THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2050 (First Term
Mathematical Analysis I
Homework IV

Questions with * will be marked.

1. Let $a > 0$ and $z_1 > 0$, $z_{n+1} := \sqrt{a + z_n}$ for all $n \in \mathbb{N}$.
   Show that the sequence is bounded (Hint: Let $\xi := \max\{1, z_1\}$. Then $z_n \leq \xi + \sqrt{a}$ for all n).
   Show that the sequence converges and find the limit.

2. Find the limits if exist:

   (i) $\lim\limits_{n\to\infty} \left(2 + \frac{1}{n}\right)^2$;

   (ii) $\lim\limits_{n\to\infty} \frac{(-1)^n}{\sqrt{n}}$;

   (iii) $\lim\limits_{n\to\infty} \frac{\sqrt{n}-1}{\sqrt{n}+2}$;

   (iv) $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n+1}$;

   (v) $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^{2n}$;

   (vi) $\lim\limits_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$.

3. Let

$$x_n := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \text{ for all } n \in \mathbb{N}.$$

   Show that (x_n) is monotone and bounded.
   Is the following argument valid:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n+1} + \lim_{n\to\infty} \frac{1}{n+2} + \cdots + \lim_{n\to\infty} \frac{1}{n+n} = 0?$$

4.* Show that $\lim x_n = x \in \mathbb{R}$ if and only if every subsequence of (x_n) has in turn a subsequence that converges to x.

5.* Let $x \in \mathbb{R}$ and (x_n) be a bounded sequence. Suppose (x_n) does not converge to x. Show that there exists a subsequence of (x_n) converges to some $x' \neq x$.