	8.805e-5

Table 5: Error in the computed solution for $a_1(t)$.

Next we test our methods on another kernel function:

$$a_2(t) = \frac{1}{\mu(1+t^2)},$$

with $\mu = 0.05$. In this case, we do not have explicit formula for the $b(t)$ of the kernel function $a_2(t)$. Tables 6-8 below give the numbers of iterations required for convergence for $a_2(t)$. We see from the tables that the performance of our proposed preconditioner is better than the others. We see from Table 9 that the error decreases like $O(h)$, $O(h^2)$ and $O(h^4)$ for the rectangular, trapezoidal and Simpson's rule respectively.

6 Concluding Remarks

We remark that the accuracy of the computed solution depends only on the quadrature rule used in discretize $\mathcal{A}_\tau$. However, the convergence rate of the preconditioned systems and the costs per iteration of the PCG method depend on how we discretize the preconditioning operators. From the numerical results, we see that it is advantageous to use higher order quadrature rule to discretize the operator equation because of accuracy concern. But to speed up the convergence rate of the method and to minimize the costs per iteration, one may need to use our proposed preconditioner rather than circulant ones.

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	τ=16			τ=32			τ=64			τ=128		
n	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$
512	8	11	72	7	11	103	7	11	139	7	11	160
1024	8	11	73	8	11	111	7	11	136	7	11	158
2048	8	11	71	8	11	111	7	11	142	7	11	156
4096	9	11	76	8	11	114	8	11	142	7	11	162
8192	9	11	77	9	11	117	8	11	148	7	11	165

Table 6: The numbers of iterations for the rectangular rule.

	τ=16			τ=32			τ=64			τ=128		
n	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$
512	8	11	71	8	11	109	7	12	141	7	12	156
1024	8	11	73	8	11	111	7	11	144	7	12	160
2048	8	11	76	8	11	115	7	11	144	7	11	164
4096	9	11	76	8	11	115	8	11	148	7	11	165
8192	9	11	76	9	11	115	8	11	151	7	11	167

Table 7: The numbers of iterations for the trapezoidal rule.

	τ=16			τ=32			τ=64			τ=128		
n	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$	$\mathcal{W}_\tau$	$\mathcal{M}_\tau$	$\mathcal{I}$
512	8	11	73	8	12	111	7	12	141	11	14	158
1024	8	11	73	8	11	111	7	12	144	7	12	160
2048	8	11	75	7	11	114	7	12	144	7	12	165
4096	9	11	75	7	11	115	7	11	148	7	12	165
8192	9	11	78	9	11	115	7	11	149	7	11	169

Table 8: The numbers of iterations for Simpson's rule.

	$\tau = 64$			$\tau = 128$		
n	Rect.	Trap.	Simp.	Rect.	Trap.	Simp.
512	29.8882	2.2306	0.036	56.2634	9.8180	1.0030
1024	15.4171	0.4993	0.0019	29.8882	2.2306	0.0360
2048	7.8301	0.1160	1.0565e-4	15.4171	0.4993	0.0019
4096	3.9457	0.0278	6.3212e-6	7.8301	0.1160	1.0565e-4
8192	1.9806	0.0068	9.8410e-7	3.9457	0.0278	6.7539e-6

Table 9: Error in the computed solution for $a_2(t)$

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