Minimization of an Edge-Preserving Regularization Functional by Conjugate Gradient Type Methods

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Summary. Recently, a powerful two-phase method for removing impulse noise has been developed. It gives a satisfactory result even for images with 90% pixels corrupted by impulse noise. However, the two-phase method is not computationally efficient, because it requires the minimization of a non-smooth functional in the second phase, which is done by a relaxation-based method. In this paper, we remove the non-smooth term from the functional, and call for the minimization of a smooth one. The minimizer is then found by using a conjugate gradient method proposed by J. Sun and J. Zhang. We prove the global convergence of the conjugate gradient type method applied to our functional. Simulation results show that our method is several times faster than the relaxation-based method when the noise ratio is high.

1 Introduction

Impulse noise is caused by malfunctioning pixels in camera sensors, faulty memory locations in hardware, or transmission in a noisy channel [2]. Let $\mathbf{x}$ denote the original image and $[s_{\min}, s_{\max}]$ denote the dynamic range of $\mathbf{x}$. The impulse noise model with noise ratio (error probability) p for a noisy image $\mathbf{y}$ is

$$y_{i,j} = \begin{cases} r_{i,j}, & \text{with probability } p, \\ x_{i,j}, & \text{with probability } 1-p, \end{cases}$$

where $x_{i,j}$ and $y_{i,j}$ are the gray levels of the original image $\mathbf{x}$ and the noisy image $\mathbf{y}$ at pixel location (i, j). There are two main models to represent impulse

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noise: the salt-and-pepper noise and the random-valued noise. For images corrupted by salt-and-pepper noise, $r_{i,j}$ can only take values $s_{\min}$ or $s_{\max}$ while for random-valued noise, $r_{i,j}$ can be any identically distributed, independent random number in $[s_{\min}, s_{\max}]$.

There are two popular types of methods for removing impulse noise. One is the median filter and its variants [7, 13]. It can detect the noise pixels accurately but it restores them poorly when the noise ratio is high. The gray levels of uncorrupted pixels are unchanged. The recovered image may loose its details and be distorted. Another procedure, the variational approach, is capable of retaining the details and the edges well but the gray level of every pixel is changed including uncorrupted ones [14].

Recently, a two-phase scheme for removing impulse noise has been proposed in [4, 5]. This scheme combines the advantages of both the median-type filters and the variational approach. In the first phase, a median-type filter is used to identify pixels which are likely to be contaminated by noise (noise candidates). In the second phase, the image is restored by minimizing a specialized regularization functional that applies only to those selected noise candidates. Therefore, the details and edges of the image can be preserved, and the uncorrupted pixels are unchanged.

The two-phase scheme is powerful even for noise ratio as high as 90%, see [4]. However, the functional to be minimized in the second phase is non-smooth, and it is costly to get the minimizer. Here we modify the functional by removing the non-smooth data-fitting term to get a smooth one. Therefore, many sophisticated methods developed for smooth optimization are applicable.

In this paper, conjugate gradient (CG) type methods are applied to minimize the smooth functional. Based on the results in [18], we apply CG methods in which the line search step is replaced by a step whose length is determined by a special formula. We prove that such CG methods are globally convergent for our minimization functional. Simulation results show that when the noise ratio is high, our method is several times faster than the relaxation method used in [4, 5].

The outline of the paper is as follows. In Section 2, we review the method presented in [4, 5]. In Section 3, we present our method. In Section 4, we give the convergence results of the method. In Section 5, simulation results are presented and finally in Section 6 we conclude the paper.

2 Review of Two Phase Methods

In this section we give a brief review on the two-phase method for removing salt-and-pepper impulse noise [4] and random-valued impulse noise [5]. The first phase is the detection of the noise pixels and the second phase is the recovering of the noise pixels detected in the first phase.

The First Phase: Detection of Noise Pixels

The first phase is the detection of the noise pixels. For salt-and-pepper noise, this is accomplished by using the adaptive median filter (AMF) [13] while for random-valued noise, it is accomplished by using the adaptive center-weighted median filter (ACWMF) [7]. Since we are concerned with accelerating the minimization procedure in the second phase, we only consider salt-and-pepper noise in the paper. The method can be applied equally well to random-valued noise.

The Second Phase: Recovering of Noise Pixels

We first give some notations. Let X be an image of size M-by-N and $\mathcal{A} = \{1, 2, 3, \ldots, M\} \times \{1, 2, 3, \ldots, N\}$ be the index set of the image X. Let $\mathcal{N} \subset \mathcal{A}$ be the set of indices of the noise pixels detected from the first phase and c be its number of elements. Let $\mathcal{V}_{i,j}$ be the set of the four closest neighbors of the pixel at position $(i, j) \in \mathcal{A}$. Let $y_{i,j}$ be the observed pixel value of the image at position (i, j). In [4], the recovering of noise pixels calls for the minimization of the functional:

$$F_\alpha(\mathbf{u}) = \sum_{(i,j)\in\mathcal{N}} \left[|u_{i,j} - y_{i,j}| + \frac{\beta}{2} \left(2 \cdot S^1_{i,j} + S^2_{i,j} \right) \right], \tag{1}$$

where

$$S^1_{i,j} = \sum_{(m,n)\in\mathcal{V}_{i,j}\setminus\mathcal{N}} \varphi_\alpha(u_{i,j} - y_{m,n}), \tag{2}$$

$$S^2_{i,j} = \sum_{(m,n)\in\mathcal{V}_{i,j}\cap\mathcal{N}} \varphi_\alpha(u_{i,j} - u_{m,n}), \tag{3}$$

