Circulant Preconditioners for Elliptic Problems*

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Abstract

We propose and analyze the use of circulant preconditioners for the solution of elliptic problems via preconditioned iterative methods such as the conjugate gradient method. Part of our motivation is to exploit the fast inversion of circulant systems via the Fast Fourier Transform (FFT). We prove that circulant preconditioners can be chosen so that the condition number of the preconditioned system can be reduced from $O(n^2)$ to $O(n)$. Numerical experiments also indicate that the preconditioned systems exhibit favorable clustering of eigenvalues. Both the computation (based on averaging of the coefficients of the elliptic operator) and the inversion (using FFT's) of the circulant preconditioners are highly parallelizable.

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1 Introduction

In this paper, we are concerned with the numerical solution of linear boundary value problems of elliptic type. After discretization, such problems reduce to the solution of linear systems of the form $Ax = b$. In this paper, we shall only consider the case where A is symmetric and positive definite. In practice, large problems of this class are often solved by iterative methods, such as the Chebychev method and the conjugate gradient method. Contrary to direct methods in which the coefficients of A are directly transformed, at each step of these iterative methods only the product of A with a given vector v is needed. Such methods are therefore ideally suited to exploit the sparsity which A possesses.

Typically, the rate of convergence of these methods depends on the condition number $\kappa(A)$ of the coefficient matrix A: the smaller $\kappa(A)$ is, the faster the convergence. Unfortunately, for elliptic problems of second order, usually $\kappa(A) = O(n^2)$, where n is the number of degrees of freedom (e.g. mesh points) in each coordinate direction, and hence grows rapidly with n. To somewhat alleviate this problem, these iterative methods are almost always used with a preconditioner M and the conjugate gradient method is applied instead to the transformed system $\tilde{A}\tilde{x} = \tilde{b}$ where $\tilde{A} = M^{-1/2}AM^{-1/2}, \tilde{x} = M^{1/2}x$ and $\tilde{b} = M^{-1/2}b$. The preconditioner M is chosen with two criteria in mind: to minimize $\kappa(M^{-1}A)$ and to allow efficient computation of the product $M^{-1}v$ for a given vector v. These last two goals are often conflicting ones and much research has gone into devising preconditioners that strike a delicate balance between the two.

One of the most popular and most successful class of preconditioners is the class of *incomplete LU factorizations*, see for instance, [10, 2]. The central idea is to factor A into approximate triangular factors L and U via an elimination process such that L and U have nonzero entries only where the corresponding element of A is nonzero. For some of these preconditioners, it can be proven that $\kappa(M^{-1}A) = O(n)$ for certain classes of elliptic problems, see [8, 11, 2]. This is a much slower growth compared to the unpreconditioned system.

One potential problem with the ILU preconditioners is that both the computation and the application of the preconditioners have limited degree of parallelism, due to the inherently sequential way in which the grid is traversed. Attempts to modify the method (e.g. by re-ordering the grid points) and to devise other more parallel methods (e.g. polynomial preconditioners) often result in a deterioration of the convergence rate.

The purpose of this paper is to propose another class of preconditioners, one that is based on averaging the coefficients of A to form a *circulant* approximation M. Part of our motivation is to exploit the fast inversion of circulant systems via the Fast Fourier Transform (FFT). We prove that circulant preconditioners can be chosen so that $\kappa(M^{-1}A) = O(n)$, just as for ILU type preconditioners. In addition, we are motivated by recent research on circulant preconditioners for Toeplitz systems [5, 7], which shows potential for favorable clustering of eigenvalues of the preconditioned system. Finally, both the computation (based on averaging of the coefficients of the elliptic operator) and the inversion (using FFT's) of our circulant preconditioners are highly parallelizable across a wide variety of architectures.

Our preliminary numerical experiments show that the circulant preconditioners are quite competitive in terms of number of iterations with the ILU preconditioners for elliptic problems with mildly varying coefficients. As is well-known, the ILU preconditioners are rather insensitive to the variation of the coefficients and for such problems they rquire much fewer number of iterations (than most known preconditioners in fact). Part of our numerical experiments are designed to study the cross-over point in this comparison.

Recently, several interesting multilevel elliptic preconditioners have been proposed in the literature [1, 3, 18, 14] which are highly parallelizable and have very attractive convergence rates. However, these preconditioners are not directly applicable when the discrete algebraic problem does not have an underlying multilevel structure. For such problems, we hope that the circulant preconditioners proposed here will offer an interesting alternative to ILU-type preconditioners on parallel computers.

The idea of circulant preconditioners has been proposed independently by Holmgren and Otto [12] for preconditioning implicit systems arising from hyperbolic problems. For such problems, the coefficient matrix A is often highly nonsymmetric and non-diagonally dominant and hence many classical preconditioning techniques are not effective (and sometimes not well-defined). For these problems, the circulant preconditioners are often the only ones that work.

We mention that it is also possible to use skew-circulant preconditioners for general Toeplitz systems. Huckle [13] has shown that skew-circulant preconditioners and combinations of skew-circulant and circulant preconditioners can be as effective as the circulant preconditioners. However, we shall limit our attention only to circulant preconditioners in this paper.

The outline of the paper is as follows. In §2, we define the circulant preconditioner and analyze a model problem in the one-dimensional case. Analysis of the spectral condition number of the preconditioned system are given in §3 for the model Laplacian operator on a square and extended to variable coefficient operators in §4. Some numerical experiments are presented in §5 to verify these theoretical bounds and to illustrate the effect of clustering of the spectrum. Extension to the case of irregular domains are discussed in §6.

2 Circulant Approximations to Elliptic Operators: The 1D Case

In this section, we derive various circulant preconditioners for elliptic operators on rectangular domains. Our basic strategy is to choose as preconditioner a matrix C which is a good approximation to the coefficient matrix A in the sense of minimizing $\|A-C\|$ in some appropriate norm. In the Frobenius norm, denoted by $\|\cdot\|_F$, this problem has a trivial solution, first noted in [7]. Let the elements of A be denoted by $a_{i,j}$ and the elements of the first row of C be denoted by $(c_1, c_2, ..., c_n)$.

Theorem 1 *The best circulant approximation C to a given n-by-n matrix A in the sense of minimizing $\|A-C\|_F$ is given by:*

$$c_i = \frac{1}{n}\sum_{j=1}^{n} a_{j,(j+i-1)\,mod\,n}. \tag{2.1}$$

Moreover, C is symmetric positive definite if A is.

The above formula has a simple graphical interpretation: c_i is simply the arithmetic average of that diagonal of A (extended to length n by wrap-around if necessary) containing the corresponding element $a_{1,i}$. For further properties of this circulant approximation to a general matrix, we refer the reader to [6].

We remark that if A is a general Toeplitz matrix, one can define other good circulant approximations to A, see for instance, [13, 16, 17]. However, we emphasize that some of these circulant approximations, such as the Strang's preconditioner [16], are not defined for general non-Toeplitz matrix.

Now consider applying the result on the best circulant approximation C to a simple elliptic problem in 1D, namely the problem:

$$-(a(x)u_x)_x = f(x) \tag{2.2}$$

on the interval $[0,1]$ with Dirichlet boundary conditions $u(0) = u_0$ and $u(1) = u_1$. Using the usual 3-point centered differencing on a uniform mesh with n interior grid points x_i's, the corresponding matrix A is a symmetric tridiagonal matrix with nonzero elements of the i-th row given by

$$(-a(x_{i-\frac{1}{2}}), a(x_{i-\frac{1}{2}}) + a(x_{i+\frac{1}{2}}), -a(x_{i+\frac{1}{2}})).$$

The best circulant approximation to A is given by

$$\begin{aligned} c_2 &= c_n = -\frac{1}{n}\sum_{j=1}^{n-1} a(x_{j+\frac{1}{2}}) \\ c_1 &= -2c_2 + \frac{1}{n}(a(x_{\frac{1}{2}}) + a(x_{n+\frac{1}{2}})), \end{aligned}$$

with all other coefficients c_i's defined to be zero. The coefficients of the circulants are therefore simple averages of the coefficient $a(x)$ over the grid points.

