Prospect and Markowitz Stochastic Dominance

Wing-Keung Wong*

Deputy Director

Risk Management Institute and Department of Economics

National University of Singapore

and

Raymond H. Chan

Associate Dean of Science

Department of Mathematics

The Chinese University of Hong Kong

Acknowledgement The authors would like to thank Professors Haim Levy, George Wu, Howard E. Thompson, Dietrich K. Fausten, Yew-Kwang Ng and Petko Kalev and the colleagues in University of Wisconsin-Madison, The University of Western Australia, Monash University, The Chinese University of Hong Kong, Hong Kong Baptist University and the University of Technology, Sydney for their valuable comments. The first author would also like to thank Professors Robert B. Miller and Howard E. Thompson for their continuous guidance and encouragement. The research is partially supported by the grants from the Chinese Unversity of Hong Kong and National University of Singapore.

Correspondence: Wing-Keung Wong, Department of Economics, National University of Singapore, 1 Arts Link Singapore 117570, Tel: (65) 6874-6014, Fax: (65) 6775-2646, Email: ecswwk@nus.edu.sg

* This paper was done when I were visiting Graduate School of Business, University of Wisconsin-Madison

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Abstract

Levy and Wiener (1998), Levy and Levy (2002, 2004) develop the Prospect and Markowitz Stochastic Dominance theory with S-shaped and reverse S-shaped utility functions for investors. In this paper, we extend their work on Prospect Stochastic Dominance theory (PSD) and Markowitz Stochastic Dominance theory (MSD) to the first three orders and link the corresponding S-shaped and reverse S-shaped utility functions to the first three orders. We also provide experiments to illustrate each case of the MSD and PSD to the first three orders and demonstrate that the higher order MSD and PSD cannot be replaced by the lower order MSD and PSD. Furthermore, we formulate the following PSD and MSD properties: hierarchy exists in both PSD and MSD relationships; arbitrage opportunities exist in the first orders of both PSD and MSD; and for any two prospects under certain conditions, their third order MSD preference will be 'the opposite of' or 'the same as' their counterpart third order PSD preference. By extending the work of Levy and Wiener and Levy and Levy, we provide investors with more tools to identify the first and third order PSD and MSD prospects and thus they could make wiser choices on their investment decision.

Keywords: Prospect stochastic dominance, Markowitz stochastic dominance, risk seeking, risk averse, S-shaped utility function, reverse S-shaped utility function

JEL Classification: D81, C91

1 Introduction

According to the von Neuman and Morgenstern (NM, 1944) expected utility theory, the functions for risk averters and risk seekers are concave and convex respectively, and both are increasing functions. Comparing the utility functions and the stochastic dominance (SD) theory has become an issue of great interest among academics. Linking the SD theory to the selection rules for risk averters under different restrictions on the utility functions include Quirk and Saposnik (1962) and Fishburn (1964). Linking the SD theory to the selection rules for risk seekers include Hammond (1974) and Stoyan (1983).

Examining the relative attractiveness of various forms of investments, Friedman and Savage (1948) claim that the strictly concave functions may not be able to explain the behavior why investors buy insurance or lottery tickets. Markowitz (1952), the first to address Friedman and Savage's concern, proposes a utility function which has convex and concave regions in both the positive and the negative domains.[1] To support Markowitz's proposed utility function, Williams (1966) reports data where a translation of outcomes produces a dramatic shift from risk aversion to risk seeking while Fishburn and Kochenberger (1979) document the prevalence of risk seeking in choices between negative prospects. Kahneman and Tversky (1979) and Tversky and Kahneman (1992) claim that the utility function is concave for gains and convex for losses, yielding an S-shaped function. They also develop a formal theory of loss aversion called prospect theory in which investors can maximize the expectation of the S-shaped utility function. It is one of the most popular decision-making theories about risk-taking and has gained much attention from economists and professionals in the financial sector.

Thereafter, a stream of papers[2] building economic or financial models on the prospect theory has been written. There have also been many empirical and experimental attempts

[1]Ng (1965) and Machina (1982) also provide other explanations to Friedman and Savage's paradox.
[2]see, for example, Shefrin and Statman (1993) and Wang and Fischbeck (2004).

to test the prospect theory, for example, the equity premium puzzle by Benartzi and Thaler (1995) and the buying strategies of hog farmers by Pennings and Smidts (2003). Most of these studies support the prospect theory. The prospect theory has also been widely applied in Economics and Finance, see for example, Myagkov and Plott (1997).

Noticing the presence of risk seeking in preferences among positive as well as negative prospects, Markowitz (1952) also proposes another type of utility functions different from the pure S-shaped utility functions used in the prospect theory. He suggests a utility which is first concave, then convex, then concave, and finally convex to explain Friedman and Savage's question about why investors buy insurance and buy lotteries tickets. Using sequential gambles technique, Thaler and Johnson (1990) obtain experimental evidence to show that prior outcomes affect subsequent behavior in a way that is contrary to the static version of the prospect theory. In particular, subjects are more risk seeking following gains and more risk averse following losses. This implies that in a dynamic context, a reverse S-shaped utility function may be more descriptive of actual behavior. Levy and Wiener (1998) further develop the theory for the reverse S-shaped utility functions for investors. Levy and Levy (2002) are the first to extend the work of Markowitz (1952) and others to develop a new criterion called Markowitz Stochastic Dominance (MSD) to determine the dominance of one investment alternative over another for all reverse S-shaped functions, and another criterion called Prospect Stochastic Dominance (PSD) to determine the dominance of one investment alternative over another for all S-shaped utility functions.

Working along similar lines as Whitmore (1970) who extends the second order SD developed by Quirk and Saposnik (1962) and others to the third order SD for risk averters, in this paper, we first extend the work of Levy and Levy to take the PSD and MSD to the first three orders SD and link the corresponding S-shaped and reverse S-shaped utility functions to the first three orders.

Another contribution of Levy and Levy to the literature is to prove the second order PSD and MSD satisfy the expected utility paradigm. Arrow (1971) first points out that an individual with unbounded utility must violate either the completeness or the continuity axiom of the expected utility theory while Machina (1982) suggests that the expected utility analysis is too theoretical and may not be empirically valid. Swalm (1966) and Barberis, Huang, and Santos (2001) mount a critique of expected utility theory. Rabin (2000) also points out that the expected utility cannot explain loss aversion which accounts for the modest-scale risk aversion for both large and small stakes typically observed in empirical studies. To circumvent this problem, Kahneman and Tversky (1979) suggest employing the certainty equivalent approach to study the negative and positive domains separately. Nonetheless, the PSD and MSD developed in Levy and Wiener (1998) and Levy and Levy (2002, 2004) bypass the above problems. Moreover, they show that both MSD and PSD satisfy the expected utility paradigm. Following Levy and Levy, another contribution of this article is to examine the compatibility of both the extended MSD and PSD with the expected utility theory and proves that both MSD and PSD of any order are consistent with the expected utility theory.

In addition, we provide experiments to illustrate each case of the MSD and PSD to the first three orders and demonstrate that the higher order MSD and PSD cannot be replaced by the lower order MSD and PSD. We also develop some other properties for the extended MSD and PSD as follows: hierarchy exists in both PSD and MSD; arbitrage opportunities exist for the first orders of both PSD and MSD; and for any two prospects under certain conditions, their third order MSD preference will be 'the opposite of' or 'the same as' their third order counterpart PSD preference. In terms of empirical analysis, our approach is superior to Levy and Levy's as the definitions of the extended PSD and MSD developed in our paper enable investors to identify the MSD and PSD prospects to the first three orders. With more information, investors can make wiser decisions with their investments. For example, when an investor has identified the first order PSD and MSD prospects, the arbitrage opportunities are revealed. In addition, by identifying the third order PSD and MSD prospects, an investor can make wiser choices about these prospects. However, Levy and Levy's approach only allows investors to identity the MSD and PSD to the second order. Without the extended PSD and MSD definitions, Levy and Levy's investors would not have as much information as ours to make their investment decisions.