φ_α is an edge-preserving function and $\mathbf{u} = [u_{i,j}]_{(i,j)\in\mathcal{N}}$ is a column vector of length c ordered lexicographically. We assume that the edge-preserving function φ_α is: (a) twice continuously differentiable, (b) $\varphi''_\alpha > 0$, and (c) even. Examples of such $\varphi_\alpha(t)$ are $\sqrt{t^2 + \alpha}$ and $\log(\cosh(\alpha t))$ where $\alpha > 0$ is a parameter, see [6] and [11]. From the above properties, we can conclude that $\varphi_\alpha(t)$ is strictly increasing with $|t|$ and coercive, i.e. $\varphi_\alpha(t) \to \infty$ as $|t| \to \infty$.

In [4], (1) is minimized by using a 1-D relaxation method. More precisely, at each iteration, we minimize (1) with respect to only one unknown while all the other unknowns are fixed. The procedure is repeated until convergence. In each iteration, a 1-D nonlinear equation is to be solved. Newton's method with special initial guess that guarantees quadratic convergence is used to solve these nonlinear equations, see [3] for detail.

3 Our Method

The function F_α in (1) is a non-smooth functional because of the $|u_{i,j} - y_{i,j}|$ term — the *data-fitting* term. In our method, we first remove this term. It is motivated by the following two facts:

1. The data-fitting term keeps the minimizer $\mathbf{u}$ close to the original image $\mathbf{y}$ so that the pixels which are uncorrupted in the original image are not altered. However, in the two-phase method the functional F_α is cleaning only the noise pixels and the uncorrupted pixels are unchanged. Hence, the data-fitting term is not required. This fact is verified numerically in [4].
2. Removing the data-fitting term will make F_α to be a smooth functional which can be minimized efficiently.

Therefore, the functional that we are minimizing in this paper is

$$\mathcal{F}_\alpha(\mathbf{u}) = \sum_{(i,j)\in\mathcal{N}} \left(2 \cdot S^1_{i,j} + S^2_{i,j}\right), \tag{4}$$

where $S^1_{i,j}$ and $S^2_{i,j}$ are the same as those defined in (2) and (3). Simulation results in Section 5 show that the minimizers of (1) and (4) attain the same signal-to-noise ratio.

The minimization methods we use to solve (4) is the conjugate gradient (CG) type method proposed in [18]. It does not need the Hessian matrix nor perform the line search. The resulting CG method can find the minimizer more efficiently by avoiding these time consuming tasks. We remark that the Hessian of (4) has not any special structure, so it is difficult to do preconditioning. Therefore, we only consider non-preconditioned CG here. We will give a very brief description of the method here.

The Minimization Algorithm

The general conjugate gradient method applied to $\min_{\mathbf{u}} \mathcal{F}_\alpha(\mathbf{u})$ has the following form. Given $\mathbf{u_0}$, let

$$\mathbf{d}_k = \begin{cases} -\mathbf{g}_k & \text{for } k = 0, \\ -\mathbf{g}_k + \beta_k \mathbf{d}_{k-1} & \text{for } k > 0, \end{cases} \tag{5}$$

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \alpha_k \mathbf{d}_k, \tag{6}$$

where $\mathbf{g}_k = \nabla\mathcal{F}_\alpha(\mathbf{u_k})$, α_k is determined by line-search and β_k is chosen so that $\mathbf{d_k}$ is the k-th conjugate direction when the function is quadratic and the line search is exact. Some of the well-known formula for β_k are:

$$\beta_k^{FR} = \frac{\|\mathbf{g}_k\|^2}{\|\mathbf{g}_{k-1}\|^2} \quad \text{(Fletcher-Reeves [10])}, \tag{7}$$

$$\beta_k^{PR} = \frac{\mathbf{g}_k^T(\mathbf{g}_k - \mathbf{g}_{k-1})}{\|\mathbf{g}_{k-1}\|^2} \quad \text{(Polak-Ribière [15])}, \tag{8}$$

$$\beta_k^{HS} = \frac{\mathbf{g}_k^T(\mathbf{g}_k - \mathbf{g}_{k-1})}{\mathbf{d}_{k-1}^T(\mathbf{g}_k - \mathbf{g}_{k-1})} \quad \text{(Hestenes-Stiefel [12])}, \tag{9}$$

$$\beta_k^{CD} = \frac{\|\mathbf{g}_k\|^2}{-\mathbf{d}_{k-1}^T\mathbf{g}_{k-1}} \quad \text{(The Conjugate Descent Method [9])}, \tag{10}$$

$$\beta_k^{DY} = \frac{\|\mathbf{g}_k\|^2}{\mathbf{d}_{k-1}^T(\mathbf{g}_k - \mathbf{g}_{k-1})} \quad \text{(Dai-Yuan [8])}. \tag{11}$$

In [18], it is proved that if $\mathcal{F}_\alpha$ satisfies the following Assumption 1 and α_k is chosen according to a special formula (see (14) below), then the resulting CG method is globally convergent.