The question now is how good this preconditioner is in the sense of minimizing $\kappa(C^{-1}A)$. As it turns out, C defined this way is not as good as some of the ILU type preconditioners asymptotically. Precisely, it can be shown (as part of a result which we shall prove later) that $\kappa(C^{-1}A) = O(n^{3/2})$.

The above situation is reminiscent of that of the *unmodified* ILU preconditioner [15]. In that situation, the bound for the condition number can be lowered to $O(n)$ if we modify the preconditioner in a simple way: [8, 11] at each step of the elimination process, we add enough to the main diagonal entry to make the row sum zero and then add a quantity of size $O(n^{-2})$. Borrowing from this idea, we can modify our circulant preconditioner C by keeping the definitions of c_2 and c_n the same and redefining c_1 as follows:

$$c_1 = -(2c_2) + \rho n^{-\alpha}, \tag{2.3}$$

where ρ is a positive constant independent of n and $\alpha > 0$. Clearly, this modified circulant matrix has each row sum equal to $\rho n^{-\alpha}$.

It turns out that this simple modification is sufficient to reduce $\kappa(C^{-1}A)$ to $O(n)$ for a suitably chosen α. We shall illustrate this for the special case of $a(x) = 1$. In this constant-coefficient case, A is a tridiagonal Toeplitz matrices given by tridiag$(-1, 2, -1)$ and C is a circulant matrix with the only three nonzero coefficients given by $c_1 = 2\beta + \rho n^{-\alpha}$ and $c_2 = c_n = -\beta$, where $\beta = \frac{n-1}{n}$. For easy reference by later discussion, we denote A and C for this constant-coefficient 1D case by A_0 and C_0 respectively.

Theorem 2 *Let* $A_0 = \text{tridiag}[-1, 2, -1]$ *and* C_0 *be the circulant matrix with the first row given by*

$$(2\beta + \frac{\rho}{n^\alpha}, -\beta, 0, \cdots, 0, -\beta), \tag{2.4}$$

where $\beta = (n-1)/n$, $\rho = O(1)$ *and* $\alpha \geq 0$. *Then we have,*

$$O(n^{\alpha-2}) \leq \lambda(C_0^{-1}A_0) \leq O(n^{\frac{\alpha}{2}}), \text{ if } \alpha \leq 2,$$

and

$$O(1) \leq \lambda(C_0^{-1}A_0) \leq O(n^{\alpha-1}), \text{ if } \alpha \geq 2.$$

As a consequence, we have:

$$\kappa(C_0^{-1}A_0) \leq O(n^{2-\frac{\alpha}{2}}), \text{ if } \alpha \leq 2,$$

and

$$\kappa(C_0^{-1}A_0) \leq O(n^{\alpha-1}), \text{ if } \alpha \geq 2.$$

The optimal value of $\kappa(C_0^{-1}A_0) \leq O(n)$ *is achieved with* $\alpha = 2$.

Proof. See Appendix.

Remark. It can be easily verified that the unmodified circulant preconditioner corresponds to the case $\rho = 2$ and $\alpha = 1$. The results of the above theorem show that in that case $\kappa(C_0^{-1}A_0) \leq O(n^{1.5})$, justifying our earlier statement.

When $\alpha = 2$, we can show furthermore that the spectrum of $C_0^{-1}A_0$ is clustered.

Corollary 1 *If $\alpha = 2$, then at most one eigenvalue of $C_0^{-1}A_0$ lies outside $[c, n/(n-1)]$, where $c = 4\pi^2/(8\pi^2 + \rho) + O(n^{-1})$.*

Proof. See Appendix.

3 Analysis for the 2D Model Problem

While so far we have discussed only 1D problems for the purpose of illustration, the results do extend to higher dimensions. Consider for example the 2D problems:

$$-(a(x,y)u_x)_x - (b(x,y)u_y)_y = f(x,y)$$

on the unit square $[0,1] \times [0,1]$ with Dirichlet boundary condition. Let the domain be discretized by using a uniform grid with n grid points in each coordinate direction, denoted by x_i and y_j. Consider the usual 5-point centered difference approximation with the grid points ordered in the x-direction first. The matrix A is an n^2-by-n^2 block tridiagonal matrix where the diagonal blocks are scalar tridiagonal matrices and the off-diagonal blocks are diagonal matrices.

We consider two choices of circulant preconditioners for A. The first is obtained by applying the circulant approximation in Theorem 1 directly to A. This preconditioner, denoted by C_P, is defined by

$$c_1 \quad = \quad 2(\overline{a} + \overline{b}) + \rho n^{-\alpha}, \tag{3.1}$$

$$c_2 = c_{n^2} = -\overline{a}, \tag{3.2}$$
$$c_{n+1} = c_{(n-1)n+1} = -\overline{b}, \tag{3.3}$$

where

$$\overline{a} = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{i=1}^{n-1} a(x_{i+\frac{1}{2}}, y_j), \tag{3.4}$$

and

$$\overline{b} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n-1} b(x_i, y_{j+\frac{1}{2}}), \tag{3.5}$$

and all other c_i's defined to be zero. Again these coefficients are simple averages of the coefficients $a(x,y)$ and $b(x,y)$ of the differential problem over the grid. We shall call C_P the *point-circulant* preconditioner for A.

For the second choice of preconditioner, we preserve the block structure of A and define a *block-circulant* preconditioner C_B as follows:

$$C_B = C^b \otimes I + I \otimes C^a, \tag{3.6}$$

where C^a and C^b are n by n circulant matrices defined by:

$$\begin{aligned}
C_1^a &= 2\bar{a} + \rho n^{-\alpha}, \\
C_2^a &= -\bar{a}, \\
C_n^a &= -\bar{a}, \\
C_1^b &= 2\bar{b} + \rho n^{-\alpha}, \\
C_2^b &= -\bar{b}, \\
C_n^b &= -\bar{b},
\end{aligned}$$

with all other diagonals of C^a and C^b defined to be zero.

We note that C_B can be inverted on a given vector using n FFTs of size n, whereas C_P requires one FFT of size n^2.

Similar circulant matrices can be defined for more general elliptic operators with more complicated difference stencils and also in higher dimensions.

We now analyze the convergence rate of our method for the special case of the discrete Laplacian on the unit square with Dirichlet boundary conditions.

The n^2-by-n^2 coefficient matrix A_c is given by

$$A_c = A_0 \otimes I + I \otimes A_0, \tag{3.7}$$

where $A_0 = \text{tridiag}[-1, 2, -1]$. In this case, $\bar{a} = \bar{b} = \beta = (n-1)/n$. In particular, the block-circulant preconditioner, denoted by C_b now, is given by

$$C_b = C_0 \otimes I + I \otimes C_0, \tag{3.8}$$

where C_0, given by (2.1) and modified by (2.3), is the circulant approximation of A_0.

For the block-circulant preconditioner, the results in the 1D case can readily be generalized.