The paper is organized as follows. We begin by introducing definitions and notations in the next section. Section 3 develops several properties for the extended MSD and PSD. Section 4 provides illustrations for MSD and PSD to the first three orders and demonstrates that the higher order MSD and PSD cannot be replaced by the lower order MSD and PSD. Section 5 concludes our findings.

2 Definitions and Notations

Let $\overline{\mathbb{R}}$ be the set of extended real numbers and $\Omega = [a, b]$ be a subset of $\overline{\mathbb{R}}$ in which $a < 0$ and $b > 0$. Let $\mathbb{B}$ be the Borel σ-field of Ω and μ be a *measure* on $(\Omega, \mathbb{B})$. We first define the functions F and F^D of the measure μ on the support Ω as

$$F(x) = \mu[a, x] \quad \text{and} \quad F^D(x) = \mu[x, b] \qquad \text{for all} \quad x \in \Omega\,. \tag{1}$$

Function F is a *probability distribution function*[3] or simply *distribution function* (DF) and μ is a *probability measure* if $\mu(\Omega) = 1$. We follow the basic probability theory that for any random variable X and for any probability measure P, there exists a unique induced probability measure μ on $(\Omega, \mathbb{B})$ and a DF F such that F satisfies (1) and

$$\mu(B) = P(X^{-1}(B)) = P(X \in B) \qquad \text{for any} \quad B \in \mathbb{B}\,.$$

[3]In this paper, the definition of F is slightly different from the 'traditional' definition of a distribution function.

An integral written in the form of $\int_A f(t)\, d\,\mu(t)$ or $\int_A f(t)\, d\,F(t)$ is a Lebesgue integral for any integrable function $f(t)$. If the integral has the same value for any set A which is equal to $(c, d]$, $[c, d)$ or $[c, d]$, then we use the notation $\int_c^d f(t)\, d\,\mu(t)$ instead. In addition, if μ is a Borel measure with $\mu(c, d] = d - c$, then we write the integral as $\int_c^d f(t)\, dt$.

Random variables, denoted by X and Y defined on Ω are considered together with their corresponding DFs F and G and their corresponding probability density functions (pdfs) f and g respectively. The following notations will be used throughout this paper:

$$\begin{aligned} \mu_F = \mu_X = E(X) = \int_a^b x\, d\,F(x), &\qquad \mu_G = \mu_Y = E(Y) = \int_a^b x\, d\,G(x)\,; \\ f(x) = F_0^A(x) = F_0^D(x), &\qquad g(x) = G_0^A(x) = G_0^D(x) \\ H_n^A(x) = \int_a^x H_{n-1}^A(y)\, dy\,, &\qquad H_n^D(x) = \int_x^b H_{n-1}^D(y)\, dy \quad n = 1, 2, 3; \end{aligned} \tag{2}$$

where $H = F$ or G.[4] In (2), $\mu_F = \mu_X$ is the mean of X whereas $\mu_G = \mu_Y$ is the mean of Y. $f = F_0^A = F_0^D$ is the pdf of X and $g = G_0^A = G_0^D$ is the pdf of Y. [5]

The definition of H_n^A in (2) can be used to develop the SD theory for risk averters (see, for example, Quirk and Saposnik 1962 and Fishburn 1964). We call this type of SD Ascending Stochastic Dominance (ASD) and H_n^A the n^{th} order ASD integral or the n^{th} order cumulative probability as H_n^A is integrated from H_{n-1}^A in ascending order from the leftmost point of downside risk. On the other hand, H_n^D can be used to develop the SD theory for risk seekers (see, for example, Meyer 1977 and Stoyan 1983). We call this type of SD Descending Stochastic Dominance (DSD) and H_n^D the n^{th} order DSD integral or the n^{th} order reversed cumulative probability as H_n^D is integrated H_{n-1}^D in descending order from the rightmost point of upside profit. Typically, risk averters prefer assets that

[4]The above definitions have been commonly used in the literature, see for example, Wong and Li (1999), Li and Wong (1999) and Anderson (2004).

[5]The notations F_0^A, F_0^D, G_0^A and G_0^D are introduced for convenience purposes. For example, without introducing the notation F_0^A, the statement $F_n^A(x) = \int_a^x F_{n-1}^A(y)\, dy$ in (2) does not hold for $n = 1$.

have a smaller probability of losing, especially in downside risk; while risk seekers prefer assets that have a higher probability of gaining, especially in upside profit. To make a choice between two assets F or G, risker averters will compare their corresponding n^{th} order ASD integrals F_n^A and G_n^A and choose F if F_n^A is smaller since it has a smaller probability of losing. On the other hand, risk seekers will compare their corresponding n^{th} order DSD integrals F_n^D and G_n^D and choose F if F_n^D is bigger since it has a higher probability of gaining.[6]

All functions are assumed to be measurable and all random variables are assumed to satisfy:[7]

$$F_1^A(a) = 0 \quad \text{and} \quad F_1^D(b) = 0. \tag{3}$$

For $H = F$ or G, we define the following functions for MSD and PSD:

$$\begin{aligned} H_1^a(x) &= H(x) = H_1^A(x), \quad H_1^d(x) = 1 - H(x) = H_1^D(x); \\ H_n^d(y) &= \int_y^0 H_{n-1}^d(t)dt, \quad y \le 0; \quad \text{and} \\ H_n^a(x) &= \int_0^x H_{n-1}^a(t)dt, \quad x \ge 0 \quad \text{for } n = 2, 3\ . \end{aligned} \tag{4}$$

In order to make the computation easier, we further define

$$H_n^M(x) = \begin{cases} H_n^A(x) & x \le 0 \\ H_n^D(x) & x > 0 \end{cases} \quad \text{and} \quad H_n^P(x) = \begin{cases} H_n^d(x) & x \le 0 \\ H_n^a(x) & x > 0\ ; \end{cases} \tag{5}$$

where $H = F$ and G and $n = 1, 2$ and 3.

As pointed out by Markowitz (1952) and many others, investors' behaviors can be different in the positive and negative domains of the return. Without loss of generality, in

[6]see Li and Wong (1999) and Anderson (2004) for more discussion.

[7]Condition (3) will hold for any random variable except a random variable with positive probability at the points of negative infinity or positive infinity.

this paper, 'upside profit' refers to the positive domain of the return and 'downside risk' the negative domain of return. We first consider the function H_n^M which is equal to H_n^A in downside risk and equal to H_n^D in upside profit. By comparing the F_n^M and G_n^M of the two assets F and G, we study whether we could choose an asset which shows a smaller probability in downside risk and a bigger probability in upside profit. Once we find F such that it has a smaller ASD integral in downside risk and a higher DSD integral in upside profit, one may believe that F has the best of both worlds – a smaller probability of losing in downside risk and a larger probability to gain in upside profit. On the other hand, in this paper we also study the properties of the function H_n^P which is equal to ASD integral (H_n^a) in upside profit and equal to the DSD integral (H_n^d) in downside risk. As shown in the next section, our paper shows that H_n^M can be used to develop the MSD theory while H_n^P can be used to develop the PSD theory.[8] The following are our definitions:

Definition 1 *Given two random variables X and Y with F and G as their respective DFs, X weakly dominates Y and F weakly dominates G in the sense of:*

a. *FMSD, denoted by $X \succeq_1^M Y$ or $F \succeq_1^M G$, if and only if $F_1^M(-x) \leq G_1^M(-x)$ and $F_1^M(x) \geq G_1^M(x)$ for each $x \geq 0$;*

b. *SMSD, denoted by $X \succeq_2^M Y$ or $F \succeq_2^M G$, if and only if $F_2^M(-x) \leq G_2^M(-x)$ and $F_2^M(x) \geq G_2^M(x)$ for each $x \geq 0$;*

c. *TMSD, denoted by $X \succeq_3^M Y$ or $F \succeq_3^M G$, if and only if $F_3^M(-x) \leq G_3^M(-x)$ and $F_3^M(x) \geq G_3^M(x)$ for each $x \geq 0$;*

where FMSD, SMSD, and TMSD stand for the first, second and third order MSD respectively. If, in addition, there exists an x in $[a,b]$ such that $F_n^M(x) < G_n^M(x)$ with $x < 0$ or $F_n^M(x) > G_n^M(x)$ with $x > 0$ for $n = 1, 2$ and 3, we say that X dominates Y and

[8]Thus, we call the function H_n^M the n^{th} order MSD integral and call the function H_n^P the n^{th} order PSD integral.