Assumption 1

1. Let $\Delta = \{\mathbf{u} \mid \mathcal{F}_\alpha(\mathbf{u}) \le \mathcal{F}_\alpha(\mathbf{u}_0)\}$. Then there exists a neighborhood Ω of Δ such that $\nabla\mathcal{F}_\alpha$ is Lipschitz continuous on Ω, i.e. there exists a Lipschitz constant $\mu > 0$ such that

$$\|\nabla\mathcal{F}_\alpha(\mathbf{u}) - \nabla\mathcal{F}_\alpha(\mathbf{v})\| \le \mu\|\mathbf{u} - \mathbf{v}\|, \qquad \forall \mathbf{u}, \mathbf{v} \in \Omega, \tag{12}$$

2. $\mathcal{F}_\alpha$ is strongly convex in Ω, i.e. there exists a $\lambda > 0$ such that

$$\left(\nabla\mathcal{F}_\alpha(\mathbf{u}) - \nabla\mathcal{F}_\alpha(\mathbf{v})\right)^T(\mathbf{u} - \mathbf{v}) \ge \lambda\|\mathbf{u} - \mathbf{v}\|^2, \qquad \forall \mathbf{u}, \mathbf{v} \in \Omega. \tag{13}$$

In that case, we choose $\{Q_k\}$ to be a sequence of c-by-c positive definite matrices such that

$$\nu_{\min}\mathbf{d}^T\mathbf{d} \le \mathbf{d}^T Q_k \mathbf{d} \le \nu_{\max}\mathbf{d}^T\mathbf{d}, \qquad \forall\ \mathbf{d} \in \mathbb{R}^c\,,$$

with $\nu_{\min} > 0$ and $\nu_{\max} > 0$. Then the step length α_k is defined as

$$\alpha_k = -\frac{\delta\,\mathbf{g}_k^T\mathbf{d}_k}{\mathbf{d}_k^T Q_k \mathbf{d}_k}, \quad \text{where} \quad \delta \in (0, \frac{\nu_{\min}}{\mu}). \tag{14}$$

If $\mathcal{F}_\alpha$ satisfies Assumption 1, the sequence $\{\mathbf{u}_k\}$ defined by (5), (6) and (14) is globally convergent for all choices of β_k in (7) – (11), see [18].

4 Convergence of the Method

The minimization of (4) is a constrained minimization problem as the minimizer must lie in the dynamic range $[s_{\min}, s_{\max}]^c = \{\mathbf{u} \in \mathbb{R}^c : s_{\min} \le u_i \le$

$s_{\max}, i = 1, \cdots, c\}$. We are going to show that it is indeed a convex unconstrained minimization problem. In fact, we show that the functional $\mathcal{F}_\alpha$ is strictly convex in $\mathbb{R}^c$ and its minimizer lies in $[s_{\min}, s_{\max}]^c$. Moreover, we show that $\mathcal{F}_\alpha$ satisfies Assumption 1, hence the CG method is globally convergent.

To show that $\mathcal{F}_\alpha$ is strictly convex we first derive some properties of the Hessian matrix. As stated before,

$$\mathcal{F}_\alpha(\mathbf{u}) = \sum_{(i,j)\in\mathcal{N}} \left(2 \cdot S^1_{i,j} + S^2_{i,j}\right).$$

Because φ_α is an even function, we get

$$\begin{aligned} & (\nabla\mathcal{F}_\alpha(\mathbf{u}))_{(i,j)\in\mathcal{N}} \\ = \; & 2 \sum_{(m,n)\in\mathcal{V}_{i,j}\setminus\mathcal{N}} \varphi'_\alpha(u_{i,j} - y_{m,n}) + 2 \sum_{(m,n)\in\mathcal{V}_{i,j}\cap\mathcal{N}} \varphi'_\alpha(u_{i,j} - u_{m,n}). \end{aligned}$$

Hence

$$\left(\nabla^2\mathcal{F}_\alpha(\mathbf{u})\right)_{((i,j),(p,q))} = \begin{cases} 2\left(R^1_{i,j} + R^2_{i,j}\right), & \text{if } (i,j) = (p,q), \\ -2\varphi''_\alpha(u_{i,j} - u_{p,q}), & \text{if } (p,q) \in \mathcal{V}_{i,j} \cap \mathcal{N}, \\ 0, & \text{otherwise}, \end{cases} \tag{15}$$

where

$$\begin{aligned} R^1_{i,j} &= \sum_{(m,n)\in\mathcal{V}_{i,j}\setminus\mathcal{N}} \varphi''_\alpha(u_{i,j} - y_{m,n}), \\ R^2_{i,j} &= \sum_{(m,n)\in\mathcal{V}_{i,j}\cap\mathcal{N}} \varphi''_\alpha(u_{i,j} - u_{m,n}). \end{aligned}$$