Theorem 3 *For the block-circulant preconditioned systems for the 2D model problem, we have*

$$O(n^{\alpha-2}) \leq \lambda(C_b^{-1}A_c) \leq O(n^{\frac{\alpha}{2}}), \text{ if } \alpha \leq 2,$$

and

$$O(1) \leq \lambda(C_b^{-1}A_c) \leq O(n^{\alpha-1}), \text{ if } \alpha \geq 2.$$

As a consequence, we have:

$$\kappa(C_b^{-1}A_c) \leq O(n^{2-\frac{\alpha}{2}}), \text{ if } \alpha \leq 2,$$

and

$$\kappa(C_b^{-1}A_c) \leq O(n^{\alpha-1}), \text{ if } \alpha \geq 2.$$

The optimal value of $\kappa(C_b^{-1}A_c) \leq O(n)$ is achieved with $\alpha = 2$.

Proof: See Appendix.

For the point-circulant preconditioned systems, we obtain a slightly larger bound on their condition numbers. For simplicity, we only consider the case where $\alpha = 2$.

Theorem 4 *Let C_p be the point-circulant preconditioner for the 2D model problem with $\alpha = 2$. Then we have $O(1) \leq \lambda(C_p^{-1}A_c) \leq O(n\log n)$ and hence $\kappa(C_p^{-1}A_c) \leq O(n\log n)$.*

Proof: See Appendix.

4 Analysis for Variable Coefficient Problems in 2D

In this section, we shall make use of the results of the previous section and extend them to variable coefficient problems. We consider elliptic equations of the form

$$-(a(x,y)u_x)_x - (b(x,y)u_y)_y = f(x,y) \tag{4.1}$$

on the unit square. We assume that the coefficients $a(x,y)$ and $b(x,y)$ satisfy

$$0 < c_{\min} \leq a(x,y), b(x,y) \leq c_{\max}$$

for some constants $c_{\min}$ and $c_{\max}$. Without loss of generality, we assume $c_{\min} \leq 1$ and $c_{\max} \geq 1$. Let A be the n^2-by-n^2 matrix obtained by discretizing (4.1) by the standard 5-point scheme on a uniform n by n grid. Define $A_{\max} = c_{\max} \cdot A_c$ and $A_{\min} = c_{\min} \cdot A_c$, where A_c is given by (3.7). We claim that both $A_{\max} - A$ and $A - A_{\min}$ are both positive semi-definite matrices.

We verify the claim for $A - A_{\min}$. Let us assume that the domain is discretized by using a uniform grid with n grid points in each coordinate direction, denoted by x_j and y_j. It is easy to see that every row in $A - A_{\min}$ has at most five nonzero entries and they are given by

$$\begin{aligned}
(A - A_{\min})_{j,j} &= a(x_{j-\frac{1}{2}}, y_j) + a(x_{j+\frac{1}{2}}, y_j) + b(x_j, y_{j-\frac{1}{2}}) \\
&\quad + b(x_j, y_{j+\frac{1}{2}}) - 4c_{\min}, \\
(A - A_{\min})_{j,j-1} &= c_{\min} - a(x_{j-\frac{1}{2}}, y_j), \\
(A - A_{\min})_{j,j+1} &= c_{\min} - a(x_{j+\frac{1}{2}}, y_j), \\
(A - A_{\min})_{j,j-n} &= c_{\min} - b(x_j, y_{j-\frac{1}{2}}), \\
(A - A_{\min})_{j,j+n} &= c_{\min} - b(x_j, y_{j+\frac{1}{2}}), \quad j = 1, \cdots, n^2,
\end{aligned}$$

where we employ the convention that $(\cdot)_{jk} = 0$ if k lies outside the range $[1, n^2]$. It is now clear that the diagonal entries of $(A - A_{\min})$ are non-negative and the off-diagonal entries are non-positive. Moreover, we have

$$(A - A_{\min})_{j,j} \geq \sum_{\substack{i=1 \\ i \neq j}}^{n^2} |(A - A_{\min})_{j,i}|.$$

Hence by the Gerschgorin Theorem, $A - A_{\min}$ is positive semi-definite. Similarly, we can show that $A_{\max} - A$ is also positive semi-definite. Thus we see that for all nonzero vectors x,

$$0 < x^* A_{\min} x \leq x^* A x \leq x^* A_{\max} x. \tag{4.2}$$

Now let C_B, $C_{\max}$ and $C_{\min}$ be the the block-circulant approximations of A, $A_{\max}$ and $A_{\min}$ respectively. Clearly, $C_{\max} = c_{\max} \cdot C_b$ and $C_{\min} = c_{\min} \cdot C_b$, where C_b, given by (3.8), is the block-circulant approximation of A_c. Consider first the matrix $C_B - C_{\min}$. By our definition of block-circulant approximations, it can be easily verified that this matrix has non-negative diagonal entries and non-positive off-diagonal entries. It therefore follows that

$$(C_B - C_{\min})_{j,j} = \sum_{\substack{i=1 \\ i \neq j}}^{n^2} |(C_B - C_{\min})_{j,i}| + \frac{2\rho(1 - c_{\min})}{n^\alpha} \geq \sum_{\substack{i=1 \\ i \neq j}}^{n^2} |(C_B - C_{\min})_{j,i}|.$$

Thus by the Gerschgorin Theorem, the matrix $C_B - C_{\min}$ is positive semi-definite. By a similar argument, so is the matrix $C_{\max} - C_B$. Hence for all nonzero vectors x, we also have,

$$0 < x^* C_{\min} x \leq x^* C_B x \leq x^* C_{\max} x.$$

Combining this result with (4.2), we get

$$0 < \frac{c_{\min}}{c_{\max}} \frac{x^* A_c x}{x^* C_b x} = \frac{x^* A_{\min} x}{x^* C_{\max} x} \leq \frac{x^* A x}{x^* C_B x} \leq \frac{x^* A_{\max} x}{x^* C_{\min} x} = \frac{c_{\max}}{c_{\min}} \frac{x^* A_c x}{x^* C_b x}.$$

Recalling the results for the constant-coefficient case, namely Theorem 3, we have the following theorem.

Theorem 5 *Let A be the 5-point discretization matrix of*

$$-(a(x,y)u_x)_x - (b(x,y)u_y)_y = f(x,y)$$

on the unit square with

$$0 < c_{\min} \le a(x,y), b(x,y) \le c_{\max}$$

for some constants $c_{\min}$ and $c_{\max}$ and let C_B be the block-circulant preconditioner of A as defined in (3.6). Then we have

$$O(n^{\alpha-2}) \le \lambda(C_B^{-1}A) \le O(n^{\frac{\alpha}{2}}), \text{ if } \alpha \le 2,$$

and

$$O(1) \le \lambda(C_B^{-1}A) \le O(n^{\alpha-1}), \text{ if } \alpha \ge 2.$$

As a consequence, we have:

$$\kappa(C_B^{-1}A) \le O(n^{2-\frac{\alpha}{2}}), \text{ if } \alpha \le 2,$$

and

$$\kappa(C_B^{-1}A) \le O(n^{\alpha-1}), \text{ if } \alpha \ge 2.$$

The optimal value of $\kappa(C_B^{-1}A) \le O(n)$ is achieved with $\alpha = 2$.

For the point-circulant preconditioned systems, using a similar argument, we have the following results. As in Theorem 4, we only consider the case where $\alpha = 2$.

Theorem 6 *Let A be the 5-point discretization matrix of*

$$-(a(x,y)u_x)_x - (b(x,y)u_y)_y = f(x,y)$$

on the unit square with

$$0 < c_{\min} \le a(x,y), b(x,y) \le c_{\max}$$

for some constants $c_{\min}$ and $c_{\max}$ and let C_P be the point-circulant preconditioner of A as defined in (3.1)-(3.3) with $\alpha = 2$. Then we have $O(1) \le \lambda(C_P^{-1}A) \le O(n \log n)$ and $\kappa(C_P^{-1}A) \le O(n \log n)$.