F dominates G in the sense of SFMSD, SSMSD, and STMSD, denoted by $X \succ_1^M Y$ *or* $F \succ_1^M G, X \succ_2^M Y$ *or* $F \succ_2^M G$*, and* $X \succ_3^M Y$ *or* $F \succ_3^M G$ *respectively, where SFMSD, SSMSD, and STMSD stand for strictly first, second and third order MSD respectively.*

Definition 2 *Given two random variables X and Y with F and G as their respective DFs, X weakly dominates Y and F weakly dominates G in the sense of:*

a. *FPSD, denoted by* $X \succeq_1^P Y$ *or* $F \succeq_1^P G$*, if and only if* $F_1^P(-x) \geq G_1^P(-x)$ *and* $F_1^P(x) \leq G_1^P(x)$ *for each* $x \geq 0$*;*

b. *SPSD, denoted by* $X \succeq_2^P Y$ *or* $F \succeq_2^P G$*, if, and only if,* $F_2^P(-x) \geq G_2^P(-x)$ *and* $F_2^P(x) \leq G_2^P(x)$ *for each* $x \geq 0$*;*

c. *TPSD, denoted by* $X \succeq_3^P Y$ *or* $F \succeq_3^P G$*, if and only if* $F_3^P(-x) \geq G_3^P(-x)$ *and* $F_3^P(x) \leq G_3^P(x)$ *for each* $x \geq 0$*;*

where FPSD, SPSD, and TPSD stand for the first, second and third order PSD respectively. If, in addition, there exists an x in $[a, b]$ *such that* $F_n^P(x) > G_n^P(x)$ *with* $x < 0$ *or* $F_n^P(x) < G_n^P(x)$ *with* $x > 0$ *for* $n = 1, 2$ *and 3, we say that X dominates Y and F dominates G in the sense of SFPSD, SSPSD, and STPSD, denoted by* $X \succ_1^P Y$ *or* $F \succ_1^P G, X \succ_2^P Y$ *or* $F \succ_2^P G$*, and* $X \succ_3^P Y$ *or* $F \succ_3^P G$ *respectively, where SFPSD, SSPSD, and STPSD stand for strictly first, second and third order PSD respectively.*

Levy and Levy (2002) define the MSD and PSD functions as:

$$H^M(x) = \begin{cases} \int_a^x H(t)\,dt & x < 0 \\ \int_x^b H(t)\,dt & x > 0 \end{cases} \quad \text{and} \quad H^P(x) = \begin{cases} \int_x^0 H(t)\,dt & x < 0 \\ \int_0^x H(t)\,dt & x > 0 \end{cases} \tag{6}$$

where $H = F$ and G. MSD and PSD are expressed in the following definition:

Definition 3

$F \succeq_{MSD} G$ *if* $F^M(x) \leq G^M(x)$ *for all x, and* $F \succeq_{PSD} G$ *if* $F^P(x) \leq G^P(x)$ *for all x.*

One can easily show that $F \succeq_{MSD} G$ if and only if $F \succeq_2^M G$ and $F \succeq_{PSD} G$ if and only if $F \succeq_2^P G$. Hence, the MSD and PSD defined in Levy and Levy are the same as the second order MSD and PSD defined in our paper.[9]

Definition 4 *$n = 1, 2, 3, U_n^A, U_n^{SA}, U_n^D$ and U_n^{SD} are the sets of the utility functions*[10] *u such that:*

$$\begin{aligned}
U_n^A(U_n^{SA}) &= \{u : (-1)^{i+1} u^{(i)} \geq (>)\, 0\,,\ i = 1, \cdots, n\}\,; \\
U_n^D(U_n^{SD}) &= \{u : u^{(i)} \geq (>)\, 0\,,\ i = 1, \cdots, n\}\,; \\
U_n^S(U_n^{SS}) &= \{u : u^+ \in U_n^A(U_n^{SA}) \quad and \quad u^- \in U_n^D(U_n^{SD})\ \}\,; \\
U_n^R(U_n^{SR}) &= \{u : u^+ \in U_n^D(U_n^{SD}) \quad and \quad u^- \in U_n^A(U_n^{SA})\ \}\,.
\end{aligned}$$

where $u^{(i)}$ is the i^{th} derivative of the utility function u, $u^+ = u$ restricted for $x \geq 0$ and $u^- = u$ restricted for $x \leq 0$.

It is noted that investors in U_n^A is risk averse while investors in U_n^D is risk seeking. Investors in U_n^R with reversed S-shaped utility functions are risk seeking for gains but risk aversion for losses while investors in U_n^S with S-shaped utility functions are risk averse for gains but risk seeking for losses. Refer to Figure 1 for the shape of utility functions in U_2^A, U_2^D, U_2^R and U_2^S and refer to Figure 2 for the shape of the first derivatives of the utility functions in U_3^A, U_3^D, U_3^R and U_3^S respectively.

One choosing between $F\ (X)$ and $G\ (Y)$ in accordance with a consistent set of preferences will satisfy the NM consistency properties. Accordingly, $F\ (X)$ is (strictly) preferred to $G\ (Y)$ if

$$\Delta Eu \equiv u(F) - u(G) \equiv u(X) - u(Y) \geq 0 (> 0), \tag{7}$$

[9] We note that Levy and Wiener (1998) and Levy and Levy (2004) define PSD as $F \succeq_{PSD} G$ if and only if $0 \leq \int_{x_1}^{x_2} [G(z) - F(z)]\, dz$ for all $x_1 \leq 0 \leq x_2$ with at least one strict inequality.

[10] We note that the theory can be easily extended to satisfy utilities defined to be non-differentiable. In this paper, we will skip the discussion of non-differentiable utilities.

where $u(F) \equiv u(X) \equiv \int_a^b u(x)dF(x)$ and $u(G) \equiv u(Y) \equiv \int_a^b u(x)dG(x)$.

There is an ongoing debate in the literature regarding the shape of the utility functions. The utility functions U_2^A advocated in the literature depict the concavity of the utility function, which is equivalent to risk aversion, according to the notion of decreasing marginal utility. The prevalence of risk aversion is the best known generalization regarding risky choices and was popular among the early decision theorists of the twentieth century (Pratt 1964, Arrow 1971).

On the other hand, Markowitz (1952) proposes a utility function which is first concave, then convex, then concave, and finally convex. The portion of this utility function that has convex and concave regions in the negative and the positive domains respectively is equivalent to U_2^S defined in our paper and forms a S-shaped utility function. Later, Kahneman and Tversky (1979) and Tversky and Kahneman (1992) formally develop the prospect theory to link up the S-shaped utility functions. Similarly, the portion that has concave and convex regions in the negative and the positive domains respectively is equivalent to U_2^R defined in our paper and forms a reverse S-shaped utility function.

Whitmore (1970) extends the second order SD to the third order SD and improves the linkage of SD to the utility functions for risk averse investors up to U_3^A. In this paper, we extend PSD and MSD to the first three orders and improve the linkage of PSD and MSD to the utility functions up to U_3^S and U_3^R. Details of these linkages are discussed in the next section. One can easily show that U_1^S and U_1^R are equivalent to U_1^A and U_1^D; all of these are simply sets of increasing utility functions. The set U_2^S containing S-shaped utility functions and the set U_n^R containing reverse S-shaped utility functions have been discussed in detail in the literature, for example, see Markowitz (1952) and Levy and Levy (2002, 2004). A utility in U_3^S is increasing with its marginal utility decreasing in the positive domain but increasing in the negative domain, and is graphically convex in both the positive and negative domains. On the other hand, a utility in U_3^R is increasing with its marginal utility increasing in the positive domain but decreasing in the negative domain, and is graphically convex in both the positive and negative domains. In order to draw a clearer picture for both the second and third orders SD, we define the following Pratt-Arrow risk aversion at ω for an individual with the utility function u:

$$r(\omega) = -\frac{u^{(2)}(\omega)}{u^{(1)}(\omega)} = -\frac{d\,\log\, u^{(1)}(\omega)}{d\,\omega}. \tag{8}$$

where $u^{(i)}$ is the i^{th} derivative of the utility function u.

