Circulant Preconditioners for Complex Toeplitz Matrices

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Abstract

We study the solution of n-by-n complex Toeplitz systems $A_n x = b$ by the preconditioned conjugate gradient method. The preconditioner C_n is the circulant matrix that minimizes $\|B_n - A_n\|_F$ over all circulant matrices B_n. We prove that if the generating function of A_n is a 2π-periodic continuous complex-valued function without any zeros, then the spectrum of the normalized preconditioned matrix $(C_n^{-1}A_n)^*(C_n^{-1}A_n)$ will be clustered around one. Hence we show that if the condition number of A_n is of $O(n^\alpha)$, the conjugate gradient method, when applied to solving the normalized preconditioned system, converges in at most $O(\alpha \log n + 1)$ steps. Thus the total complexity of the algorithm is $O(\alpha n \log^2 n + n \log n)$.

Abbreviated Title. Complex Toeplitz Systems.

Key Words. Toeplitz matrix, circulant matrix, preconditioned conjugate gradient method, generating function.

AMS(MOS) Subject Classifications. 65F10, 65F15

1 Introduction.

In this paper, we discuss the solutions of n-by-n complex Toeplitz systems $A_n x = b$ by the preconditioned conjugate gradient method. A matrix $A_n = (a_{j,k})$ is said to be Toeplitz if $a_{j,k} = a_{j-k}$, i.e. A_n is constant along its diagonals. Toeplitz matrices occur in a variety of applications, especially in signal processing and control theory. Existing direct methods for dealing with them include the Levinson-Trench-Zohar $O(n^2)$ algorithms [20], and a variety of $O(n \log^2 n)$ algorithms such as the one by Ammar and Gragg [1]. The stability properties of these direct methods for symmetric positive definite matrices are discussed in Bunch [2].

An n-by-n matrix B_n is said to be circulant if it is Toeplitz and its diagonals b_j satisfy $b_{n-j} = b_{-j}$ for $0 < j \leq n-1$. Circulant matrices can always be diagonalized by a Fourier matrix, i.e.

$$B_n = F_n \Lambda_n F_n^*, \tag{1}$$

where Λ_n is diagonal and

$$[F_n]_{jk} = \frac{1}{\sqrt{n}} e^{\frac{-2\pi ijk}{n}}, \quad 0 \leq j, k < n,$$

see Davis [13]. The idea of using the preconditioned conjugate gradient method with circulant preconditioners B_n for solving positive definite Toeplitz systems was first proposed by Strang [19]. Instead of solving $A_n x = b$, we solve the preconditioned system $B_n^{-1} A_n x = B_n^{-1} b$ by the conjugate gradient method with B_n being a circulant matrix.

The number of operations per iteration in the preconditioned conjugate gradient method depends mainly on the work of computing the matrix-vector multiplication $B_n^{-1} A_n y$, see for instance Golub and van Loan [14]. For any vector y, since $B_n^{-1} y = F_n^* \Lambda_n^{-1} F_n y$, the product $B_n^{-1} y$ can be found efficiently by the Fast Fourier Transform in $O(n \log n)$ operations. Likewise, the product $A_n y$ can also be computed by the Fast Fourier Transform by first embedding A_n into a $2n$-by-$2n$ circulant matrix. The multiplication thus requires $O(2n \log(2n))$ operations. It follows that the total operations per iteration is of order $O(n \log n)$.

In order to compete with direct methods, the circulant matrix B_n should be chosen such that the conjugate gradient method converges sufficiently fast when applied to solving the preconditioned system $B_n^{-1}A_nx = B_n^{-1}b$. It is well-known that the method converges fast if $B_n^{-1}A_n$ has a clustered spectrum, i.e. $B_n^{-1}A_n$ is of the form $I_n + U_n + V_n$ where I_n is the identity matrix, U_n is a matrix of low rank and V_n is a matrix of small ℓ_2 norm.

Several circulant preconditioners have been proposed and analyzed, see for instance, Chan and Strang [3], Chan [4, 5], Chan, Jin and Yeung [8], Ku and Kuo [17], Tyrtyshinkov [21] and Huckle [16]. The convergence rate analysis of these circulant preconditioners depends on an assumption that the diagonals of the Toeplitz matrix A_n are Fourier coefficients of a given function called the generating function. One typical convergence result is that if the generating function is a positive 2π-periodic continuous *real-valued* function, then the spectrum of the preconditioned system $C_n^{-1}A_n$ is clustered around one, see Chan and Yeung [9]. Here C_n is the T. Chan [12] circulant preconditioner which is defined to be the minimizer of $||B_n - A_n||_F$ in Frobenius norm over all circulant matrices B_n. It follows that the preconditioned conjugate gradient method, when applied to solving the preconditioned system, converges superlinearly. Hence the number of iterations required for convergence is independent of the size of the matrix A_n. In particular, the system $A_nx = b$ can be solved in $O(n \log n)$ operations.