Finally, we note that the application of the circulant preconditioners require $O(n^2 \log n)$ flops, which is slightly more expensive than the $O(n^2)$ flops for the ILU-type preconditioners. However, the FFTs can be computed in $O(\log n)$ parallel steps with $O(n^2)$ processors whereas the ILU preconditioners require at least $O(n)$ steps regardless of how many processors are available.

5 Numerical Experiments

In this section, we compare the performance of our method to the *modified* incomplete LU (MILU) preconditioner [2]. In these preliminary tests, we shall mainly compare the number of iterations, rather than the actual computing time. The equation we used is

$$\frac{\partial}{\partial x}[(1+\epsilon e^{x+y})\frac{\partial u}{\partial x}] + \frac{\partial}{\partial y}[(1+\frac{\epsilon}{2}\sin(2\pi(x+y)))\frac{\partial u}{\partial y}] = f(x,y),$$

on the unit square and where ϵ is a parameter. We discretize the equation using the standard 5-point scheme. Both the right hand side and the initial guess are chosen to be random vectors and are the same for the different methods. Computations are done with double precision on a VAX 6420 and the iterations are stopped when $||r^j||_2/||r^0||_2 < 10^{-6}$. Here r^j is the residual at the jth step and $||(x_1, \cdots, x_n)^*||_2^2 = \sum_{i=1}^n x_i^2$. The block- and the point-circulant preconditioners we used are defined in §3 and §4. The parameters we choose for our experiments are $\rho = 1$ and $\alpha = 2$ for both the circulant and the MILU preconditioners.

Since the circulant preconditioners are based on averaging of these coefficients over the grid points, their performance will deteriorate as the variation in the coefficients increase. To somewhat alleviate this potential problem, we first symmetrically scale A by its diagonal before applying the circulant preconditioners. This technique has also proven to be very useful when used in conjunction with other kinds of preconditioners. In our experiments, we apply diagonal scaling to all methods.

Tables 1a-1b show the number of iterations required for convergence for different choices of ϵ. The data for the preconditioned iterations are also plotted in Figures 1a-1d.

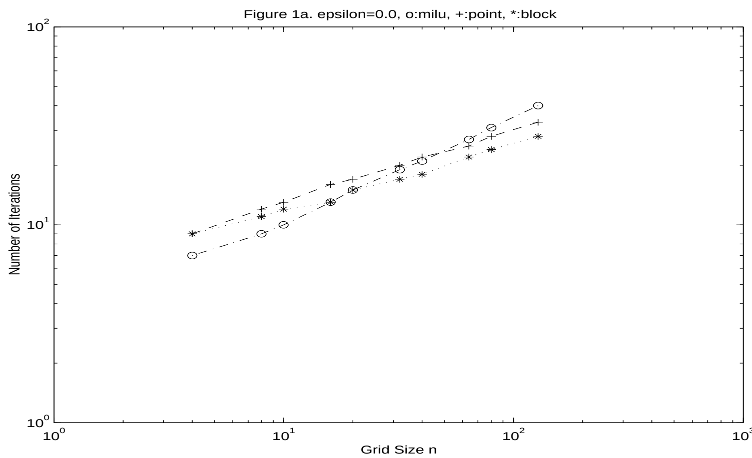


Figure 1a. epsilon=0.0, o:milu, +:point, *:block
Number of Iterations
Grid Size n

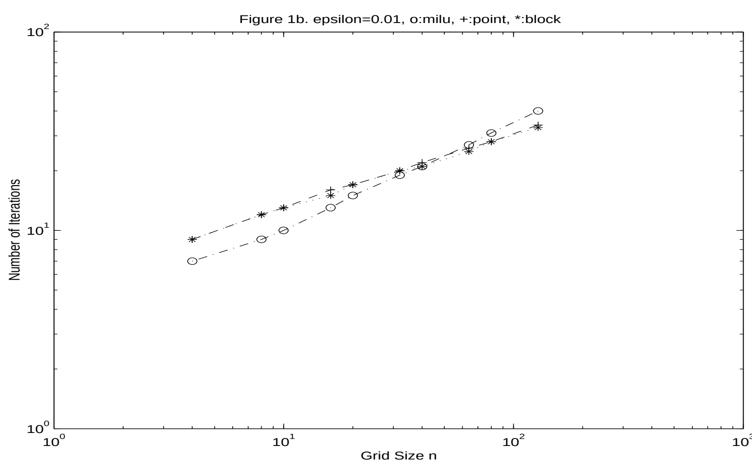
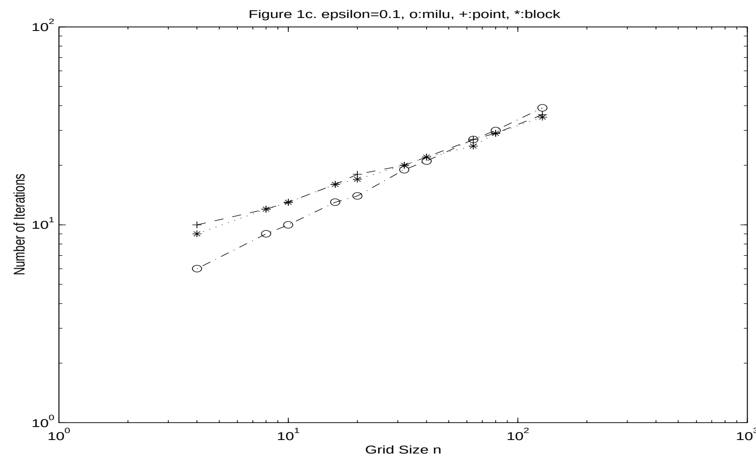
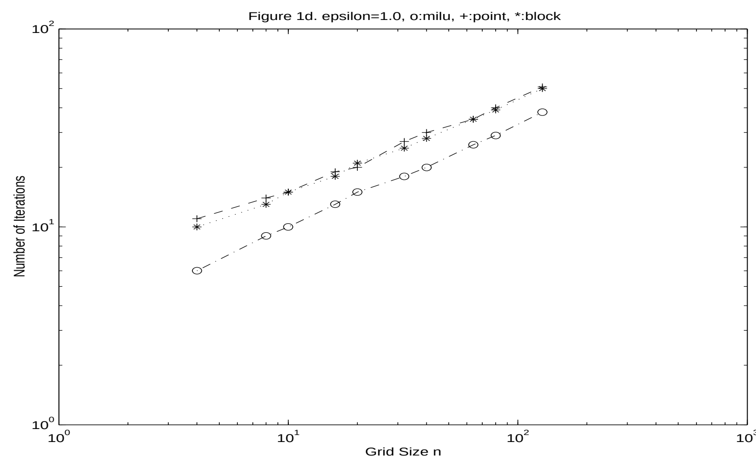


Figure 1b. epsilon=0.01, o:milu, +:point, *:block
Number of Iterations
Grid Size n





We see that for small values of ϵ (e.g. $\epsilon \leq 0.01$), the performance of the circulant preconditioner seems to be better than that of MILU. However, the MILU method is less sensitive to the changes in ϵ, and for larger values of ϵ (e.g. $\epsilon > 1.0$), MILU requires less number of iterations than the circulant preconditioners, at least for the values of n used in our experiments. We also observe that the number of iterations for the circulant preconditioners grows with a rate slightly slower than the predicted $O(\sqrt{n})$ growth of MILU. Therefore, the circulant preconditioners appear more competitive with MILU as n increases. In all cases, the number of iterations grows slower than as predicted by Theorems 5 and 6.