Consider another matrix $\mathcal{G}_\alpha$ of size MN-by-MN defined by

$$\begin{aligned} & (\mathcal{G}_\alpha)_{((i,j),(p,q))} \\ \triangleq \; & \begin{cases} 2\left(R^1_{i,j} + R^2_{i,j}\right), & \text{if } (i,j) = (p,q) \in \mathcal{N}, \\ 2\left(T^1_{i,j} + T^2_{i,j}\right), & \text{if } (i,j) = (p,q) \notin \mathcal{N}, \\ -2\varphi''_\alpha(y_{i,j} - u_{p,q}), & \text{if } (i,j) \notin \mathcal{N}, (p,q) \in \mathcal{N} \text{ and } (p,q) \in \mathcal{V}_{i,j}, \\ -2\varphi''_\alpha(u_{i,j} - y_{p,q}), & \text{if } (i,j) \in \mathcal{N}, (p,q) \notin \mathcal{N} \text{ and } (p,q) \in \mathcal{V}_{i,j}, \\ -2\varphi''_\alpha(u_{i,j} - u_{p,q}), & \text{if } (i,j) \in \mathcal{N}, (p,q) \in \mathcal{N} \text{ and } (p,q) \in \mathcal{V}_{i,j}, \\ -2\varphi''_\alpha(y_{i,j} - y_{p,q}), & \text{if } (i,j) \notin \mathcal{N}, (p,q) \notin \mathcal{N} \text{ and } (p,q) \in \mathcal{V}_{i,j}, \\ 0, & \text{otherwise}, \end{cases} \end{aligned}$$

where

$$\begin{aligned} T^1_{i,j} &= \sum_{(m,n)\in\mathcal{V}_{i,j}\setminus\mathcal{N}} \varphi''_\alpha(y_{i,j} - y_{m,n}), \\ T^2_{i,j} &= \sum_{(m,n)\in\mathcal{V}_{i,j}\cap\mathcal{N}} \varphi''_\alpha(y_{i,j} - u_{m,n}). \end{aligned}$$

Then since $\varphi''_\alpha > 0$, $\mathcal{G}_\alpha$ has exactly the same graph as the 2D Laplacian, and thus is irreducible. In addition, $\mathcal{G}_\alpha$ has row sum being zero, except on rows corresponding to pixels on the boundary and in that case the row sum is strictly greater than zero. Hence $\mathcal{G}_\alpha$ is irreducibly diagonally dominant and so by Corollary 1.22 of [19], $\mathcal{G}_\alpha$ is positive definite.

Now, note that $\nabla^2\mathcal{F}_\alpha(\mathbf{u})$ is a principal sub-matrix of $\mathcal{G}_\alpha$, formed by deleting the rows and columns in $\mathcal{G}_\alpha$ corresponding to the pixels not in $\mathcal{N}$. Thus $\nabla^2\mathcal{F}_\alpha(\mathbf{u})$ is also positive definite.

We summarize the results below:

Theorem 1. *For any given* $\mathbf{u} \in \mathbb{R}^c$, *the matrix* $\nabla^2\mathcal{F}_\alpha(\mathbf{u})$ *defined in* (15) *is positive definite, i.e.,*

$$\lambda_{\min}(\nabla^2(\mathcal{F}_\alpha(\mathbf{u}))) > 0,$$

where $\lambda_{\min}(\nabla^2(\mathcal{F}_\alpha(\mathbf{u})))$ *is the minimal eigenvalue of* $\nabla^2(\mathcal{F}_\alpha(\mathbf{u}))$.

Theorem 2. *The functional* $\mathcal{F}_\alpha$ *given in* (4) *has only one local minimum which is also the global minimum. The global minimizer* $\mathbf{u}^*$ *of* $\mathcal{F}_\alpha$ *is always within the dynamic range, i.e.* $\mathbf{u}^* \in [s_{\min}, s_{\max}]^c$.

Proof. By Theorem 1, $\mathcal{F}_\alpha$ is strictly convex. Then, a local minimum of $\mathcal{F}_\alpha$ is also a global minimum and there exists at most one global minimum, see Proposition B.10 [1].