The main aim of this paper is to study the solution of Toeplitz system $A_nx = b$ for A_n generated by *complex-valued* functions. We note that such A_n are in general complex non-Hermitian matrices whereas A_n generated by real-valued functions are Hermitian Toeplitz matrices. Since A_n is not positive-definite, the conjugate gradient method in general does not converge when applied to the system $A_nx = b$. Clearly one can consider the normalized system $A_n^*A_nx = A_n^*b$, but the numerical results in §5 show that the convergence rate is usually poor.

In this paper, we consider applying the conjugate gradient method to the following normalized preconditioned system

$$(C_n^{-1}A_n)^*(C_n^{-1}A_n)x = (C_n^{-1}A_n)^*C_n^{-1}b.$$

We show that if the generating function of A_n is a 2π-periodic continuous complex-valued function without any zeros, then the spectrum of the iteration matrix $(C_n^{-1}A_n)^*(C_n^{-1}A_n)$ is clustered around one. From that we get a bound on the convergence rate of the method that depends on the condition number $\kappa(A_n)$ of A_n. More precisely, we show that if $\kappa(A_n) = O(n^\alpha)$, then the number of iterations required for convergence is at most $O(\alpha \log n)$ where $\alpha > 0$. By noting that the number of operations per iteration in the conjugate gradient method is of $O(n \log n)$, the total complexity of the algorithm is therefore of $O(n \log^2 n)$. In the case when $\alpha = 0$, i.e. A_n is well-conditioned, the method converges in $O(1)$ steps. Hence the complexity is reduced to $O(n \log n)$.

We note that symmetric positive definite Toeplitz systems can be solved in $O(n \log^2 n)$ operations by superfast direct Toeplitz solvers, see Ammar and Gragg [1] for instance. However, these methods are in general not applicable to complex non-Hermitian Toeplitz matrices. We remark that Ku and Kuo [18] have also considered solving non-symmetric Toeplitz matrix systems by preconditioned conjugate gradient method. In their paper, A_n is assumed to be generated by complex-valued rational function in the Wiener class which happens to be a sub-class of the class of 2π-periodic continuous functions considered in this paper.

Numerical examples in §5 will show that the requirements on f, namely that f has no zeros and $\kappa(A_n) = O(n^\alpha)$ are indispensible in order to get the said convergence rate. In particular, this implies that circulant preconditioners cannot be used for indefinite Toeplitz systems such as the one generated by $f(\theta) = \sin\theta$. We note however that in Chan [6] and Chan and Tang [11], we have proved that if f is nonnegative with only countable zeros, (e.g. $f(\theta) = \sin^2\theta$), then band-Toeplitz preconditioners can be used to speed up the convergence rate.

The outline of the paper is as follows. In §2, we obtain bounds for the spectra of A_n and C_n in terms of the generating function of A_n. In §3, we show that the spectrum of $(C_n^{-1}A_n)^*(C_n^{-1}A_n)$ is clustered around 1. In §4, we give the bound for the number of iterations required for convergence. Finally, numerical examples and concluding remarks are given in §5 and §6 respectively.

2 The Spectra of A_n and C_n.

For simplicity, we denote by $\mathcal{C}_{2\pi}$ the Banach space of all 2π-periodic continuous complex-valued functions equipped with the supremum norm $||\cdot||_\infty$. For all $f \in \mathcal{C}_{2\pi}$, let

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta, \ k = 0, \pm 1, \pm 2, \cdots,$$

be the Fourier coefficients of f. Let $A_n[f]$ be the n-by-n complex Toeplitz matrix with the (j,k)th entry given by a_{j-k}. The function f is called the generating function of the matrices $A_n[f]$.

We will use f_R and f_I to denote respectively the real and imaginary parts of the function f. We remark that $A_n[f_R]$ and $A_n[f_I]$ are both Hermitian matrices and

$$A_n[f] = A_n[f_R] + iA_n[f_I]. \tag{2}$$

The following Lemma gives the relation between $||f||_\infty$ and the ℓ_2 norm of $A_n[f]$.

