Upper/Lower Limits

Let (x_n) be a bounded seq.

(*) $\limsup_n x_n := \inf\{\sup\{x_m : m \geq n\} : n \in \mathbb{N}\}$

where "inf" and "sup" on the RHS do exist in $\mathbb{R}$ as (x_n) is bounded. Let $a, b \in \mathbb{R}$ be s.t. $a \leq x_n \leq b \ \forall n$;

let $T_n := \{x_m : m \geq n\}$ (so $T_n \downarrow_n$)

$t_n := \sup T_n \leq b$ (so $t_n \downarrow_n$ and $a \leq t_n \leq b \ \forall n$)

Thus $\limsup_n x_n := \inf\{t_n : n \in \mathbb{N}\} = \lim_n t_n$

by the MCT.

<u>Th</u> 1. Let (x_n) be bounded and $l \in \mathbb{R}$. $\circlearrowright$:

(i) $l = \limsup_n x_n$

(ii) l has the properties that, $\forall \varepsilon > 0$,

(a) $x_n < l + \varepsilon$ holds eventually in the sense that $\exists N \in \mathbb{N}$ such that $x_n < l + \varepsilon \ \forall n \geq N$; (#)

(b) $l - \varepsilon < x_n$ holds frequently in the sense that

$l - \varepsilon < x_n$ for infinitely many n

i.e. $\forall N \in \mathbb{N} \ \exists n > N$ s.t.

$l - \varepsilon < x_n$. (##)

Proof. Nothing but from definitions. Let us show (ii) $\Rightarrow$ (i) while that for (i) $\Rightarrow$ (ii) is left as exercize.

Let N be as in (ii a). Then, by (#),

$$\sup\{x_n : n \geq N\} \leq l + \varepsilon$$

and it follows from (*) that $\limsup_n x_n \leq l + \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, this implies that $\limsup_n x_n \leq l$.

On the other hand, by (ii b), one has from (##) that

$$l - \varepsilon < x_n \leq \sup\{x_m : m \geq N\}$$

for all $N \in \mathbb{N}$. Taking infimum over N, it follows from (*) that

$$l - \varepsilon \leq \limsup_n x_n$$

and consequently $l \leq \limsup_n x_n$ as $\varepsilon > 0$ is arbitrary.

Th 2. Let (x_n) be as in Th 1, and let

$$S := \{x \in \mathbb{R} : \exists \text{ a subseq } x_{n_k} \to x \ \underset{(\text{of } (x_n))}{} \}$$

Then

$$\limsup_n x_n = \max S, \text{ the largest ele of } S$$

Proof of Th2. Let $l := \limsup_n x_n$. Applying Th 1 (ii) with $\varepsilon := 1$, $\exists N_1 \in \mathbb{N}$ s.t.

$$x_n < l+1 \quad \forall n \geq N_1$$

and, for this N_1, $\exists\ n_1 > N_1$ s.t.

$$l-1 < x_{n_1}$$

and consequently

$$l-1 < x_{n_1} < l+1 .$$

Similarly, $\exists\ N_2 \in \mathbb{N}$ s.t.

$$x_n < l + \frac{1}{2} \quad \forall n \geq N_2$$

and, for this $\max\{N_2, n_1\}$, $\exists$ $n_2 > N_2, n_1$ s.t.

$$l - \frac{1}{2} < x_{n_2}$$

and consequently

$$l - \frac{1}{2} < x_{n_2} < l + \frac{1}{2} .$$

Inductively, we have $n_1 < n_2 < n_3 < \cdots$, such that

$$l - \frac{1}{k} < x_{n_k} < l + \frac{1}{k} \quad \forall k .$$

This implies that (x_{n_k}) is a subsequence of (x_n) with limit l; thus $\limsup_n x_n \in S$.

Conversely, let (x_{n_k}) be a convergent subseq of (x_n) with limit x. Let T_m denote the m-tail of (x_n), and $t_m = \sup T_m$ one has

$$x_{n_k} \leq \sup T_{n_k} \leq \sup T_k = t_k$$

(as $T_{n_k} \subseteq T_k$ since $n_k \geq k$) it follows by passing to the limits (as $k \to +\infty$) that

$$x \leq \lim_k t_k = \limsup_n x_n .$$

QED.

Ex. Do corresponding results for $\liminf x_n$

$$\stackrel{\text{def}}{=} \sup\{\inf T_n : n \in \mathbb{N}\}$$

$$= \lim_{n\to\infty} \left(\inf\{x_m : m \geq n\} \right),$$ limit of increasing seq.