To show that the global minimum exists, consider the box

$$S = \{\mathbf{u} \in \mathbb{R}^{\mathbf{c}} \mid a \le u_i \le b,\ i = 1, \ldots, c\}$$

with $a < s_{\min}$ and $s_{\max} < b$. Since S is compact and $\mathcal{F}_\alpha$ is continuous and strictly convex, there exists the global minimizer $\mathbf{u}^* = \left(u^*_{i,j}\right)_{(i,j)\in\mathcal{N}}$ of $\mathcal{F}_\alpha$ over S. Now we show that $\mathbf{u}^*$ lies in the interior of S, and hence $\mathbf{u}^*$ is the global minimizer of $\mathcal{F}_\alpha$ over $\mathbb{R}^c$. To this end, note that if $\mathbf{u}^*$ belongs to the boundary of S, then there exists a point $\mathbf{u}$ in the interior of S with $\mathcal{F}_\alpha(\mathbf{u}) < \mathcal{F}_\alpha(\mathbf{u}^*)$. Indeed, we define

$$u_{i,j} = \begin{cases} s_{\max}, & s_{\max} < u^*_{i,j} \le b, \\ s_{\min}, & a \le u^*_{i,j} < s_{\min}, \\ u^*_{i,j}, & \text{otherwise.} \end{cases} \tag{16}$$

Then we have

$$\begin{cases} |u_{i,j} - u_{p,q}| \le |u^*_{i,j} - u^*_{p,q}|, & (p,q) \in \mathcal{V}_{i,j} \cap \mathcal{N}, \\ |u_{i,j} - y_{p,q}| \le |u^*_{i,j} - y_{p,q}|, & (p,q) \in \mathcal{V}_{i,j} \setminus \mathcal{N}. \end{cases} \tag{17}$$

Since at least one of the $u^*_{i,j}$ is on the boundary of S and all the $y_{p,q}$ are in $[s_{\min}, s_{\max}]$, we can conclude that at least one of the equalities in (17) is a strict inequality. Since $\mathcal{F}_\alpha$ is a sum of terms of the form $\varphi_\alpha(v - w)$ and $\varphi_\alpha(v - w)$ is strictly increasing w.r.t the difference $|v - w|$, $\mathcal{F}_\alpha(\mathbf{u}) < \mathcal{F}_\alpha(\mathbf{u}^*)$. Hence $\mathbf{u}^*$ cannot be the minimizer of (4) over S. Thus the minimizer $\mathbf{u}^*$ must be in the interior of S, and it is therefore also the global minimizer of $\mathcal{F}_\alpha$ in $\mathbb{R}^c$.

Finally, to show that $\mathbf{u}^* \in [s_{\min}, s_{\max}]^c$, we proceed as above. In particular, if some components of $\mathbf{u}^*$ are outside $[s_{\min}, s_{\max}]$, we define a new point $\mathbf{u}$ as in (16). Then again we will have a contradiction that $\mathcal{F}_\alpha(\mathbf{u}) < \mathcal{F}_\alpha(\mathbf{u}^*)$. □

Theorem 2 shows that the minimization problem can be viewed as an unconstrained minimization problem. Next we show that $\mathcal{F}_\alpha$ satisfies the Assumption 1.

Theorem 3. *Let $\{\mathbf{u}_k\}$ be the sequence generated by the conjugate gradient method. Then, the function $\mathcal{F}_\alpha$ defined in* (4) *satisfies* (12) *and* (13).

Proof. Since φ_α is continuous and coercive, $\mathcal{F}_\alpha(\mathbf{u}) \to \infty$ as $\|\mathbf{u}\| \to \infty$. To show this, we proceed by contradiction and suppose that $\mathcal{F}_\alpha(\mathbf{u})$ is bounded for $\|\mathbf{u}\| \to \infty$. Note that if there is one noisy pixel $|u_{i,j}| \to \infty$ having at least one non-noisy neighbor, then $S^1_{i,j} \to \infty$ and consequently $\mathcal{F}_\alpha(\mathbf{u}) \to \infty$. Therefore, if $\mathcal{F}_\alpha(\mathbf{u})$ is bounded for $\|\mathbf{u}\| \to \infty$ we conclude that for each noisy pixel $|u_{i,j}| \to \infty$ all its neighbors are noisy and tend to infinity at the same rate as $|u_{i,j}|$. Repeating this argument for each of such neighbors, we conclude that all the pixels are noisy, i.e. $\mathcal{A} \equiv \mathcal{N}$ which is impossible.

Since $\mathcal{F}_\alpha(\mathbf{u}) \to \infty$ as $\|\mathbf{u}\| \to \infty$, given the initial guess $\mathbf{u}_0$, the level set $\Delta = \{\mathbf{u} \mid \mathcal{F}_\alpha(\mathbf{u}) \le \mathcal{F}_\alpha(\mathbf{u}_0)\}$ must be bounded. Let $(u_0)_{k,l}$ be an arbitrary component of $\mathbf{u}_0$, and

$$z = \max\left\{ |(u_0)_{k,l}|, \max_{(i,j)\in\mathcal{V}_{k,l}} |(u_0)_{(i,j)}| \right\}.$$

Then we define a new vector $\mathbf{w}$ by replacing the entry $(u_0)_{k,l}$ by $w_{k,l} = 1+3z$. Then, for any neighbors v of $(u_0)_{k,l}$ we have

$$\begin{aligned} |(u_0)_{k,l} - v| &< 1 + (|v| - v) + |(u_0)_{k,l}| + |v| \\ &= 1 + |(u_0)_{k,l}| + 2|v| - v \le 1 + 3z - v = |w_{k,l} - v|, \end{aligned}$$

and consequently, $\mathcal{F}_\alpha(\mathbf{u}_0) < \mathcal{F}_\alpha(\mathbf{w})$. Therefore,

$$\Delta \subseteq \Omega \equiv \{\mathbf{u} \mid \mathcal{F}_\alpha(\mathbf{u}) < \mathcal{F}_\alpha(\mathbf{w})\}.$$