Lemma 1 *Let $f \in \mathcal{C}_{2\pi}$. Then we have*

$$||A_n[f]||_2 \leq 2||f||_\infty, \quad n = 1, 2, \cdots. \tag{3}$$

Proof: Clearly f_R and f_I are continuous real-valued functions. Hence we have

$$||A_n(f_R)||_2 \leq ||f_R||_\infty \quad \text{and} \quad ||A_n(f_I)||_2 \leq ||f_I||_\infty, \tag{4}$$

see for instance, Grenander and Sezgö [15]. Therefore, by (2)

$$||A_n[f]||_2 \leq ||A_n(f_R)||_2 + ||A_n(f_I)||_2 \leq ||f_R||_\infty + ||f_I||_\infty \leq 2||f||_\infty. \quad \square$$

Let $C_n[f]$ be the n-by-n circulant preconditioner of $A_n[f]$ as defined in T. Chan [12], i.e. $C_n[f]$ is the minimizer of $||A_n[f] - B_n||_F$ over all circulant matrices B_n. We note that the (j,ℓ)th entry of $C_n[f]$ is given by the diagonal $c_{j-\ell}$ where

$$c_k = \begin{cases} \dfrac{(n-k)a_k + ka_{k-n}}{n} & 0 \leq k < n, \\ c_{n+k} & 0 < -k < n, \end{cases} \tag{5}$$

see Chan, Jin and Yeung [7].

We now give a simple formula for the eigenvalues $\lambda_j(C_n[f])$ of $C_n[f]$ in terms of the Fejér kernel

$$\hat{F}_k(\theta) = \frac{1}{k}\left\{\frac{\sin(\frac{k}{2}\theta)}{\sin(\frac{1}{2}\theta)}\right\}^2, \quad k = 1, 2, \cdots.$$

The following Lemma was proved in Chan and Yeung [10] for the case where f is real-valued.

Lemma 2 *Let $f \in \mathcal{C}_{2\pi}$. Then*

$$\lambda_j(C_n[f]) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\hat{F}_n(\frac{2\pi j}{n} - \phi)d\phi \equiv (f * \hat{F}_n)(\frac{2\pi j}{n}), \;\; 0 \le j < n. \quad (6)$$

Proof: By (1), it is clear that

$$\lambda_j(C_n[f]) = \lambda_j(C_n[f_R]) + i\lambda_j(C_n[f_I]), \quad 0 \le j < n.$$

Hence by noting that (6) holds for real-valued functions, we have

$$\lambda_j(C_n[f]) = \left\{(f_R + if_I) * \hat{F}_n\right\}(\frac{2\pi j}{n}) = (f * \hat{F}_n)(\frac{2\pi j}{n}), \quad 0 \le j < n. \quad \Box$$

The following Lemma gives the bounds for $||C_n[f]||_2$ and $||C_n^{-1}[f]||_2$.

Lemma 3 *Let $f \in \mathcal{C}_{2\pi}$. Then we have*

$$||C_n[f]||_2 \le 2||f||_\infty, \quad n = 1, 2, \cdots. \quad (7)$$

If moreover f has no zeros, i.e.

$$|f|_{\min} \equiv \min_{\theta \in [-\pi,\pi]} |f(\theta)| > 0,$$

then for all sufficiently large n, we also have

$$||C_n^{-1}[f]||_2 \le 2||\frac{1}{f}||_\infty. \quad (8)$$

Proof: Since $C_n[f_R]$ and $A_n[f_R]$ are Hermitian, we have

$$||C_n[f_R]||_2 \le ||A_n[f_R]||_2,$$

see for instance, Chan, Jin and Yeung [7]. Hence by (4) we have

$$||C_n[f_R]||_2 \le ||A_n[f_R]||_2 \le ||f_R||_\infty.$$

Similarly, we get

$$||C_n[f_I]||_2 \le ||A_n(f_I)||_2 \le ||f_I||_\infty.$$

It follows that

$$||C_n[f]||_2 \le ||C_n[f_R]||_2 + ||C_n[f_I]||_2 \le ||f_R||_\infty + ||f_I||_\infty \le 2||f||_\infty.$$

To get the bound for $||C_n^{-1}[f]||_2$, we note that by (6), we have

$$\begin{aligned}
\min_j |\lambda_j(C_n[f])| &= \min_j |(f * \hat{F}_n)(\frac{2\pi j}{n})| \\
&= \min_j |f(\frac{2\pi j}{n}) + (f * \hat{F}_n - f)(\frac{2\pi j}{n})| \\
&\ge |f|_{\min} - ||f * \hat{F}_n - f||_\infty, \quad 0 \le j < n.
\end{aligned}$$

Since $f * \hat{F}_n$ tends to f uniformly, see for instance Zygmund [24], we see that for n sufficiently large,

$$\min_j |\lambda_j(C_n[f])| \ge \frac{1}{2}|f|_{\min}, \tag{9}$$

or

$$\max_j |\lambda_j(C_n^{-1}[f])| \le \frac{2}{|f|_{\min}} = 2||\frac{1}{f}||_\infty.$$