ϵ	0.0				0.01			
n	No	Block	Point	MILU	No	Block	Point	MILU
4	9	9	9	7	12	9	9	7
8	23	11	12	9	23	12	12	9
10	26	12	13	10	30	13	13	10
16	43	13	16	13	47	15	16	13
20	53	15	17	15	57	17	17	15
32	82	17	20	19	89	20	20	19
40	101	18	22	21	106	21	22	21
64	157	22	25	27	171	25	26	27
80	194	24	28	31	215	28	28	31
128	307	28	33	40	333	33	34	40

TABLE 1a: Number of iterations for different systems.

ϵ	0.1				1.0			
n	No	Block	Point	MILU	No	Block	Point	MILU
4	13	9	10	6	14	10	11	6
8	26	12	12	9	28	13	14	9
10	31	13	13	10	34	15	15	10
16	49	16	16	13	51	18	19	13
20	59	17	18	14	61	21	20	15
32	89	20	20	19	99	25	27	18
40	118	22	22	21	122	28	30	20
64	175	25	27	27	195	35	35	26
80	228	29	29	30	246	39	40	29
128	366	35	36	39	395	50	51	38

TABLE 1b: Number of iterations for different systems.

Tables 2 and 3 show the eigenvalue distributions of the preconditioned systems for $\epsilon = 0.0$ and 0.1 respectively. In the table, the eigenvalues are ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \lambda_{n-1} \leq \lambda_n$. We see that for the point- and the block-circulant preconditioned systems, they have one outlying eigenvalue λ_n. The rest are in a relatively small interval. In Figures 2 and 3, we plot the eigenvalue distributions, leaving out the rightmost eigenvalue λ_n. The clustering effect is similar to that of MILU.

	No			MILU		
n	λ_1	λ_{n-1}	λ_n	λ_1	λ_{n-1}	λ_n
4	0.191	1.559	1.809	0.844	1.312	1.332
8	0.0630	1.853	1.940	0.878	2.114	2.117
16	0.0170	1.958	1.983	0.912	3.874	3.885

TABLE 2a: Eigenvalue Distribution for $\epsilon = 0.0$

	Block			Point		
n	λ_1	λ_{n-1}	λ_n	λ_1	λ_{n-1}	λ_n
4	0.730	1.500	2.522	0.759	1.723	4.386
8	0.609	2.150	5.132	0.643	2.356	9.045
16	0.553	3.602	10.380	0.575	3.889	18.347

TABLE 2b: Eigenvalue Distribution for $\epsilon = 0.0$

	No			MILU		
n	λ_1	λ_{n-1}	λ_n	λ_1	λ_{n-1}	λ_n
4	0.192	1.589	1.808	0.845	1.302	1.331
8	0.0606	1.863	1.939	0.878	2.106	2.114
16	0.0171	1.961	1.983	0.912	3.856	3.864

TABLE 3a: Eigenvalue Distribution for $\epsilon = 0.1$

	Block			Point		
n	λ_1	λ_{n-1}	λ_n	λ_1	λ_{n-1}	λ_n
4	0.730	1.568	2.528	0.752	1.761	4.400
8	0.604	2.300	5.142	0.637	2.471	9.067
16	0.543	3.912	10.394	0.561	4.207	18.377

TABLE 3b: Eigenvalue Distribution for $\epsilon = 0.1$

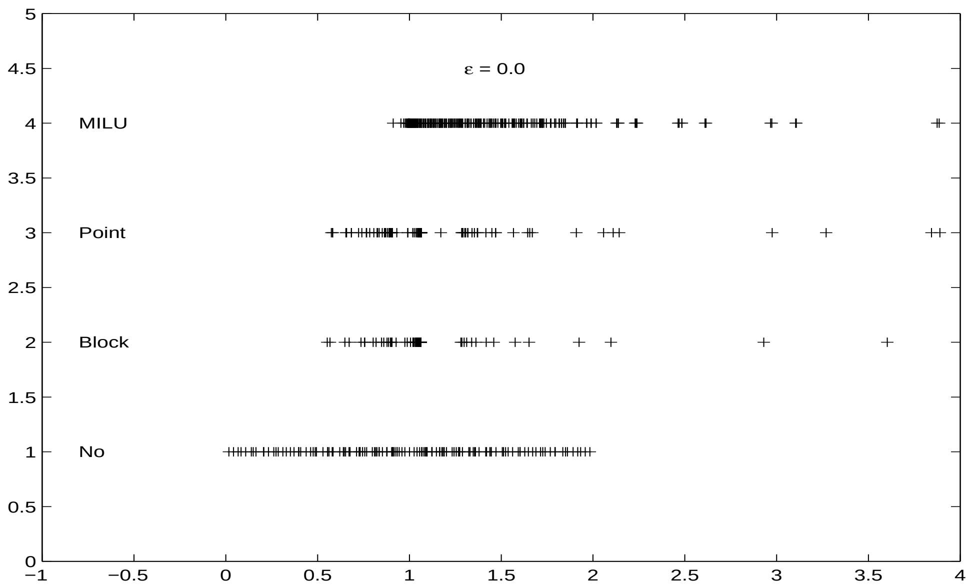


Figure 2. Spectra of the preconditioned systems for $n = 16$.

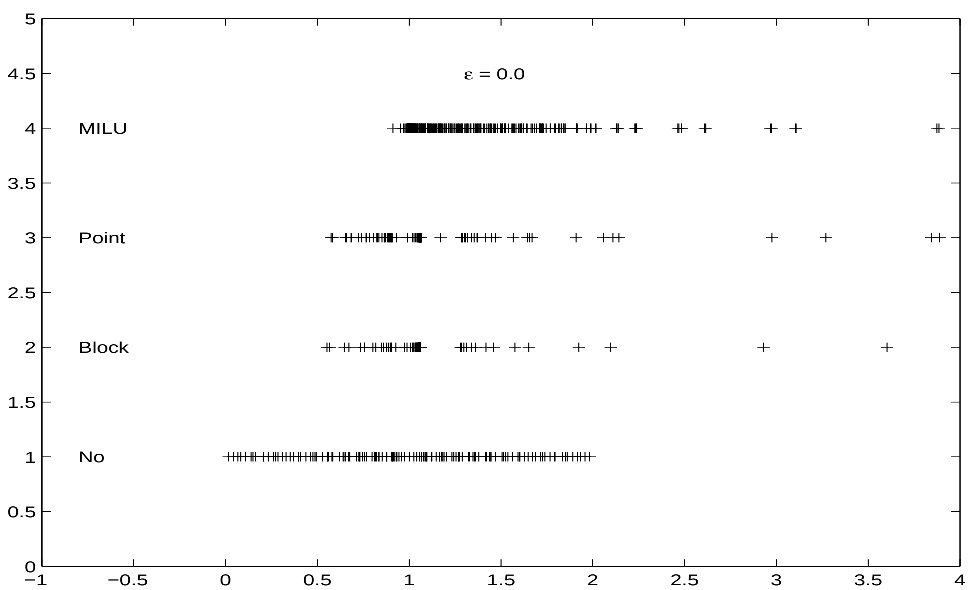


Figure 3. Spectra of the preconditioned systems for $n = 16$.

To summarize, we make the following observations from the numerical results:

1. The circulant preconditioners seem to grow slower than $O(\sqrt{n})$ in number of iterations, which is the asymptotic rate for the MILU preconditioner and also slower than the bounds in Theorems 5 and 6.
2. For small variation of coefficients ($\epsilon \leq 0.1$ in our test problem), the circulant preconditioners seem to be competitive with the MILU preconditioner in number of iterations.
3. For large variation of coefficients ($\epsilon > 1.0$), MILU requires fewer number of iterations.
4. The circulant-preconditioned systems exhibit clustering of the eigenvalues around 1, similar to MILU.

6 Extensions and Remarks

We first discuss several ways for extending the idea of circulant preconditioners for solving more general elliptic problems.