By the continuity of $\mathcal{F}_\alpha$, Ω is an open set and its closure is

$$\bar{\Omega} = \{\mathbf{u} \mid \mathcal{F}_\alpha(\mathbf{u}) \le \mathcal{F}_\alpha(\mathbf{w})\}.$$

Repeating the argument in the first paragraph of this proof, we see that the closure $\bar{\Omega}$ is also bounded. Moreover,

$$\|\nabla^2 \mathcal{F}_\alpha(\mathbf{u})\| \le \sup_{\mathbf{v}\in\bar{\Omega}} \|\nabla^2 \mathcal{F}_\alpha(\mathbf{v})\| = \max_{\mathbf{v}\in\bar{\Omega}} \|\nabla^2 \mathcal{F}_\alpha(\mathbf{v})\|, \quad \text{for all } \mathbf{u} \in \Omega,$$

since $\nabla^2 \mathcal{F}_\alpha(\mathbf{v})$ is a continuous function of $\mathbf{v}$ on the bounded and closed set $\bar{\Omega}$, hence the supremum can be attained in $\bar{\Omega}$. So by Theorem 9.19 of [16], we have the desired result (12) by taking $\mu = \max_{\mathbf{v} \in \bar{\Omega}} \|\nabla^2 \mathcal{F}_\alpha(\mathbf{v})\|$.

By Taylor's expansion on $\mathcal{F}_\alpha$, we have

$$\mathcal{F}_\alpha(\mathbf{u}) = \mathcal{F}_\alpha(\mathbf{v}) + \nabla \mathcal{F}_\alpha(\mathbf{v})^T (\mathbf{u} - \mathbf{v}) + \frac{1}{2} (\mathbf{u} - \mathbf{v})^T \nabla^2 \mathcal{F}_\alpha(\bar{\mathbf{u}}) (\mathbf{u} - \mathbf{v}), \quad (18)$$

and

$$\mathcal{F}_\alpha(\mathbf{v}) = \mathcal{F}_\alpha(\mathbf{u}) + \nabla \mathcal{F}_\alpha(\mathbf{u})^T (\mathbf{v} - \mathbf{u}) + \frac{1}{2} (\mathbf{v} - \mathbf{u})^T \nabla^2 \mathcal{F}_\alpha(\bar{\mathbf{v}}) (\mathbf{v} - \mathbf{u}), \quad (19)$$

where $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ lie on the line segment with end-points $\mathbf{u}$, $\mathbf{v} \in \Omega$. Adding up (18) and (19) and rearranging, we have

$$(\nabla \mathcal{F}_\alpha(\mathbf{u}) - \nabla \mathcal{F}_\alpha(\mathbf{v}))^T (\mathbf{u} - \mathbf{v}) = \frac{1}{2} (\mathbf{u} - \mathbf{v})^T \left(\nabla^2 \mathcal{F}_\alpha(\bar{\mathbf{u}}) + \nabla^2 \mathcal{F}_\alpha(\bar{\mathbf{v}})\right) (\mathbf{u} - \mathbf{v}).$$

Note that for a positive definite matrix A,

$$\mathbf{x}^T A \mathbf{x} \geq \lambda_{\min}(A) \mathbf{x}^T \mathbf{x} = \lambda_{\min}(A) \|\mathbf{x}\|^2,$$

where $\lambda_{\min}(A)$ is the smallest eigenvalue of A. Hence, together with the result of Theorem 1, we have:

$$\begin{aligned} & (\nabla \mathcal{F}_\alpha(\mathbf{u}) - \nabla \mathcal{F}_\alpha(\mathbf{v}))^T (\mathbf{u} - \mathbf{v}) \\ \geq \quad & \frac{1}{2} \left(\lambda_{\min}(\nabla^2 \mathcal{F}_\alpha(\bar{\mathbf{u}})) + \lambda_{\min}(\nabla^2 \mathcal{F}_\alpha(\bar{\mathbf{v}}))\right) \|\mathbf{u} - \mathbf{v}\|^2 \\ \geq \quad & \frac{1}{2} \cdot 2 \cdot \inf_{\mathbf{z} \in \bar{\Omega}} \lambda_{\min}(\nabla^2 \mathcal{F}_\alpha(\mathbf{z})) \|\mathbf{u} - \mathbf{v}\|^2 \\ = \quad & \lambda \|\mathbf{u} - \mathbf{v}\|^2, \end{aligned}$$

where $\lambda \equiv \inf_{\mathbf{z} \in \bar{\Omega}} \lambda_{\min}(\nabla^2 \mathcal{F}_\alpha(\mathbf{z}))$. Since $\lambda_{\min}(\nabla^2 \mathcal{F}_\alpha(\mathbf{z}))$ is a continuous function of $\mathbf{z}$ on the closed and bounded set $\bar{\Omega}$ (see Corollary 4.10 in [17]), we have $\lambda = \lambda_{\min}(\nabla^2 \mathcal{F}_\alpha(\mathbf{z}_0))$ for some $\mathbf{z}_0 \in \bar{\Omega}$. By Theorem 1, $\lambda > 0$. This proves (13). □

We conclude by providing a global convergence result of the CG method applying to (4).