By (1), we see that

$$\lambda_j(C_n^{-1*}[f]C_n^{-1}[f]) = |\lambda_j(C_n^{-1}[f])|^2, \quad 0 \le j < n. \tag{10}$$

Therefore we have

$$||C_n^{-1}[f]||_2 = \max_j |\lambda_j(C_n^{-1*}[f]C_n^{-1}[f])|^{1/2} \le 2||\frac{1}{f}||_\infty. \quad \square$$

3 The Spectrum of the Iteration Matrix.

In this section, we show that the spectrum of the normalized preconditioned matrix

$$(C_n^{-1}[f]A_n[f])^*(C_n^{-1}[f]A_n[f])$$

is clustered around 1. We first show that $A_n[f] - C_n[f]$ can be written as the sum of a low rank matrix and a small norm matrix.

Theorem 1 *Let $f \in \mathcal{C}_{2\pi}$. Then for all $\epsilon > 0$, there exist N and $M > 0$, such that for all $n > N$,*

$$A_n[f] - C_n[f] = U_n[f] + V_n[f] \tag{11}$$

where

$$\text{rank } U_n[f] \leq 2M \tag{12}$$

and

$$||V_n[f]||_2 \leq \epsilon. \tag{13}$$

Proof: Let $f \in \mathcal{C}_{2\pi}$. Then for any $\epsilon > 0$, by Weierstrass theorem, there exists a trigonometric polynomial

$$p_M(\theta) = \sum_{k=-M}^{M} \rho_k e^{ik\theta}$$

such that

$$\|f - p_M\|_\infty \leq \epsilon \ . \tag{14}$$

For all $n > 2M$, we write

$$\begin{aligned} C_n[f] - A_n[f] &= C_n[f - p_M] - A_n[f - p_M] + C_n[p_M] - A_n[p_M] \\ &= C_n[f - p_M] - A_n[f - p_M] - W_n + U_n \end{aligned} \tag{15}$$

where by (5), we see that W_n and U_n are Toeplitz matrices given by

$$\begin{bmatrix} 0 & \frac{1}{n}\rho_{-1} & \cdots & \frac{M}{n}\rho_{-M} & 0 & \cdots & 0 \\ \frac{1}{n}\rho_1 & 0 & \frac{1}{n}\rho_{-1} & \ddots & \frac{M}{n}\rho_{-M} & \ddots & \\ \vdots & \ddots & & & & \ddots & 0 \\ \frac{M}{n}\rho_M & & & & & & \frac{M}{n}\rho_{-M} \\ 0 & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \frac{M}{n}\rho_M & & \frac{1}{n}\rho_1 & 0 & \frac{1}{n}\rho_{-1} \\ 0 & \cdots & 0 & \frac{M}{n}\rho_M & \cdots & \frac{1}{n}\rho_1 & 0 \end{bmatrix} \tag{16}$$

and

$$\begin{bmatrix} 0 & \cdots & 0 & \frac{n-M}{n}\rho_M & \cdots & \frac{n-1}{n}\rho_1 \\ \vdots & \ddots & & \ddots & \ddots & \vdots \\ 0 & & & & & \frac{n-M}{n}\rho_M \\ \frac{n-M}{n}\rho_{-M} & & & & & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ \frac{n-1}{n}\rho_{-1} & \cdots & \frac{n-M}{n}\rho_{-M} & 0 & \cdots & 0 \end{bmatrix} \tag{17}$$

respectively. It is clear from (17) that

$$\text{rank } U_n \leq 2M \ . \tag{18}$$

We will show that the first three terms in the right hand side of (15) are matrices of small norm. We note that by (3), (7) and (14),

$$\begin{aligned} ||C_n[f - p_M] - A_n[f - p_M]||_2 &\leq ||C_n[f - p_M]||_2 + ||A_n[f - p_M]||_2 \\ &\leq 2||f - p_M||_\infty + 2||f - p_M||_\infty \leq 4\epsilon. \end{aligned} \tag{19}$$

It remains to estimate $||W_n||_2$. For all $|k| \leq M$, we first note that

$$\begin{aligned} |\rho_k| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} p_M(t) e^{-ikt} dt \right| \\ &\leq \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (p_M(t) - f(t)) e^{-ikt} dt \right| + \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right| \\ &\leq ||f - p_M||_\infty + ||f||_\infty \leq \epsilon + ||f||_\infty. \end{aligned}$$