With the definition of risk aversion, one can easily show the relationship between risk aversion and the sets of utility functions defined in Definition 4. For example, if $u \in U_2^S$, then its risk aversion will be positive in the positive domain and negative in the negative domain. Similarly, if $u \in U_2^R$, then its risk aversion will be negative in the position domain and positive in the negative domain. In addition, if the risk aversion is positively decreasing in the positive domain and negatively decreasing in the negative domain, then the utility function belongs to $u \in U_3^S$. On the other hand, if the risk aversion is negatively decreasing in the positive domain but positively decreasing in the negative domain, then the utility function belongs to $u \in U_3^R$. Investors with utility u is well-known to have Decreasing Absolute Risk Aversion (DARA) behavior if $u^{(1)} > 0$, $u^{(2)} < 0$ and $u^{(3)} > 0$, see for example, Falk and Levy (1989). We can say that investors with utility functions $u \in U_3^S$ have DARA behavior in the positive domain and investors with utility functions $u \in U_3^R$ have DARA behavior in the negative domain.

Let us turn to the empirical evidence on the (reverse) S-shaped utility functions. It is well-known that under the expected utility theory, convexity of utility is equivalent to risk seeking while concavity is equivalent to risk aversion. Empirical measurements generally corroborate with the concavity in the utility for gains, for example, see Fishburn and Kochenberger (1979). However, the behavior of gamblers reveals convexity for gains (Friedman and Savage 1948). For the utility for losses, some studies find convexity while some find concavity. For example, Pennings and Smidts (2003) find convex utility for losses for the majority of cases and concave utility for losses for a sizable minority of subjects. Despite the study of Myagkov and Plott (1997), no conclusive evidence in favor of convex utility for losses is provided, which would have supported the reverse S-shaped utility functions. However, Wu and Gonzalez (1996) propose to use preference 'ladders' to test for concavity and convexity of the weighting function. They validate the findings of an S-shaped weighting function, concave with probability up to .4, and convex beyond that probability. Nevertheless, using sequential gambles technique, Thaler and Johnson (1990) obtain experimental evidence to show that prior outcomes affect subsequent behavior in a way that subjects are more risk seeking following gains and more risk averse following losses. This supports the reverse S-shaped utility function behavior.

Finally, we note that in the prospect theory developed by Kahneman and Tversky (1979), the S-shaped utility function is called the value function as it is attuned to the evaluation of changes or differences of wealth rather than the evaluation of absolute magnitudes. In this paper, we simply call it utility function as we do not restrict its applications to total wealth or the changes or differences of wealth.[11] In addition, prospect theory assumes loss aversion which reflects the observed behavior that agents are more sensitive to losses than to gains, resulting in the value functions for losses are usually restricted to be steeper than their shapes for gains.[12] In another words, the investors are downside risk averse and could be measured by loss aversion.[13] In Definition 4, we do not include this restriction in the definition of U_2^S. However, U_2^S is a more general class of S-shaped utility functions containing all the value functions with this restriction and hence the theory of loss aversion and value function satisfy the theory developed in this paper.

[11]Levy and Wiener (1998) define U_p and Levy and Levy (2002) define V_{KT} as the class of all prospect theory value S-shaped functions with an inflection point at $x = 0$. This is the same as our U_2^S. They also define V_M as the class of all Markowitz utility functions which are reverse S-shaped, with an inflection point at $x = 0$. This is the same as our U_2^R.

[12]see, for example, Barberis, Huang and Santos (2001) and Wakker (2003).

[13]see for example, Rabin (2000) and Kobberling and Wakker (2005).

3 Theory

In this section we develop some basic properties for MSD and PSD. We first introduce the basic theorem linking the MSD (PSD) of the first three orders to investors with reverse S-shaped utility functions (S-shaped utility functions) to the first three orders:

Theorem 1 [14] *Let X and Y be random variables with probability distribution functions F and G respectively. Suppose u is a utility function. For $n = 1, 2$ and 3, we have*

a. $F \succeq_n^M (\succ_n^M) G$ *if and only if* $u(F) \geq (>) u(G)$ *for any u in* U_n^R (U_n^{SR})*, and*

b. $F \succeq_n^P (\succ_n^P) G$ *if and only if* $u(F) \geq (>) u(G)$ *for any u in* U_n^S (U_n^{SS})*.*

See appendix for the proof of Theorem 1. The SD results for risk averters and risk seekers similar to the above theorem have been well explored. Levy and Wiener (1998) and Levy and Levy (2002, 2004) extend the theory of SD by developing the second order PSD and MSD theories and link them to the second order S-shaped and reverse S-shaped utility functions. They also prove that both the second order MSD and PSD satisfy the expected utility paradigm. In this paper, we extend their work and link PSD and MSD of any order to the S-shaped and reverse S-shaped utility functions. We also extend Levy and Levy's results to examine the compatibility of the MSD and PSD of any order with the expected utility theory and prove that the MSD and PSD of any order are consistent with the expected utility paradigm as shown in the above theorem.

In the following corollary, FPSD and FMSD are equivalent, and both of them coincide with the traditional firt order SD (FSD).

[14] As most of the established properties of SD require the "strict" form but not the "weak" form of SD, from now on, we will only discuss the "strict" form of SD in our paper. Thus, for $n = 1, 2$ and 3, we will use "$\succ_n^X$" to represent both "$\succ_n^X$" and "$\succeq_n^X$" for $X = M$ and P, and U_n^Y to represent both U_n^Y and U_n^{SY} for $Y = S$ and R if no confusion occurs.

Corollary 1 *For any random variables X and Y, $X \succ_1^M Y$ if and only if $X \succ_1^P Y$ if and only if X stochastically dominates Y in the sense of the FSD.*

The proof of Corollary 1 is straightforward.[15] Incorporating this into the Arbitrage versus SD theorem in Jarrow (1986) will yield the following corollary:

Corollary 2 *Under the condition that the market is complete, for any random variables X and Y, $X \succ_1^M Y$ and/or $X \succ_1^P Y$ if and only if there is an arbitrage opportunity between X and Y such that one will increase one's wealth as well as one's utility if one shifts the investments from Y to X.*

The safety-first rule is first introduced by Roy (1952) for decision making under uncertainty. It stipulates choosing an alternative that provides a target mean return while minimizing the probability of the return falling below some threshold of disaster. Bawa (1978) takes the idea and examines the relationships between the SD and generalized safety-first rules for arbitrage distributions. Thereafter, Jarrow (1986) discovers the existence of the arbitrage opportunities in the SD rules. He defines a 'complete' market as 'an economy where all contingent claims on the primary assets trade.' His Arbitrage versus SD theorem says that when the market is complete, X stochastically dominates Y in the sense of FSD if and only if there is an arbitrage opportunity between X and Y. As $X \succ_1^M Y$ is equivalent to $X \succ_1^P Y$ (see Corollary 1), both are equivalent to X stochastically dominates Y in the sense of FSD. Thus, Corollary 2 holds when the Arbitrage versus SD theorem in Jarrow is applied.

Whitmore (1970) extends the second order SD to the third order SD for risk averters and thereafter many academics demonstrate the usefulness of the third order SD. In addition, Hammond (1974) generalizes the SD theory to the n-order for any integer n.

[15] The proof is available on request.