Theorem 4. *Let $\{\mathbf{u}_k\}$ be the sequence generated by the conjugate gradient method with α_k given in* (14). *Then, for any choice of β_k in* (7)–(11), *$\{\mathbf{u}_k\}$ converges to the global minimum of $\mathcal{F}_\alpha$.*

Proof. By Theorem 9 of [18], $\lim_{k \to \infty} \|\nabla \mathcal{F}_\alpha(\mathbf{u}_k)\| = 0$. Hence, all the limit points of $\{\mathbf{u}_k\}$ are stationary points of $\mathcal{F}_\alpha$. By Theorem 2, the thesis follows. □

5 Simulation

Throughout the simulations, we use MATLAB 7.01 (R14) on a PC equipped with Intel Pentium 4 CPU 3.00 GHz and 1,024 MB RAM memory. Our test images are the 512-by-512 *goldhill* and *lena* images. To assess the restoration performance qualitatively, we use the PSNR (peak signal to noise ratio, see [2]) defined as

$$\text{PSNR} = 10\,\log_{10} \frac{255^2}{\frac{1}{MN}\sum_{i,j}(x^r_{i,j} - x_{i,j})^2},$$

where $x^r_{i,j}$ and $x_{i,j}$ denote the pixel values of the restored image and the original image respectively.

We emphasize that in this paper, we are concerned with the speed of solving the minimization problem in the second phase of the two-phase method, i.e. minimizing the functional $\mathcal{F}_\alpha$. We report the time required for the whole denoising process and the PSNR of the recovered image. In order to test the speed of the algorithms more fairly, the experiments are repeated 10 times and the average of the 10 timings is given in the tables. The stopping criteria of the minimization phase is set

$$\frac{\|\mathbf{u}^k - \mathbf{u}^{k-1}\|}{\|\mathbf{u}^k\|} \le 10^{-4} \quad \text{and} \quad \frac{|\mathcal{F}_\alpha(\mathbf{u}^k) - \mathcal{F}_\alpha(\mathbf{u}^{k-1})|}{\mathcal{F}_\alpha(\mathbf{u}^k)} \le 10^{-4}.$$

The potential function is $\varphi_\alpha(t) = \sqrt{t^2+\alpha}$ with $\alpha = 100$.

For the conjugate gradient type method, we choose Q_k in (14) to be the identity matrix. To choose μ in Assumption 1, we must have $\mu \ge \max_{\mathbf{v}\in\bar{\Omega}} \|\nabla^2\mathcal{F}_\alpha(\mathbf{v})\|$. By (15) and the fact that $\nabla^2\mathcal{F}_\alpha(\mathbf{v})$ is symmetric, we have

$$\|\nabla^2\mathcal{F}_\alpha(\mathbf{v})\| \le \|\nabla^2\mathcal{F}_\alpha(\mathbf{v})\|_\infty \le 16\sup_t \varphi''_\alpha(t), \quad \forall \mathbf{v}\in\bar{\Omega}.$$

Therefore, we choose

$$\mu = 16\sup_t \varphi''_\alpha(t) = \frac{16}{\sqrt{\alpha}},$$

and hence δ in (14) is chosen as

$$\delta = \frac{\sqrt{\alpha-1}}{16} = \frac{\sqrt{99}}{16} < \frac{1}{\mu} = \frac{\sqrt{\alpha}}{16} = \frac{5}{8}.$$

In Table 1, we compare the five nonlinear CG type methods defined in (7) – (11), which are denoted by FR, PR, HS, CD and DY respectively. We see that PR is the most efficient one among the five methods. Therefore, we take PR as a representative of the CG type methods in the following tests. Next, we show the advantages of PR method over the 1D relaxation method applied to the functional (1) as discussed in [4]. The results are given in Table 2. One sees from Table 2 that the CG type method is faster than the relaxation method when the noise ratio is larger than 50% for both test images. When the noise ratio is 90%, the CG method is about three times faster than the relaxation-based method, i.e. about 60%–70% saving in CPU time. Moreover, we note that the PSNR values attained by the minimizers of (1) and (4) are almost exactly the same.