Hence we see from (16) that

$$\begin{aligned} \|W_n\|_\infty &= \|W_n\|_1 \\ &= \frac{M}{n}|\rho_{-M}| + \cdots + \frac{2}{n}|\rho_{-2}| + \frac{1}{n}|\rho_{-1}| \\ &\qquad + \frac{1}{n}|\rho_1| + \frac{2}{n}|\rho_2| + \cdots + \frac{M}{n}|\rho_M| \\ &\leq \frac{2}{n}(1 + 2 + \cdots + M)(\epsilon + ||f||_\infty). \end{aligned}$$

Therefore, we have

$$\|W_n\|_2 \leq (\|W_n\|_\infty \|W_n\|_1)^{1/2} \leq \frac{1}{n} M(M+1)(\epsilon + ||f||_\infty) \ .$$

Thus if we let

$$N \equiv \max\{M(M+1)(1+\frac{||f||_\infty}{\epsilon}), 2M\} = M(M+1)(1+\frac{||f||_\infty}{\epsilon}),$$

then for all $n \geq N$, we have $\|W_n\|_2 \leq \epsilon$. Combining this estimate with (18) and (19), we see that for all $n \geq N$, $C_n[f] - A_n[f]$ is the sum of a matrix of ℓ_2 norm less than 5ϵ and a matrix of rank less than $2M$. □

We now consider the spectrum of $C_n^{-1}[f]A_n[f] - I_n$ where I_n is the n-by-n identity matrix. Using (11), (8) and the fact that

$$C_n^{-1}[f]A_n[f] - I_n = C_n^{-1}[f](A_n[f] - C_n[f]) = C_n^{-1}[f]U_n[f] + C_n^{-1}[f]V_n[f],$$

we have the following immediate Corollary.

Corollary 1 *Let $f \in \mathcal{C}_{2\pi}$. If f has no zeros, then for all $\epsilon > 0$, there exist N and $M > 0$, such that for all $n > N$,*

$$C_n^{-1}[f]A_n[f] - I_n = \tilde{U}_n[f] + \tilde{V}_n[f] \tag{20}$$

where rank $\tilde{U}_n[f] \leq 2M$ *and* $||\tilde{V}_n[f]||_2 \leq \epsilon$.

We now show that the spectrum of the normalized preconditioned matrix

$$(C_n^{-1}[f]A_n[f])^*(C_n^{-1}[f]A_n[f])$$

is clustered around 1.

Theorem 2 *Let $f \in \mathcal{C}_{2\pi}$. If f has no zeros, then for all $\epsilon > 0$, there exist N and $M > 0$, such that for all $n > N$, at most M eigenvalues of the matrix*

$$(C_n^{-1}[f]A_n[f])^*(C_n^{-1}[f]A_n[f]) - I_n$$

have absolute values larger than ϵ.

Proof: By (20), we have

$$\begin{aligned} & (C_n^{-1}[f]A_n[f])^*(C_n^{-1}[f]A_n[f]) \\ = \; & (I_n + \tilde{U}_n[f] + \tilde{V}_n[f])^*(I_n + \tilde{U}_n[f] + \tilde{V}_n[f]) \\ = \; & I_n + \hat{U}_n[f] + \hat{V}_n[f] \end{aligned}$$

where

$$\hat{U}_n[f] = \tilde{U}_n[f]^*(I_n + \tilde{U}_n[f] + \tilde{V}_n[f]) + (I_n + \tilde{V}_n[f]^*)\tilde{U}_n[f]$$

and

$$\hat{V}_n[f] = \tilde{V}_n[f] + \tilde{V}_n[f]^* + \tilde{V}_n[f]^*\tilde{V}_n[f].$$

Then by Corollary 1, we see that rank $\hat{U}_n[f] \le 4M$ and $||\hat{V}_n[f]||_2 \le 3\epsilon$. Since now we have

$$(C_n^{-1}[f]A_n[f])^*(C_n^{-1}[f]A_n[f]) - I_n = \hat{U}_n[f] + \hat{V}_n[f]$$

and both $\hat{U}_n[f]$ and $\hat{V}_n[f]$ are Hermitian, by applying Cauchy's interlace theorem, see Wilkinson [23], we conclude that at most $4M$ eigenvalues of the matrix

$$(C_n^{-1}[f]A_n[f])^*(C_n^{-1}[f]A_n[f]) - I_n$$

have absolute values larger than 3ϵ. □

4 Convergence Rate.

In this section, we analyze the convergence rate of the conjugate gradient method when applied to solving the normalized preconditioned system

$$(C_n^{-1}[f]A_n[f])^*(C_n^{-1}[f]A_n[f])x = (C_n^{-1}[f]A_n[f])^*C_n^{-1}[f]b. \tag{21}$$

We show that the method converges in at most $O(\alpha \log n + 1)$ steps where $O(n^\alpha)$ is the condition number $\kappa(A_n[f])$ of $A_n[f]$. We begin by deriving a lower bound for the singular values of $C_n^{-1}[f]A_n[f]$.