First, we discuss how to apply the idea of circulant preconditioners for problems on irregular domains. It should be obvious that the circulant approximation we use is sensitive to the ordering of the grid points. The regularity of the coefficient of the matrix A for the natural ordering on rectangular domains, which plays a fundamental role in the successful performance that we have observed so far, is not naturally present for irregular domains. We now describe an embedding technique which does maintain the regularity of the rectangular case. The main idea, which is similar to one used in the Capacitance Matrix method [4], is to embed the irregular grid, say Ω, in an inscribing rectangular grid S. A natural ordering of the grid points of S is then used. For grid points in Ω, the difference stencil and right hand side are chosen to match those of the corresponding problem defined on Ω. In addition, the difference operator must be chosen so that there is no coupling with grid points in $S \setminus \Omega$. For the grids points in $S \setminus \Omega$, we can use an artificially chosen elliptic operator and right hand side, as their choice do not affect the solution in Ω. The circulant approximation (which is defined on the embedding domain S) is then obtained by the averaging procedure defined in Theorem 1. Note that in this approach, the iteration is carried out on the whole domain S. Of course, the quality of the circulant approximation will depend on the operator we choose on $S \setminus \Omega$. Intuitively, one should choose it to be as close to the operator on Ω as possible.

We now make some general remarks on the application of the circulant preconditioners. First, circulant preconditioners can be applied to more general discretizations (e.g. higher order finite elements) and problems other than second order elliptic problems with Dirichlet type boundary conditions. As mentioned in the introduction, the possibility of applying them to nonsymmetric linear systems arising from discretizations of hyperbolic systems is particularly attractive, because many of the classical preconditioners (e.g. ILU) either are not well-defined or do not perform very well for these problems, primarily due to the non-diagonal-dominance of the coefficient matrix. Some promising preliminary numerical results have been reported in [12]. Finally, the type of boundary conditions may also affect the performance of the circulant preconditioners, which should work better for problems with periodic boundary conditions.

We would like to make a final comment on the relationship of circulant preconditioners to preconditioning by approximations by separable elliptic operators (and the use of fast direct solvers (FDS)). Both derive their efficiency from that of the Fast Fourier Transform (FFT). For problems on regular domains, it is possible for the FDS method to produce a spectrally equivalent preconditioner to the original operator [9] (although this does not necessarily mean it is a more efficient method for a problem with a given size). Unfortunately, for problems on irregular domains, the separable preconditioner itself cannot be directly solved efficiently via FDSs. The usual approach is the capacitance matrix method, in which an embedding of the irregular domain within a regular one is also made. The coefficient matrix S of the separable approximation to A on the embedded domain can be written as: $S = B + UV^T$, where B is a separable operator on the regular embedded domain and U and V are low rank matrices. In the capacitance matrix approach, the system with S is solved using the Woodbury formula and at each step the necessary application of B^{-1} is computed by the FDS. Thus, this approach consists of a two-step process: preconditioning A by

S and then computing $S^{-1}v$ via repeated applications of B^{-1}. The circulant preconditioner approach can be viewed as directly solving the system $Ax = b$ by the preconditioned conjugate gradient method with a circulant preconditioner B without going through a separable approximation first. In some sense, one can view the circulant proconditioner approach as a way of extending the FDS to irregular domains by using the main tools of the FDS (i.e. FFT) to define a preconditioner.

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8 Appendix

Proof of Theorem 2: In the constant-coefficient case,

$$A_0 = \mathrm{tridiag}[-1, 2, -1], \tag{8.1}$$

and C_0, constructed according to (2.1) and (2.3), is given by

$$C_0 = \beta * \{A_0 - e_1 e_n^* - e_n e_1^*\} + \frac{\rho}{n^\alpha} I \tag{8.2}$$

where $\beta = (n-1)/n = O(1)$ and e_j is the j-th unit vector.

To compute $\lambda_{\min}(C_0^{-1} A_0)$, we first note that for all n-vectors x,

$$x^* C_0 x = \beta x^* A_0 x + \beta x^* (e_1 e_1^* + e_n e_n^*) x - \beta x^* (e_1 + e_n)(e_1 + e_n)^* x + \frac{\rho}{n^\alpha} x^* x.$$

Since the matrices $(e_1 + e_n)(e_1 + e_n)^*$ and $A_0 - (e_1 e_1^* + e_n e_n^*)$ are positive semi-definite, we have

$$x^* C_0 x \le 2\beta x^* A_0 x + \frac{\rho}{n^\alpha} x^* x. \tag{8.3}$$

Using the fact that $x^*x \leq O(n^2)x^*A_0x$ and $\rho = O(1)$, we see that

$$(2\beta + O(n^{2-\alpha}))^{-1} \leq \lambda_{\min}(C_0^{-1}A_0). \tag{8.4}$$

To compute $\lambda_{\max}(C_0^{-1}A_0)$, we note from (8.2) that for all n-vectors x,

$$\beta x^*A_0x = x^*C_0x + \frac{\beta}{2}x^*(e_1+e_n)(e_1+e_n)^*x - \frac{\beta}{2}x^*(e_1-e_n)(e_1-e_n)^*x - \frac{\rho}{n^\alpha}x^*x,$$

where the last two terms on the right hand side are always non-positive. Thus

$$\beta x^*A_0x \leq x^*C_0x + \frac{\beta}{2}x^*ee^*x, \tag{8.5}$$

where $e = e_1 + e_n$. Next we claim that for all nonzero n-vectors x,

$$\frac{x^*ee^*x}{x^*C_0x} \leq ||C_0^{-1/2}ee^*C_0^{-1/2}||_2 \leq O(n^{\alpha/2}) + O(n^{\alpha-1}). \tag{8.6}$$

Substituting this into (8.5), we have

$$\lambda_{\max}(C_0^{-1}A_0) \leq O(n^{\alpha/2}) + O(n^{\alpha-1}).$$

Theorem 2 now follows by combining this with (8.4).

It remains to prove (8.6). We note that for all nonzero vectors x,

$$\frac{x^*ee^*x}{x^*C_0x} \leq ||C_0^{-1/2}ee^*C_0^{-1/2}||_2 = e^*C_0^{-1}e.$$

Since C_0 is a circulant matrix, $C_0 = F\Lambda F^*$, where

$$F = [\frac{1}{\sqrt{n}}e^{2\pi ijk/n}]_{0\leq j\leq n-1, 0\leq k\leq n-1},$$

is the Fourier matrix and Λ is the diagonal matrix containing the eigenvalues of C_0. It can easily be shown that

$$[\Lambda]_{j,j} = \lambda_j(C_0) = \frac{\rho}{n^\alpha} + 4\beta\sin^2\theta_j,$$

where $\theta_j = \pi j/n$, $0 \leq j \leq n-1$. Hence

$$e^*C_0^{-1}e \quad = \quad e^*F\Lambda^{-1}F^*e$$

$$
\begin{aligned}
&= \frac{4}{n}\sum_{j=0}^{n-1}\frac{\cos^2\theta_j}{\rho/n^\alpha+4\beta\sin^2\theta_j} \\
&= \frac{4}{n}\frac{n^\alpha}{\rho}+\frac{8}{n}\sum_{j=1}^{n/2-1}\frac{\cos^2\theta_j}{\rho/n^\alpha+4\beta\sin^2\theta_j} \\
&\le O(n^{\alpha-1})+\frac{2}{n\beta}\sum_{j=n/n^{\alpha/2}}^{n/2-1}\frac{\cos^2\theta_j}{\sin^2\theta_j}+\frac{8}{n}\sum_{j=1}^{n/n^{\alpha/2}}\frac{\cos^2\theta_j}{\rho/n^\alpha} \\
&\le O(n^{\alpha-1})+\frac{2}{\beta\pi}\int_{\pi/n^{\alpha/2}}^{\pi/2}\cot^2\theta d\theta+\frac{8}{n}\cdot\frac{n}{n^{\alpha/2}}\cdot\frac{n^\alpha}{\rho} \\
&\le O(n^{\alpha-1})+\frac{2}{\beta\pi}\cot(\frac{\pi}{n^{\alpha/2}})+O(n^{\alpha/2}) \\
&\le O(n^{\alpha-1})+O(n^{\alpha/2}). \quad \square
\end{aligned}
$$