Table 1. Comparison of the conjugate gradient type methods for *goldhill* image

Noise Ratio	Time					PSNR
	FR	PR	HS	CD	DY	
30%	39.0	28.9	30.6	29.8	44.7	36.0
50%	58.4	43.4	44.8	44.3	67.4	32.7
70%	77.7	59.3	60.3	60.4	90.5	29.8
90%	184	152	153	153	217	26.1

Table 2. Comparison of the conjugate gradient type method with the relaxation-based method

Noise	*goldhill*				*lena*			
	Relaxation		PR		Relaxation		PR	
Ratio	Time	PSNR	Time	PSNR	Time	PSNR	Time	PSNR
30%	35.5	36.0	28.9	36.0	35.7	36.4	49.2	36.5
50%	71.7	32.7	43.4	32.7	85.4	32.9	78.3	33.0
70%	130	29.8	59.3	29.8	133	29.7	81.1	29.8
90%	453	26.1	152	26.1	500	25.3	185	25.4

Finally, Figures 1 and 2 show the results obtained by (i) the adaptive median filter (AMF), (ii) the two-phase schemes solved by 1D relaxation [4], and (iii) the two-phase schemes solved by the conjugate gradient method.

6 Conclusion

In this paper, we give an efficient CG algorithm to minimize the regularization functional in the two-phase impulse removal proposed in [4]. In its original form, the regularization functional is not differentiable because of its non-smooth data-fitting term. We modify it by removing the data-fitting term. Then an efficient CG method, where the line search rule is replaced by a pre-determined step length strategy, is applied to minimize the new functional. Based on the results in [18], global convergence of the algorithm is established. This variant of the two-phase method gives an output having the same visual

(a) (b) (c)

(d) (e)

Fig. 1. Restoration results of different algorithms: (a) Original *Goldhill* image, (b) Corrupted *Goldhill* image with 70% salt-and-pepper noise (6.9 dB), (c) Adaptive median fiter (26.1 dB), (d) Two-phase method with relaxation (29.8 dB), and (e) Two-phase method with conjugate gradient using (8) for β_k (29.8 dB).

quality as the original method. With slight modification, the CG algorithm can also be applied equally well to random-valued impulse noise (cf. [5]).

Regarding future research directions, we note that in the CG algorithm we are allowed to select a sequence of $\{Q_k\}$ (see (14)) and they are chosen to be the identity in our computations. It would be interesting to define $\{Q_k\}$ according to the Hessian of the objective functional, or further, to perform some preconditioning for the CG algorithm. Preconditioning is not straightforward as the Hessian does not have any special structure. Also here the second order derivative of $\varphi_\alpha(t)$ is only required in the convergence analysis and not in the computation. One may hope to relax the twice continuously differentiable assumption on $\varphi_\alpha(t)$ to only continuously differentiable. This may extend the method to more potential functions such as $\varphi_\alpha(t) = |t|^{1+\epsilon}, \epsilon > 0$, which is known to produce better restored images.

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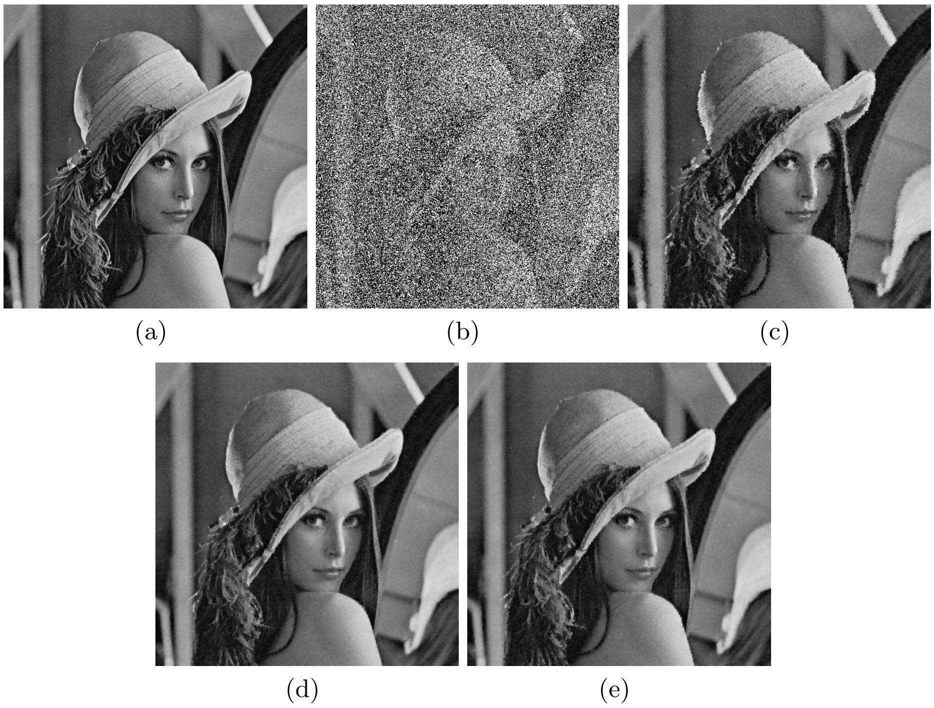

Fig. 2. Restoration results of different algorithms: (a) Original *Lena* image, (b) Corrupted *Lena* image with 70% salt-and-pepper noise (6.7 dB), (c) Adaptive median fiter (25.8 dB), (d) Two-phase method with relaxation (29.7 dB), and (e) Two-phase method with conjugate gradient using (8) for β_k (29.8 dB).

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