Both the MSD and PSD theories can be extended to any order in similar ways. However, we focus our discussion up to the first three orders in this paper as the first three orders SD are of most importance in theory as well as empirical applications. However, the first order PSD and MSD are not very interesting concepts because the curvature of the utility function does not influence the first-order SD criterion which only requires the utility function to be increasing. In this paper, we mainly discuss the higher order cases. But before doing so, we will show the hierarchy relationships for MSP and PSD in the following corollary:

Corollary 3 *For any random variables X and Y, for $n = 1$ and 2, we have:*

$$\textit{if } X \succ_n^M Y, \textit{ then } X \succ_{n+1}^M Y \,; \textit{ and if } X \succ_n^P Y, \textit{ then } X \succ_{n+1}^P Y .$$

The proof of Corollary 3 is straightforward. The results of this corollary suggest that practitioners report the MSD and PSD results to the lowest order in empirical analysis.

Levy and Levy (2002) show that it is possible for MSD to be 'the opposite' of PSD in their second orders and that F dominates G in SPSD, but G dominates F in SMSD. In the following corollary, we extend their results to include MSD and PSD to the second and third orders:

Corollary 4 *For any random variables X and Y, if F and G have the same mean which is finite, then we have*

$$F \succ_2^M G \quad \textit{if and only if} \quad G \succ_2^P F \,; \textit{and} \tag{9}$$

if, in addition, either $F \succ_2^M G$ or $G \succ_2^P F$ holds, we have

$$F \succ_3^M G \quad \textit{and} \quad G \succ_3^P F \,. \tag{10}$$

The proof of (9) follows the paper by Levy and Levy, while (10) follows Corollary 3. However, there are cases when distributions F and G have the same mean and do not satisfy (9) yet satisfying (10) as shown in the following example:

Example 1: Consider the distribution functions

$$F(t)=\begin{cases} 0 & -1\le t\le -7/8, \\ 1/6 & -7/8\le t\le -3/4, \\ 2(t+1)/3 & -3/4\le t\le -1/2, \\ 1/3 & -1/2\le t\le -1/4, \\ 1/2 & -1/4\le t\le 0, \\ 1-G(-t) & 0\le t\le 1, \end{cases} \quad \text{and} \quad G(t)=\begin{cases} 2(t+1)/3 & -1\le t\le -3/4, \\ 1/6 & -3/4\le t\le -5/8, \\ 1/3 & -5/8\le t\le -1/2, \\ 1/2+t/3 & -1/2\le t\le 0, \\ 1-F(-t) & 0\le t\le 1. \end{cases}$$

In this example, one can easily show that there is no SMSD and no SPSD dominance but $F \succ_3^M G$ and $G \succ_3^P F$.[16] The above corollary provides the conditions in which F is 'the opposite' of G and the above example shows that there exist pairs of distributions which are 'opposites' in the third order but not in the second order. On the other hand, we find that under some regularities, F becomes 'the same' as G in the sense of TMSD and TPSD as shown in the corollary below:

Corollary 5 *If F and G satisfy*

$$F_2^A(0)=G_2^A(0),\ F_3^A(0)=G_3^A(0),\ F_2^a(b)=G_2^a(b),\ \text{and}\ F_3^a(b)=G_3^a(b), \tag{11}$$

then

$$F \succ_3^M G \quad \textit{if and only if} \quad F \succ_3^P G\,.$$

The proof of Corollary 5 is straightforward.[17] One should note that the assumptions in

[16]The working is available on request.
[17]The proof is available on request.

(11) are very restrictive. In fact, if some of the assumptions are not satisfied, there exists F and G such that $G \succ_3^P F$ but neither $F \succ_3^M G$ nor $G \succ_3^M F$ holds, as shown in the following example:

Example 2: Consider

$$F(t) = \begin{cases} 4(t+1)/5 & -1 \le t \le -3/4, \\ 2t/5 + 1/2 & -3/4 \le t \le -1/4, \\ (4t+3)/5 & -1/4 \le t \le 0, \\ 1 - G(-t) & 0 \le t \le 1, \end{cases} \quad \text{and} \quad G(t) = \begin{cases} 0 & -1 \le t \le -3/4, \\ 2/5 & -3/4 \le t \le 0, \\ 1 - F(-t) & 0 \le t \le 1. \end{cases}$$

In this example, one can easily show that we do not have $F \succ_3^M G$ or $G \succ_3^M F$ but we have $G \succ_3^P F$.[18] The above corollary and example show that under some regularities, F is 'the same' as G in the sense of TMSD and TPSD. One may wonder whether this 'same direction property' could appear in FMSD vs FPSD and SMSD vs SPSD. In the following corollary, we show that this is possible.

Corollary 6 *If the random variable $X = p + qY$ and if $p + qx > x$ for all $x \in [a, b]$, then we have $X \succ_n^M Y$ and $X \succ_n^P Y$ for $n = 1$, 2 and 3.*

The proof of the above corollary is trivial. As shown by Levy and Levy (2002), MSD is generally not 'the opposite' of PSD. In other words, if F dominates G in PSD, it does not necessarily mean that G dominates F in MSD. This is easy to see because having a higher mean is a necessary condition for dominance by both rules. Therefore, if F dominates G in the sense of PSD, and F has a higher mean than G, G cannot possibly dominate F in the sense of MSD. The above corollary goes one step further and shows that they could

[18] The working is available on request.

be 'the same' in the sense of MSD and PSD. In addition, we derive the following corollary to show the relationship between the first order MSD and PSD.

Using the results in Theorem 1, we can call a person a first-order-MSD (FMSD) investor if his/her utility function u belongs to U_1^R, and a first-order-PSD (FPSD) investor if his/her utility function U belongs to U_1^S. A second-order-MSD (SMSD) risk investor, a second-order-PSD (SPSD) risk investor, a third-order-MSD (TMSD) risk investor and a third-order-PSD (TPSD) risk investor can be defined in the same way. From Definition 4 and the definition of risk aversion defined in (8), one can tell that the risk aversion of a SPSD investor is positive in the positive domain and negative in the negative domain and a SMSD investor's risk aversion is negative in the positive domain and positive in the negative domain. If one's risk aversion is positive and decreasing in the positive domain and negative and decreasing in the negative domain, then one is a TPSD investor; but the reverse is not true. Similarly, if one's risk aversion is negative and decreasing in the positive domain and positive and decreasing in the negative domain, then one is a TMSD investor. We summarize these results in the following corollary:

Corollary 7 *For an investor with an increasing utility function u and risk aversion r,*

a. *s/he is a SPSD (SMSD) investor if and only if her/his risk aversion r is positive (negative) in the positive domain and negative (positive) in the negative domain;*

b. *if her/his risk aversion r is always decreasing and is positive (negative) in the positive domain and negative (positive) in the negative domain, then s/he is a TPSD (TMSD) investor.*

The proof of Corollary 7 is straightforward.[19] Corollary 7 states the relationships

[19]The proof is available on request.

between different types of investors and their risk aversions. We note that the converse of (b) is not true.

4 Illustration

In this section we illustrate each case of MSD and PSD to the first three orders by using examples from Levy and Levy (2002) and modifying them. We first use Task III of Experiment 3 in Levy and Levy (2002) which is a replication of the tasks in Kahneman and Tversky (1979). In the experiment, \$10,000 is invested in either stock F or Stock G with the following dollar gain one month later and with probabilities f and g respectively, as shown in Table 1.

We use the MSD and PSD integrals H_n^M and H_n^P for $H = F$ and G and $n = 1, 2$ and 3 as defined in (5). To make the comparison easier, we define their differentials

$$GF_n^M = G_n^M - F_n^M \quad \text{and} \quad GF_n^P = G_n^P - F_n^P \tag{12}$$

for $n = 1, 2$ and 3 and present the results of the MSD and PSD integrals with their differentials for the first three orders in Tables 2 and 3.