Proof of Corollary 1: We first observe that for any n-vectors x,

$$x^*x \le (4\sin^2\frac{\pi}{n+1})^{-1}x^*A_0x \le \frac{(n+1)^2}{4\pi^2}x^*A_0x.$$

Thus by (8.3), we have

$$\lambda_{\min}(C_0^{-1}A_0) \ge (2\beta+\frac{\rho}{4\pi^2}+O(\frac{1}{n}))^{-1} \ge \frac{4\pi^2}{8\pi^2\beta+\rho}+O(\frac{1}{n}).$$

Next we rewrite (8.2) as

$$
\begin{aligned}
C_0^{-\frac{1}{2}}A_0C_0^{-\frac{1}{2}} &= \frac{1}{\beta}I-\frac{\rho}{\beta n^2}C_0^{-1}-C_0^{-\frac{1}{2}}(e_1-e_n)(e_1-e_n)^*C_0^{-\frac{1}{2}} \\
&\quad +C_0^{-\frac{1}{2}}(e_1+e_n)(e_1+e_n)^*C_0^{-\frac{1}{2}}.
\end{aligned}
$$

Notice that the second and the third terms in the right hand side are negative semi-definite matrices. Hence the matrix formed by the first three terms in the right hand side will have eigenvalues $\lambda \le 1/\beta$. Since the last term in the right hand side is a rank one matrix, by Cauchy interlace theorem, see [10, p.269], at most one eigenvalue of $C_0^{-\frac{1}{2}}A_0C_0^{-\frac{1}{2}}$ has value greater than $1/\beta$. Since $C_0^{-\frac{1}{2}}A_0C_0^{-\frac{1}{2}}$ and $C_0^{-1}A_0$ are similar, the Corollary follows. $\square$

Proof of Theorem 3: For the constant-coefficient case, we have

$$A_c = A_0 \otimes I + I \otimes A_0,$$

and its block-circulant approximation is given by

$$C_b = C_0 \otimes I + I \otimes C_0.$$

Here A_0 and C_0 are given by (8.1) and (8.2) respectively. By (8.3), we have for any n^2-vector x,

$$\begin{aligned} x^* C_b x &= x^*(C_0 \otimes I)x + x^*(I \otimes C_0)x \\ &\leq 2\beta[x^*(A_0 \otimes I)x + x^*(I \otimes A_0)x] + \frac{2\rho}{n^\alpha} x^*(I \otimes I)x \\ &= 2\beta x^* A_c x + \frac{2\rho}{n^\alpha} x^* x. \end{aligned}$$

Since $x^* x \leq O(n^2) x^* A_c x$ for all vectors x, we have

$$(2\beta + O(n^{2-\alpha}))^{-1} \leq \lambda_{\min}(C_b^{-1} A_c).$$

To find $\lambda_{\max}(C_b^{-1} A_c)$, we note that by (8.5), we have

$$\begin{aligned} \beta x^* A_c x &= x^*(\beta A_0 \otimes I)x + x^*(I \otimes \beta A_0)x \\ &\leq x^* C_b x + \frac{\beta}{2} x^*[(I \otimes ee^*) + (ee^* \otimes I)]x, \end{aligned} \tag{8.7}$$

where $e = e_1 + e_n$. Since $(C_0 \otimes I)$ is positive definite, we have for all nonzero vectors x, $x^* C_b x \geq x^*(I \otimes C_0)x$. By (8.6), we then have

$$\begin{aligned} \frac{x^*(I \otimes ee^*)x}{x^* C_b x} &\leq \frac{x^*(I \otimes ee^*)x}{x^*(I \otimes C_0)x} = \|(I \otimes C_0)^{-\frac{1}{2}}(I \otimes ee^*)(I \otimes C_0)^{-\frac{1}{2}}\|_2 \\ &= \|(I \otimes C_0^{-\frac{1}{2}})(I \otimes ee^*)(I \otimes C_0^{-\frac{1}{2}})\|_2 = \|I \otimes (C_0^{-\frac{1}{2}} ee^* C_0^{-\frac{1}{2}})\|_2 \\ &= \|C_0^{-\frac{1}{2}} ee^* C_0^{-\frac{1}{2}}\|_2 \leq O(n^{\alpha-1}) + O(n^{\alpha/2}). \end{aligned}$$

Similarly, we have

$$\frac{x^*(ee^* \otimes I)x}{x^* C_b x} \leq O(n^{\alpha-1}) + O(n^{\alpha/2}).$$

Thus by (8.7),

$$\lambda_{\max}(C_b^{-1}A_c) \le O(n^{\alpha-1}) + O(n^{\alpha/2}). \quad \Box$$

Proof of Theorem 4: We first observe in this case,

$$C_p = \beta A_c - \beta L_{4n} + \frac{\rho}{n^2} I, \tag{8.8}$$

where L_{4n} is a symmetric matrix of rank $4n$ given by

$$\begin{aligned} L_{4n} &= e_1 e_{n^2}^* + \sum_{j=1}^{n} e_j e_{n^2-n+j}^* + \sum_{j=1}^{n-1} e_{jn} e_{jn+1}^* \\ &\quad + e_{n^2} e_1^* + \sum_{j=1}^{n} e_{n^2-n+j} e_j^* + \sum_{j=1}^{n-1} e_{jn+1} e_{jn}^*. \end{aligned}$$

By the Gerschgorin Theorem [10], we can easily check that $A_c + L_{4n}$ is a positive semi-definite matrix. Thus for any n^2-vector x, $-x^* L_{4n} x \le x^* A_c x$. Since $x^* x \le O(n^2) x^* A_c x$, we have, by (8.8)

$$x^* C_p x \le (2\beta + O(1)) x^* A_c x,$$

for any vector x. Thus $\lambda_{\min}(C_p^{-1} A_c) \ge O(1)$.

Next we claim that $\lambda_{\max}(C_p^{-1} A_c) \le O(n \log n)$. By (8.8),

$$C_p^{-\frac{1}{2}} A_c C_p^{-\frac{1}{2}} = \frac{1}{\beta}(I - \frac{\rho}{n^2} C_p^{-1}) + C_p^{-\frac{1}{2}} L_{4n} C_p^{-\frac{1}{2}}. \tag{8.9}$$

Let

$$\tilde{L}_{4n} \equiv \sum_{j=1}^{n} e_j e_j^* + \sum_{j=n^2-n+1}^{n^2} e_j e_j^* + \sum_{j=1}^{n} e_{jn} e_{jn}^* + \sum_{j=0}^{n-1} e_{jn+1} e_{jn+1}^*,$$

and

$$M \equiv \tilde{L}_{4n} - L_{4n}$$

then it is straightforward to check that

$$\begin{aligned} M &= \sum_{j=1}^{n}(e_j - e_{n^2-n+j})(e_j - e_{n^2-n+j})^* \\ &\quad + \sum_{j=1}^{n-1}(e_{jn} - e_{jn+1})(e_{jn} - e_{jn+1})^* + (e_1 - e_{n^2})(e_1 - e_{n^2})^*, \end{aligned}$$

which is clearly a positive semi-definite matrix.