Lemma 4 *Let $f \in \mathcal{C}_{2\pi}$. If f has no zeros, then there exists a constant $\tilde{c} > 0$ such that for n sufficiently large, we have*

$$||A_n[f]||_2 > \tilde{c}.$$

Hence we have

$$||A_n^{-1}[f]C_n[f]||_2 \le \frac{||C_n[f]||_2}{||A_n[f]||_2}\kappa(A_n[f]) \le c \cdot \kappa(A_n[f]) \tag{22}$$

for some constant $c > 0$.

Proof: By (11), we have

$$A_n^*[f]A_n[f] = C_n^*[f]C_n[f] + X_n + Y_n, \tag{23}$$

where

$$X_n = U_n[f]^*(C_n[f] + U_n[f] + V_n[f]) + (C_n^*[f] + V_n^*[f])U_n[f]$$

and

$$Y_n = C_n^*[f]V_n[f] + V_n^*[f]C_n[f] + V_n^*[f]V_n[f].$$

Let us analyze each term in the right hand side of (23). By (12), we see that rank $X_n \leq 4M$. By (13) and (7), we get

$$||Y_n||_2 \leq 2||C_n[f]||_2||V_n[f]||_2 + ||V_n[f]||_2^2 \leq 4\epsilon||f||_\infty + \epsilon^2.$$

Hence for ϵ small enough, we have

$$||Y_n||_2 \leq \frac{1}{8}|f|_{\min}^2.$$

Finally by (9) and (10),

$$\lambda_j(C_n^*[f]C_n[f]) \geq \frac{1}{4}|f|_{\min}^2, \quad 0 \leq j < n.$$

Thus by applying Cauchy's interlace theorem to (23), we conclude that at most $4M$ eigenvalues of $A_n^*[f]A_n[f]$ have values less than

$$\frac{1}{4}|f|_{\min}^2 - \frac{1}{8}|f|_{\min}^2 = \frac{1}{8}|f|_{\min}^2.$$

Therefore

$$||A_n[f]||_2^2 = \lambda_{\max}(A_n^*[f]A_n[f]) \geq \frac{1}{8}|f|_{\min}^2,$$

for $n > 4M$. Equation (22) now follows directly from (7) and the fact that $\kappa(A_n) = ||A_n||_2||A_n^{-1}||_2$. □

To obtain the number of iterations for convergence, we need the following Lemma by van der Vorst [22].

Lemma 5 *Let x be the solution to $G^*Gx = G^*b$ and x_j be the jth iterant of the ordinary conjugate gradient method applied to this normal equation. If the eigenvalues $\{\delta_k\}$ of G^*G are such that*

$$0 < \delta_1 \le \dots \le \delta_p \le b_1 \le \delta_{p+1} \le \dots \le \delta_{n-q} \le b_2 \le \delta_{n-q+1} \le \dots \le \delta_n,$$

then

$$\frac{||G(x-x_j)||_2}{||G(x-x_0)||_2} \le 2\left(\frac{b-1}{b+1}\right)^{j-p-q} \cdot \max_{\delta\in[b_1,b_2]} \left\{ \prod_{k=1}^{p} \left(\frac{\delta-\delta_k}{\delta_k}\right) \prod_{k=n-q+1}^{n} \left(\frac{\delta_k-\delta}{\delta_k}\right) \right\}. \tag{24}$$

Here

$$b \equiv \left(\frac{b_2}{b_1}\right)^{\frac{1}{2}} \ge 1.$$

We remark that equation (24) can be derived from the following standard error estimate of the conjugate gradient method:

$$\frac{||G(x-x_j)||_2}{||G(x-x_0)||_2} \le \min_{P_j} \max_{k=1,\cdots,n} |P_j(\delta_k)|,$$

see Golub and van Loan [14]. Here P_j is any jth degree polynomial with constant term 1. By passing linear polynomials through the outlying eigenvalues δ_k, $1 \le k \le p$ and $n-q+1 \le k \le n$, and using a $(j-p-q)$th degree Chebyshev polynomial to minimize the error in the interval $[\delta_{p+1}, \delta_{n-q}]$ we get (24).