In this example, Levy and Levy conclude that $F \succeq_{MSD} G$ but $G \succeq_{PSD} F$ while our results show that $F \succ_n^M G$ and $G \succ_n^P F$ for $n = 2$ and 3. From Corollary 3, we know that hierarchy exists in both MSD and PSD such that $F \succ_2^M G$ implies $F \succ_3^M G$ while $G \succ_2^P F$ implies $G \succ_3^P F$. Hence, one only has to report the lowest SD order. Our findings shows that $F \succ_2^M G$ and $G \succ_2^P F$, same as the findings in Levy and Levy. Our approach has no advantage over Levy and Levy's in this example. However, Levy and Levy's approach can only detect the second order MSD and PSD while our approach, by incorporating the extended PSD and MSD, enables investors to compare MSD and PSD to any order.

In order to show the superiority of our approach, we modify the above experiment by adjusting the probabilities f and g for investments F and G respectively. Reported in Tables 4–6 are all other orders of both MSD and PSD. For simplicity, we only report the differentials GF_n^M and GF_n^P and skip reporting their integrals. For easy comparison, we also report the MSD and PSD computation based on Levy and Levy's formula:

$$GF^M = G^M - F^M \quad \text{and} \quad GF^P = G^P - F^P \ . \tag{13}$$

Note that Levy and Levy define $F \succeq_{MSD} G$ if $GF^M(x) \geq 0$ for all x and $F \succeq_{PSD} G$ if $GF^P(x) \geq 0$ for all x with some strict inequality.

In Table 4, if one adopts Levy and Levy's approach, one will conclude that $F \succeq_{MSD} G$ and $F \succeq_{PSD} G$. However, if one applies our approach, one will conclude that $F \succ_1^M G$ and $F \succ_1^P G$, which is different from the conclusion drawn from Levy and Levy's approach. From Corollary 3, we know that hierarchy exists in both MSD and PSD such that $F \succ_1^M G$ implies $F \succ_2^M G$ while $G \succ_1^P F$ implies $G \succ_2^P F$. Hence, one only has to report the lowest SD order. However, reporting the first order MSD and PSD obtained by using our approach should be more appropriate.

In Table 5, if one uses Levy and Levy's approach, one will conclude $G \succeq_{PSD} F$ and conclude that neither F nor G dominate each other in the sense of MSD. However, if one applies our approach, one will conclude that $G \succ_2^P F$ but $F \succ_3^M G$, which is different from the conclusion drawn from Levy and Levy's approach. Similarly, in Table 6, if one uses Levy and Levy's approach, one will conclude $G \succeq_{MSD} F$ and conclude that neither F and G dominates each other in the sense of PSD. However, if one applies our approach, one will conclude that $F \succ_2^M G$ but $G \succ_3^P F$, which is different from the conclusion drawn from Levy and Levy's approach. Thus, our approach reveals more information on both MSD and PSD.

The results from our illustrations are more informative for investors than Levy and Levy's because we identify the MSD and PSD prospects for the first three orders while Levy and Levy only identify MSD and PSD for the second order, which may not truly present the MSD and PSD nature of these prospects. As our approach can provide investors with more information about investments opportunities, our approach could enable investors to make wiser decisions on investments. For example, in Table 4, using Levy and Levy's approach, SMSD and SPSD (also TMSD and TPSD) investors will choose to invest on F rather than G and will increase their expected utilities but not their wealth when shifting their investments from G to F. For FMSD and FPSD investors, they will not be able to obtain any useful information at all. However, if investors adopt our approach, it will be a completely different story. FMSD, SMSD, TMSD, FPSD, SPSD and TPSD investors will choose to invest on F rather than G and all of them will increase their expected utilities as well as their wealth when shifting their investments from G to F. What's more, our approach enables investors to identify that there is an arbitrage opportunity between F and G and one could long F and short G and making good profit.

Furthermore, Levy and Levy's approach will not be able to reveal any TMSD or TPSD prospect, while ours will enable investors to identify them, which in turn provides useful information for the TMSD and TPSD investors. If the approach by Levy and Levy is applied, one will conclude neither MSD nor PSD. For the TMSD and TPSD investors, they will not know about the relationships between these prospects and will miss these investment opportunities. For example, referring to Table 5, TMSD investors will not be able to decide which prospect to invest if they apply Levy and Levy's approach. However, if they apply our approach, they will invest in F rather than G and if they have invested in G, our approach will tell them that they will increase their expected utilities if they shift their investments from G to F. Similar conclusion can be made by TPSD investors about the investment choices presented in Table 6.

5 Concluding Remarks

In this paper, we extend the MSD and PSD theory by first defining the MSD and PSD of the first three orders and link them to the corresponding S-shaped and reverse S-shaped utility functions to the first three orders. We then provide experiments to illustrate each case of the MSD and PSD to the first three orders and develop some properties for the extended MSD and PSD.

Prospect theory is a paradigm challenging the expected utility theory. The main controversy is the prospect theory's S-shaped value function which describes preferences. This has been discussed in our paper in detail and our conclusion is that it is consistent with the expected utility theory. The next allegation is that the prospect theory invalidates the expected utility theory as being "subjectively distorted probabilities" (Levy and Wiener 1998)[20]. This was later corrected by what is now known as Cumulative Prospect Theory, see Starmer (2000) for the review of the subject. We suggest incorporating the Bayesian approach (Matsumura, Tsui and Wong, 1990) and distribution-free statistics (Wong and Miller, 1990) into the subjective probability (Anscombe and Aumann, 1963; Machina and Schmeidler, 1992) to estimate the subjectively distorted probabilities. Prospect theory will satisfy the Bayesian expected utility maximization. Thus, the problem that the prospect theory violates the expected utility theory could be circumvented.

The advantage of the SD approach is that we have a decision rule which holds for all utility functions of certain classes. Specifically, PSD (MSD) of any order is a criterion which is valid for all S-shaped (reverse S-shaped) utility functions of the corresponding order. Moreover, the SD rules for S-shaped and reverse S-shaped utility functions can be employed with mixed prospects. We note that in our paper we do not restrict the

[20]We note that Wu, Wu and Abdellaoui (2005) develop a critical test of the two prospect theories based on their respective probability tradeoff consistency conditions.

S-shaped utility functions to be steeper than their shapes for gains as the restricted set in value functions defined by Kahneman and Tversky (1979). However, the class of S-shaped utility functions defined in our paper is more general and contains all the value functions with this restriction. Wakka (2003) claims that some examples in Levy and Levy (2002) violate this curvature restriction on value function posited by prospect theory. In this paper, as we follow Levy and Levy (2002)'s definition of S-shaped utility function without this curvature restriction. Our examples could be set without this curvature restriction. However, one could easier show that all examples with this curvature restriction will fit our theory well.

The MSD and PSD developed by Levy and Wiener (1998), Levy and Levy (2002, 2004) and the extensions in our paper only link to the S-shaped and reverse S-shaped utility functions. These utility functions are simplified version of the utility functions proposed by Markowitz (1952) which have convex and concave regions in both the positive and the negative domains. Empirical studies reveal a more complex behavior. For example, people are mostly risk averse to prospects yielding a best outcome with a low probability but they are mostly risk seeking to prospects yielding a worst outcome with a low probability (Starmer 2000 and Luce 2000). Further research includes extension of the MSD and PSD to link to this more complicated patterns of behavior.

These days, it is popular to apply SD to explain financial theories and anomalies, for example, McNamara (1998), Post and Levy (2005), Fong, Wong and Lean (2005) and Broll, Wahl and Wong (2006). Some apply the Levy and Levy approach to study risk averse and risk seeking behaviors. For example, Post and Levy (2005) study risk seeking behaviors in order to explain the cross-sectional patterns of stock returns and suggest that the reverse S-shaped utility functions can explain stock returns, with risk aversion to losses and risk seeking for gains reflecting investors' twin desire for downside protection in bear markets and upside potential in bull markets. Using the second order MSD and

PSD introduced by Levy and Levy can only detect investment opportunities for SMSD and SPSD investors. We recommend financial analysts to apply the approach introduced in this paper and examine the MSD and PSD relationships of different orders so that they can detect opportunities for MSD and PSD investors of different orders.