Rewrite (8.9) as

$$C_p^{-\frac{1}{2}} A_c C_p^{-\frac{1}{2}} = \frac{1}{\beta}(I - \frac{\rho}{n^2} C_p^{-1}) - C_p^{-\frac{1}{2}} M C_p^{-\frac{1}{2}} + C_p^{-\frac{1}{2}} \tilde{L}_{4n} C_p^{-\frac{1}{2}}, \qquad (8.10)$$

For $j = 1, 2, \cdots, n^2$, since

$$\lambda_j(C_p) = 4\beta \sin^4(\frac{\pi j}{n^2}) + 4\beta \sin^2(\frac{\pi j}{n}) + \frac{\rho}{n^2}, \qquad (8.11)$$

we have

$$\frac{\rho}{n^2} + 8\beta \geq \lambda_j(C_p) \geq \frac{\rho}{n^2}.$$

Therefore,

$$\lambda_{\max}(\frac{1}{\beta}(I - \frac{\rho}{n^2} C_p^{-1})) \leq \frac{1}{\beta}(1 - \frac{\rho}{\rho + 8\beta n^2}) \leq O(1). \qquad (8.12)$$

Since $C_p^{-\frac{1}{2}} M C_p^{-\frac{1}{2}}$ is a positive semi-definite matrix, to get a bound for $\lambda_{\max}(C_p^{-1} A_c)$, it remains to estimate the 2-norm of the last term in (8.10). We notice that for all $j = 1, \cdots, n^2$,

$$\|C_p^{-\frac{1}{2}} e_j e_j^* C_p^{-\frac{1}{2}}\|_2 = \|e_j^* C_p^{-1} e_j\|_2 = [C_p^{-1}]_{jj},$$

the j-th diagonal entry of C_p^{-1}. Since C_p^{-1} is circulant and positive definite, $[C_p^{-1}]_{jj} = d$ for all j, where d is some positive constant. Thus

$$||C_p^{-\frac{1}{2}} \tilde{L}_{4n} C_p^{-\frac{1}{2}}||_2 \leq 4nd. \qquad (8.13)$$

Next we estimate d. By the Trace Theorem and (8.11), we have

$$d = \frac{1}{n^2} \sum_{j=1}^{n^2} \frac{1}{\lambda_j(C_p)} = \frac{1}{n^2} \sum_{j=1}^{n^2} \left(\frac{\rho}{n^2} + 4\beta \sin^2(\frac{\pi j}{n^2}) + 4\beta \sin^2(\frac{\pi j}{n}) \right)^{-1}.$$

Since

$$\begin{aligned}\lambda_{n^2-j}(C_p) &= \frac{\rho}{n^2} + 4\beta\sin^2(\pi - \frac{\pi j}{n^2}) + 4\beta\sin^2(n\pi - \frac{\pi j}{n}) \\ &= \frac{\rho}{n^2} + 4\beta\sin^2(\frac{\pi j}{n^2}) + 4\beta\sin^2(\frac{\pi j}{n}) = \lambda_j(C_p),\end{aligned}$$

for $j = 1, \cdots, n^2/2$, we see that

$$d \le \frac{2}{n^2}\sum_{j=1}^{n^2/2}\left(\frac{\rho}{n^2} + 4\beta\sin^2(\frac{\pi j}{n^2}) + 4\beta\sin^2(\frac{\pi j}{n})\right)^{-1} + \frac{2}{\rho}. \tag{8.14}$$

We now compute the summation in (8.14) by partitioning the interval $[1, \frac{n^2}{2}]$ into n subintervals of length $\frac{1}{2}n$.

Let $k = 0, \cdots, \frac{n}{2}-1$. We first consider the case when $kn+1 \le j \le kn+\frac{n}{2}$. Since

$$0 \le \frac{\pi j}{n^2} \le \frac{\pi(kn+n/2)}{n^2} \le \frac{\pi(k+1)n}{n^2} \le \pi/2,$$

we see that

$$4\beta\sin^2(\frac{\pi j}{n^2}) \ge 16\beta\frac{j^2}{n^4}.$$

Similarly, if we let $\ell = j - kn$, then $\pi\ell/n \le \pi/2$, and we have

$$4\beta\sin^2(\frac{\pi\ell}{n}) \ge 16\beta\frac{\ell^2}{n^2}.$$

Thus using the substitution $\ell = j - kn$, we have

$$\begin{aligned}&\sum_{j=kn+1}^{kn+n/2}\left(\frac{\rho}{n^2} + 4\beta\sin^2(\frac{\pi j}{n^2}) + 4\beta\sin^2(\frac{\pi j}{n})\right)^{-1} \\ \le\ & \sum_{\ell=1}^{n/2}\left(\frac{\rho}{n^2} + 16\beta\frac{(\ell+kn)^2}{n^4} + 4\beta\sin^2\left[\frac{\pi(\ell+kn)}{n}\right]\right)^{-1} \\ =\ & \sum_{\ell=1}^{n/2}\left(\frac{\rho}{n^2} + 16\beta(\frac{\ell^2}{n^4} + \frac{2\ell kn}{n^4} + \frac{k^2}{n^2}) + 4\beta\sin^2(\frac{\pi\ell}{n})\right)^{-1} \\ \le\ & \sum_{\ell=1}^{n/2}\left(\frac{\rho}{n^2} + 16\beta\frac{k^2}{n^2} + 16\beta\frac{\ell^2}{n^2}\right)^{-1} \\ \le\ & n^2\sum_{\ell=0}^{n/2}\left(\rho + 16\beta(k^2+\ell^2)\right)^{-1}.\end{aligned} \tag{8.15}$$

For $kn+\frac{1}{2}n+1 \le j \le kn+n$, we let $\ell = j - kn$ and use the same argument as above, we have

$$\begin{aligned}
&\sum_{j=kn+\frac{1}{2}n+1}^{kn+n} \left(\frac{\rho}{n^2} + 4\beta\sin^2(\frac{\pi j^2}{n^2}) + 4\beta\sin^2(\frac{\pi j}{n})\right)^{-1} \\
\le\ & \sum_{\ell=\frac{1}{2}n+1}^{n} \left(\frac{\rho}{n^2} + 16\beta\frac{k^2}{n^2} + 4\beta\sin^2(\frac{\pi\ell}{n})\right)^{-1} \\
=\ & \sum_{\ell=0}^{\frac{1}{2}n-1} \left(\frac{\rho}{n^2} + 16\beta\frac{k^2}{n^2} + 4\beta\sin^2(\frac{\pi\ell}{n})\right)^{-1} \\
\le\ & n^2\sum_{\ell=0}^{n/2} (\rho + 16\beta(k^2+\ell^2))^{-1}.
\end{aligned}$$

Combining this inequality with (8.15), we see that (8.14) becomes

$$d \le 4\sum_{k=0}^{n/2-1}\sum_{\ell=0}^{n/2} (\rho + 16\beta(k^2+\ell^2))^{-1} + \frac{2}{\rho} = O(\log n).$$

Hence by (8.13), $\|C_p^{-\frac{1}{2}}\tilde{L}_{4n}C_p^{-\frac{1}{2}}\|_2 \le O(n\log n)$. Applying this result and (8.12) to (8.10) and noting that $C_p^{-\frac{1}{2}}MC_p^{-\frac{1}{2}}$ and $C_p^{-\frac{1}{2}}A_cC_p^{-\frac{1}{2}}$ are positive semi-definite, we see that

$$\lambda_{\max}(C_p^{-1}A_c) = \|C_p^{-\frac{1}{2}}A_cC_p^{-\frac{1}{2}}\|_2 \le O(n\log n). \quad \Box$$

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