Notice that for $\delta \in [b_1, b_2]$, we always have

$$0 \le \frac{\delta_k - \delta}{\delta_k} \le 1, \quad n-q+1 \le k \le n.$$

Thus (24) can be simplified to

$$\frac{||G(x-x_j)||_2}{||G(x-x_0)||_2} \le 2\left(\frac{b-1}{b+1}\right)^{j-p-q} \cdot \max_{\delta\in[b_1,b_2]} \prod_{k=1}^{p} \left(\frac{\delta-\delta_k}{\delta_k}\right). \tag{25}$$

In our case, we have $G = C_n^{-1}[f]A_n[f]$. By Theorem 2, we can choose $b_1 = 1-\epsilon$ and $b_2 = 1+\epsilon$. Then p and q are constants that depend only on ϵ but not on n. By choosing $\epsilon < 1$, we have

$$\frac{b-1}{b+1} = \frac{1-\sqrt{1-\epsilon^2}}{\epsilon} < \epsilon.$$

In order to use (25), we need a lower bound for δ_k for $1 \le k \le p$. By (22), we see that for n sufficiently large,

$$||G^{-1}||_2 = ||A_n^{-1}[f]C_n[f]||_2 \le c\kappa(A_n[f]) \le cn^{\alpha},$$

for some constant c that does not depend on n. Hence

$$\delta_k \ge \min_{\ell} \delta_\ell = \frac{1}{||G^{-1}||_2^2} \ge cn^{-2\alpha}, \quad 1 \le k \le n.$$

Thus for $1 \le k \le p$ and $\delta \in [1-\epsilon, 1+\epsilon]$, we have,

$$0 \le \frac{\delta - \delta_k}{\delta_k} \le cn^{2\alpha}.$$

Hence (25) becomes

$$\frac{||G(x-x_j)||_2}{||G(x-x_0)||_2} < c^p n^{2p\alpha} \epsilon^{j-p-q}.$$

Therefore given arbitrary tolerance $\tau > 0$, an upper bound for the number of iterations required to make

$$\frac{||G(x-x_j)||_2}{||G(x-x_0)||_2} < \tau$$

is given by

$$j_0 \equiv p + q - \frac{p \log c + 2\alpha p \log n - \log \tau}{\log \epsilon} = O(\alpha \log n + 1).$$

Since by using FFT, the matrix-vector product

$$(C_n^{-1}[f]A_n[f])^*(C_n^{-1}[f]A_n[f])v$$

can be done in $O(n \log n)$ operations for any vector v, the cost per iteration of the conjugate gradient method is also of $O(n \log n)$. Thus we conclude that the work of solving (21) to a given accuracy τ is $O(n \log^2 n)$ when $\alpha > 0$.

When $\alpha = 0$, i.e. $\kappa(A_n[f]) = O(1)$, the number of iterations required for convergence is of $O(1)$. Hence the complexity of the algorithm reduces to $O(n \log n)$. We remark that in this case, one can show further that the method converges superlinearly for the normalized preconditioned system due to the clustering of the singular values, see Chan and Strang [3] or Chan [5] for details. In contrast, the method converges just linearly for the normalized system $A_n^*[f]A_n[f]x = A_n^*[f]b$.

5 Numerical Results.

In this section, we test the convergence rate of the normalized preconditioned systems with generating functions in $\mathcal{C}_{2\pi}$. Six different generating functions were tested. They are

(a) $a_j = (|j|+1)^{-1.1} + i(|j|+1)^{-1.1}, \quad j = 0, \pm 1, \pm 2, \cdots,$

(b) $a_j = \begin{cases} (|j|+1)^{-1.1} & j \geq 0, \\ i(|j|+1)^{-1.1} & j < 0, \end{cases}$

(c) $a_j = \begin{cases} (|j|+1)^{-1.1} + i(|j|+1)^{-1.1} & j \neq 0, \\ 0 & j = 0, \end{cases}$

(d) $a_j = \begin{cases} (|j|+1)^{-1.1} & j > 0, \\ 0 & j = 0, \\ i(|j|+1)^{-1.1} & j < 0, \end{cases}$

(e) $a_j = \begin{cases} 2 & j = 0, \\ -1 & |j| = 1, \\ 0 & |j| > 1, \end{cases}$

(f) $a_j = \begin{cases} \frac{1}{5}\pi^4 & j = 0, \\ 4(-1)^j(\frac{\pi^2}{j^2} - \frac{6}{j^4}) & |j| > 0. \end{cases}$

Since the sequences a_j are absolutely summable, it follows that the corresponding generating functions are continuous. Tables 1-3 show the number of iterations required to solve the systems

$$A_n[f]^* A_n[f] x = A_n^*[f] b$$

and

$$(C_n^{-1}[f]A_n[f])^*(C_n^{-1}[f]A_n[f])x = (C_n^{-1}[f]A_n[f])^* C_n^{-1}[f]b.$$

The stopping criterion we used is $||r_q||_2/||r_0||_2 < 10^{-7}$, where r_q is the residual vector after q iterations. The right hand side b is the vector of all ones and the zero vector is our initial guess. The computations are done by using 8-byte arithmetic on a Vax 6420.