One of the main reasons for the recent popularity of the SD in financial literature is that Post (2003) and Kuosmanen (2004) have developed means to test if it is possible to construct a dominating portfolio from an infinitely large set of diversified portfolios. As they increase the appeal of the SD approach in finance, the present paper shows that the FSD test by Kuosmanen (2004) is consistent with the first-order PSD and MSD as well. Development of portfolio efficiency tests for the second- and third-order PSD and MSD would be interesting challenges for future research.

Appendices

We only prove the necessary condition for both Parts A and B of Theorem 1. The sufficient condition can be proved by contradiction. Huang and Litzenberger (1988) and others have proved the sufficeint condition of SD for risk averters. One could easily modify their proofs to obtain the proof of sufficeint conditions in both Parts A and B of Theorem 1 of our paper.

Appendix A – Proof of Part a of Theorem 1:

Levy and Levy (2002) have proved the second order of Part A of Theorem 1. Suppose $[a, b]$ is the support with negative a and positive b, we modify and extend their proof to include the first three orders of the MSD as follows:

$$
\begin{aligned}
\Delta Eu &\equiv u(F) - u(G) \equiv \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) \\
&= [F(x) - G(x)]u(x)\big|_a^b - \int_a^b [F(x) - G(x)]u^{(1)}(x)\,dx \\
&= \int_a^b [G(x) - F(x)]u^{(1)}(x)\,dx \\
&= \int_a^0 [G(x) - F(x)]u^{(1)}(x)\,dx + \int_0^b [G(x) - F(x)]u^{(1)}(x)\,dx \\
&= \int_a^0 [G_1^A(x) - F_1^A(x)]u^{(1)}(x)\,dx + \int_0^b [F_1^D(x) - G_1^D(x)]u^{(1)}(x)\,dx \qquad (14) \\
&= \int_a^0 u^{(1)}(x)\,d\,[G_2^A(x) - F_2^A(x)] - \int_0^b u^{(1)}(x)\,d\,[F_2^D(x) - G_2^D(x)] \\
&= [G_2^A(x) - F_2^A(x)]u^{(1)}(x)\big|_a^0 - \int_a^0 [G_2^A(x) - F_2^A(x)]u^{(2)}(x)\,dx - \\
&\quad [F_2^D(x) - G_2^D(x)]u^{(1)}(x)\big|_0^b + \int_0^b [F_2^D(x) - G_2^D(x)]u^{(2)}(x)\,dx \\
&= B_1 + \int_a^0 [F_2^A(x) - G_2^A(x)]u^{(2)}(x)\,dx + \int_0^b [F_2^D(x) - G_2^D(x)]u^{(2)}(x)\,dx \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
&= B_1 + \int_a^0 u^{(2)}(x)\, d\,[F_3^A(x) - G_3^A(x)] - \int_0^b u^{(2)}(x)\, d\,[F_3^D(x) - G_3^D(x)] \\
&= B_1 + [F_3^A(x) - G_3^A(x)]u^{(2)}(x)\Big|_a^0 - \int_a^0 [F_3^A(x) - G_3^A(x)]u^{(3)}(x)\, dx - \\
&\quad [F_3^D(x) - G_3^D(x)]u^{(2)}(x)\Big|_0^b + \int_0^b [F_3^D(x) - G_3^D(x)]u^{(3)}(x)\, dx \\
&= B_1 + B_2 + \int_a^0 [G_3^A(x) - F_3^A(x)]u^{(3)}(x)\, dx + \int_0^b [F_3^D(x) - G_3^D(x)]u^{(3)}(x)\, dx \quad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= [G_2^A(0) - F_2^A(0) + F_2^D(0) - G_2^D(0)]u^{(1)}(0) \quad \text{and} \\
B_2 &= [F_3^A(0) - G_3^A(0) + F_3^D(0) - G_3^D(0)]u^{(2)}(0)\ . \qquad (17)
\end{aligned}
$$

From (14), we have if $F \succ_1^M G$ then $F_1^D(x) \geq G_1^D(x)$ for $x \geq 0$ and $F_1^A(x) \leq G_1^A(x)$ for $x \leq 0$. If $u \in U_1^R$ then $u^{(1)} \geq 0$. Hence $\Delta Eu = u(F) - u(G) \geq 0$.

If $F \succ_2^M G$, then $F_2^D(x) \geq G_2^D(x)$ for $x \geq 0$ and $F_2^A(x) \leq G_2^A(x)$ for $x \leq 0$. If in addition, $u \in U_2^S$ then $u^{(1)} \geq 0$, $u^{(2)}(x) \geq 0$ for $x \geq 0$ and $u^{(2)}(x) \leq 0$ for $x \leq 0$. From (17), $B_1 \geq 0$, and hence from (15), $\Delta Eu = u(F) - u(G) \geq 0$.

If $F \succ_3^M G$, then $F_3^D(x) \geq G_3^D(x)$ for $x \geq 0$ and $F_3^A(x) \leq G_3^A(x)$ for $x \leq 0$. If in addition, $u \in U_3^R$ then $u^{(1)} \geq 0$, $u^{(2)}(x) \geq 0$ for $x \geq 0$, $u^{(2)}(x) \leq 0$ for $x \leq 0$, $u^{(2)}(0) = 0$ and $u^{(3)} \geq 0$. From (17), we have $B_2 = 0$ and hence from (16), $\Delta Eu = u(F) - u(G) \geq 0$.

Appendix B – Proof of Part b of Theorem 1:

Levy and Levy (2002) have proved the second order of Part B of Theorem 1. Suppose $[a, b]$ is the support with negative a and positive b, we modify and extend their proof to include the first three orders of the PSD as follows:

$$\begin{aligned}
\Delta Eu &\equiv u(F) - u(G) \equiv \int_a^b u(x)dF(x) - \int_a^b u(x)dG(x) \\
&= \int_a^0 [G(x) - F(x)]u^{(1)}(x)\,dx + \int_0^b [G(x) - F(x)]u^{(1)}(x)\,dx \\
&= \int_a^0 [F_1^d(y) - G_1^d(y)]u^{(1)}(y)\,dy + \int_0^b [G_1^a(x) - F_1^a(x)]u^{(1)}(x)\,dx \qquad (18) \\
&= \int_a^0 u^{(1)}(y)\,d\,[G_2^d(y) - F_2^d(y)] + \int_0^b u^{(1)}(x)\,d\,[G_2^a(x) - F_2^a(x)] \\
&= [G_2^d(y) - F_2^d(y)]u^{(1)}(y)\big|_a^0 + \int_a^0 [F_2^d(y) - G_2^d(y)]u^{(2)}(y)\,dy \\
&\quad + [G_2^a(x) - F_2^a(x)]u^{(1)}(x)\big|_0^b + \int_0^b [F_2^a(x) - G_2^a(x)]u^{(2)}(x)\,dx \\
&= B_2 + \int_a^0 [F_2^d(y) - G_2^d(y)]u^{(2)}(y)\,dy + \int_0^b [F_2^a(x) - G_2^a(x)]u^{(2)}(x)\,dx \qquad (19) \\
&= B_2 + \int_a^0 u^{(2)}(y)\,d\,[G_3^d(y) - F_3^d(y)] + \int_0^b u^{(2)}(x)\,d\,[F_3^A(x) - G_3^A(x)] \\
&= B_2 + [G_3^d(y) - F_3^d(y)]u^{(2)}(y)\big|_a^0 + \int_a^0 [F_3^d(y) - G_3^d(y)]u^{(3)}(y)\,dx \\
&\quad + [F_3^a(x) - G_3^a(x)]u^{(2)}(x)\big|_0^b + \int_0^b [G_3^a(x) - F_3^a(x)]u^{(3)}(x)\,dx \\
&= B_2 + B_3 + \int_a^0 [F_3^d(y) - G_3^d(y)]u^{(3)}(y)\,dy + \int_0^b [G_3^a(x) - F_3^a(x)]u^{(3)}(x)\,dx \qquad (20)
\end{aligned}$$

where

$$
\begin{aligned}
B_2 &= u^{(1)}(a)[F_2^d(a) - G_2^d(a)] + u^{(1)}(b)[G_2^a(b) - F_2^a(b)] \\
&\quad + u^{(1)}(0)[G_2^d(0) - F_2^d(0)] + u^{(1)}(0)[F_2^a(0) - G_2^a(0)] \quad \text{and} \\
B_3 &= u^{(2)}(a)[F_3^d(a) - G_3^d(a)] + u^{(2)}(b)[F_3^a(b) - G_3^a(b)] \qquad (21) \\
&\quad + u^{(2)}(0)[G_3^d(0) - F_3^d(0)] + u^{(2)}(0)[G_3^a(0) - F_3^a(0)]
\end{aligned}
$$