We see that for the normalized preconditioned systems, the number of iterations required for convergence indeed depends on the condition number of A_n. If A_n is well-conditioned, as is in the cases (a) and (b), then the number of iterations remains constant when n increases. Therefore the total complexity of the algorithm is $O(n \log n)$ in these cases. However, if A_n is not well-conditioned, as is in the cases (c) and (d), we see that the number of iterations does increase with n.

Sequences (e) and (f) are the Fourier coefficients of functions $f(\theta) = 4\sin^2\theta$ and $f(\theta) = \theta^4$ respectively and they both have a zero in $[-\pi, \pi]$. In case (e), the matrix A_n is the 1-dimensional discrete Laplacian and is known to have $\kappa(A_n) = O(n^2)$. In case (f), $\kappa(A_n) = O(n^4)$, see Chan [6]. For case (e), the normalized preconditioned system still converges in an $O(\log n)$ fashion while for case (f), the number of iterations increases faster than $O(n)$. Thus the convergence rate of our method does depend on whether f has a zero or not.

As for the time comparison, we report that in case (a) with $n = 1024$, it requires about 32.57 seconds to solve the original normalized system and about 6.05 seconds to solve the normalized preconditioned systems. For case (c) with $n = 1024$ again, it requires about $1,149.57$ seconds to solve the original normalized system and about 13.75 seconds to solve the normalized preconditioned systems. Thus there is about five to eighty times saving in speed when preconditioning is employed.

In Figures 1 and 2, we depict the spectra of the iteration matrices in cases (b) and (d) with $n = 64$. In the figures, the eigenvalues of the matrices are ordered as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

We note that the spectra of the normalized preconditioned matrices indeed are clustered around 1.

n	(a)		(b)	
	$A_n^*A_n$	$(C_n^{-1}A_n)^*(C_n^{-1}A_n)$	$A_n^*A_n$	$(C_n^{-1}A_n)^*(C_n^{-1}A_n)$
16	9	5	11	7
32	15	5	16	8
64	22	5	22	9
128	31	5	29	9
256	41	6	35	9
512	53	6	40	9
1024	62	6	45	9

Table 1. Number of Iterations for Different Generating Functions

n	(c)		(d)	
	$A_n^*A_n$	$(C_n^{-1}A_n)^*(C_n^{-1}A_n)$	$A_n^*A_n$	$(C_n^{-1}A_n)^*(C_n^{-1}A_n)$
16	9	9	18	15
32	20	10	41	18
64	45	13	101	19
128	115	12	266	19
256	318	14	715	24
512	857	13	1853	26
1024	2280	17	4665	25

Table 2. Number of Iterations for Different Generating Functions

n	(e)		(f)	
	$A_n^*A_n$	$(C_n^{-1}A_n)^*(C_n^{-1}A_n)$	$A_n^*A_n$	$(C_n^{-1}A_n)^*(C_n^{-1}A_n)$
16	8	9	20	9
32	22	11	101	21
64	74	14	934	63
128	238	18	> 5000	191
256	850	24	> 5000	739
512	3264	32	> 5000	1904

Table 3. Number of Iterations for Different Generating Functions

6 Concluding Remarks

In this paper, we have considered solution of complex Toeplitz systems $A_n x = b$ where A_n is generated by 2π-periodic complex-valued continuous function. The system is solved by conjugate gradient method applied to the preconditioned system

$$(C_n^{-1}A_n)^*(C_n^{-1}A_n)x = (C_n^{-1}A_n)^*C_n^{-1}b,$$

where C_n is the T. Chan circulant preconditioner. We show that if (i) f has no zeros and (ii) $\kappa(A_n) = O(n^\alpha)$, then the number of iterations required for convergence is at most $O(\alpha \log n + 1)$. Hence the total complexity of the algorithm is of $O(\alpha n \log^2 n + n \log n)$.

We emphasize that from the examples given in §5, we cannot remove neither condition (i) nor (ii) on f in order that the method still converges in $O(\alpha \log n + 1)$ steps. We further remark that these two conditions are mutually exclusive. In fact, if $f(\theta) = e^{i\theta}$, then f has no zeros but $A_n[f]$ is singular for all n. On the other hand, if $f(\theta) = 4\sin^2\theta$, then $A_n[f]$ is the 1-dimensional discrete Laplacian with $\kappa(A_n[f]) = O(n^2)$.

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