As $a \leq 0$, $F_2^d(a) \geq G_2^d(a)$. Similarly $G_2^a(b) \geq F_2^a(b)$ as $b \geq 0$. Since $u^{(1)}(a)$, $u^{(1)}(b)$ are nonnegative; $H_2^d(0) = H_2^a(0) = 0$ for $H = F$ and G, we see that $B_2 \geq 0$. Also, as $a \leq 0$, $F_3^d(a) \geq G_3^d(a)$ and also we have $u^{(2)}(a) \geq 0$. Similarly $G_2^a(b) \geq F_2^a(b)$ as $b \geq 0$, but we have $u^{(2)}(b) \leq 0$. In addition, $H_3^d(0) = H_3^a(0) = 0$ for $H = F$ and G, We see that $B_3 \geq 0$.

Hence, from (18), if $X \succ_1^P Y$ or $F \succ_1^P G$, then we have $\Delta Eu = u(F) - u(G) \geq 0$; from (19), we have if $X \succ_2^P Y$ or $F \succ_2^P G$, then $\Delta Eu = u(F) - u(G) \geq 0$; and from (20), we have if $X \succ_3^P Y$ or $F \succ_3^P G$, then $\Delta Eu = u(F) - u(G) \geq 0$.

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Table 1 : The Distributions for Investments F and G

Investment F		Investment G	
Gain	Probability (f)	Gain	Probability (g)
-1,500	$\frac{1}{2}$	-3,000	$\frac{1}{4}$
4,500	$\frac{1}{2}$	3,000	$\frac{3}{4}$

Table 2 : The MSD Integrals and their Differentials for F and G

Gain	First Order			Second Order			Third Order		
X	F_1^M	G_1^M	GF_1^M	F_2^M	G_2^M	GF_2^M	F_3^M	G_3^M	GF_3^M
-3	0	0.25	0.25	0	0	0	0	0	0
-1.5	0.5	0.25	-0.25	0	0.375	0.375	0	0.28125	0.28125
0^-	0.5	0.25	-0.25	0.75	0.75	0	0.5625	1.125	0.5625
0^+	0.5	0.75	0.25	2.25	2.25	0	5.0625	3.375	-1.6875
3	0.5	0.75	0.25	0.75	0	-0.75	0.5625	0	-0.5625
4.5	0.5	0	-0.5	0	0	0	0	0	0

Table 3 : The PSD Integrals and their Differentials for F and G

Gain	First Order			Second Order			Third Order		
X	F_1^P	G_1^P	GF_1^P	F_2^P	G_2^P	GF_2^P	F_3^P	G_3^P	GF_3^P
-3	1	1	0	2.25	2.25	0	2.8125	3.375	0.5625
-1.5	1	0.75	-0.25	0.75	1.125	0.375	0.5625	0.84375	0.28125
0^-	0.5	0.75	0.25	0	0	0	0	0	0
0^+	0.5	0.25	-0.25	0	0	0	0	0	0
3	0.5	1	0.5	1.5	0.75	-0.75	2.25	1.125	-1.125
4.5	1	1	0	2.25	2.25	0	5.0625	3.375	-1.6875

Table 4 : The MSP and PSD Differentials for F and G : Case 2

Gain	probability		MSD			PSD			Levy and Levy	
X	f	g	GF_1^M	GF_2^M	GF_3^M	GF_1^P	GF_2^P	GF_3^P	GF^M	GF^P
-3	0	0.25	0.25	0	0	0	-0.45	-0.45	0	0.45
-1.5	0.2	0	0.05	0.375	0.28125	-0.25	-0.075	-0.05625	0.375	0.075
0^-	0	0	0.05	0.45	0.9	-0.05	0	0	0.45	0
0^+	0	0	-0.05	-1.35	-4.785	0.05	0	0	1.35	0
3	0	0.75	-0.05	-1.2	-0.9	0.8	0.15	0.225	1.2	0.15
4.5	0.8	0	-0.8	0	0	0	1.35	1.35	0	1.35

Table 5 : The MSP and PSD Differentials for F and G : Case 3

Gain	probability		MSD			PSD			Levy and Levy	
X	f	g	GF_1^M	GF_2^M	GF_3^M	GF_1^P	GF_2^P	GF_3^P	GF^M	GF^P
-3	0	0.25	0.25	0	0	0	0.075	0.73125	0	-0.075
-1.5	0.55	0	-0.3	0.375	0.28125	-0.25	0.45	0.3375	0.375	-0.45
0^-	0	0	-0.3	-0.075	0.50625	0.3	0	0	-0.075	0
0^+	0	0	0.3	0.225	-1.18125	-0.3	0	0	-0.225	0
3	0	0.75	0.3	-0.675	-0.50625	0.45	-0.9	-1.35	0.675	-0.9
4.5	0.45	0	-0.45	0	0	0	-0.225	-2.19375	0	-0.225

Table 6 : The MSP and PSD Differentials for F and G : Case 4

Gain	probability		MSD			PSD			Levy and Levy	
X	f	g	GF_1^M	GF_2^M	GF_3^M	GF_1^P	GF_2^P	GF_3^P	GF^M	GF^P
-3	0	0.25	0.25	0	0	0	-0.15	0.225	0	0.15
-1.5	0.4	0	-0.15	0.375	0.28125	-0.25	0.225	0.16875	0.375	-0.225
0^-	0	0	-0.15	0.15	0.625	0.15	0	0	0.15	0
0^+	0	0	0.15	-0.45	-2.7	-0.15	0	0	0.45	0
3	0	0.75	0.15	-0.9	-0.625	0.6	-0.45	-0.675	0.9	-0.45
4.5	0.6	0	-0.6	0	0	0	0.45	-0.675	0	0.45

Figure 1: Functions in U_2^A, U_2^D, U_2^S and U_2^R

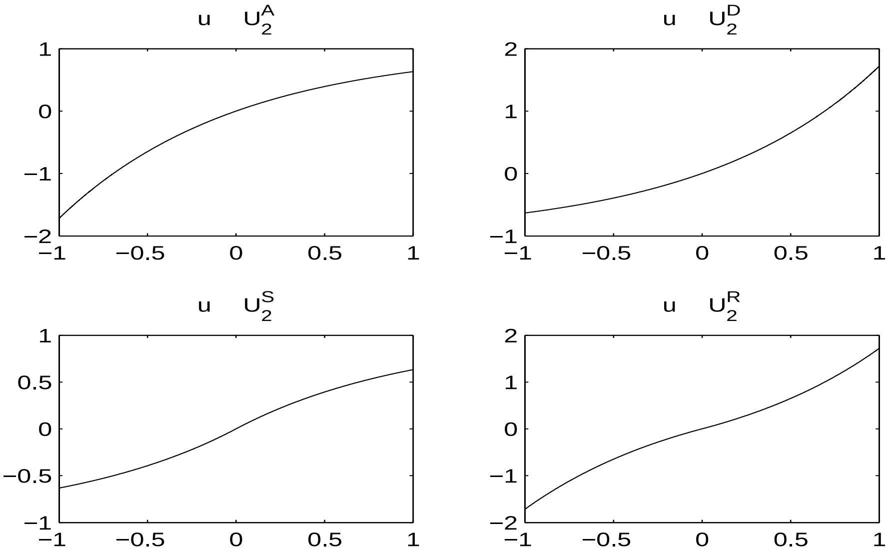

Figure 2: Derivatives of Functions in U_3^A, U_3^D, U_3^S and U_3^R