MATH2050B 1920 Quiz 1
TA's solutions[1] to selected problems

Q1. Give definition and its negation for each of the following:

(i) (x_n) converges to $x (\in \mathbb{R})$

(ii) (x_n) converges to $-\infty$

(iii) (x_n) is Cauchy (a Cauchy sequence)

Solution.

(i) **Definition:** $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n > N$.
Negation: $\exists \epsilon > 0$ such that $\forall N \in \mathbb{N}$, $\exists n > N$ with $|x_n - x| \geq \epsilon$.

(ii) **Definition:** $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $x_n < M$ for all $n > N$.
Negation: $\exists M \in \mathbb{R}$ such that $\forall N \in \mathbb{N}$, $\exists n > N$ with $x_n \geq M$.

(iii) **Definition:** $\forall \epsilon > 0$, $\exists N \in \mathbb{R}$ such that $|x_n - x_m| < \epsilon$ for all $n, m > N$.
Negation: $\exists \epsilon > 0$ such that $\forall N \in \mathbb{N}$, $\exists n, m > N$ with $|x_n - x_m| \geq \epsilon$.

Q2. State (without proof):

(i) Monotone Convergence Theorem

(ii) Order-Preserving Theorem and Squeeze Theorem for sequences

(iii) Nested Interval Theorem

(iv) Bolzano-Weierstrass Theorem

(v) Cauchy Criterion Theorem

Solution. Let (x_n), (y_n), (z_n) be sequences of real numbers

(i) Suppose that (x_n) is bounded and is monotone, then (x_n) is convergent. Moreover if (x_n) is increasing, then $\lim x_n = \sup\{x_n : n = 1, 2, \dots\}$. (respectively for decreasing sequence)

(ii) (Order-preserving theorem)Suppose that $(x_n)\,(y_n)$ are convergent and $x_n \leq y_n$ for all n. Then $\lim x_n \leq \lim y_n$.

(Squeeze Theorem)Suppose that (x_n) and (y_n) are convergent with $x_n \leq z_n \leq y_n$ and $\lim x_n = \lim y_n = L$. Then z_n is convergent and $\lim z_n = L$.

(iii) Suppose that I_n's are intervals, $I_n = [a_n, b_n]$ such that $I_1 \supset I_2 \supset I_3 \supset \dots$, then $\bigcap I_n \neq \varnothing$.

(iv) A bounded sequence has a convergent subsequence.

(v) A sequence is convergent iff it is Cauchy.

[1] please kindly send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

Q3. Use the definition in ϵ-N terminology, show:

(i) If $\lim x_n = x \in \mathbb{R}$, $\lim y_n = y$ $(y_n, y \in \mathbb{R} \setminus \{0\} \forall n)$then

$$\lim \frac{x_n}{y_n} = \frac{x}{y}.$$

(ii) If $\lim a_n = 3$, then $\lim \frac{a_n^2+1}{a_n-2} = 10$.

Solution. (i): We claim that there is N so that $|\frac{1}{y_n}| \leq |\frac{2}{y}|$ for all $n > N$.

Since $|\frac{y}{2}| > 0$, so there is N such that $|y_n - y| < |\frac{y}{2}|$ for all $n > N$. By triangle inequality, $|y| - |y_n| \leq |y_n - y| \leq |\frac{y}{2}|$. So for all $n > N$:

$$|\frac{y}{2}| \leq |y_n|.$$

Hence the claim is proved. Next we claim that x_n/y_n is convergent to x/y:

Let $\epsilon > 0$, by convergence of (x_n) and (y_n), there is $M \in \mathbb{N}$ such that for all $n > M$:

$$|x_n - x| < \frac{\epsilon}{2}\frac{|y|}{2}$$

and

$$|y_n - y| < \frac{\epsilon}{2}\frac{|y|^2}{2|x|+1}.$$

Hence for all $n > \max(N, M)$:

$$\begin{aligned}
|\frac{x_n}{y_n} - \frac{x}{y}| &= |\frac{x_n y - x y_n}{y_n y}| \\
&= |\frac{x_n y - xy + xy - x y_n}{y_n y}| \\
&\leq \frac{|y| \cdot |x_n - x|}{|y_n y|} + \frac{|x| \cdot |y_n - y|}{|y_n y|} \\
&\leq \frac{2}{|y|}|x_n - x| + \frac{2|x|}{|y|^2}|y_n - y| \\
&< \epsilon.
\end{aligned}$$

(ii): First observe

$$|\frac{a_n^2+1}{a_n-2} - 10| = |\frac{a_n^2 - 10a_n + 21}{a_n - 2}| = |\frac{(a_n-7)(a_n-3)}{a_n-2}|.$$

Because $a_n \to 3$, so $a_n - 2 \to 1$. (Easy to see using definition) By a similar argument as in (i), there is N so that for all $n > N$:

$$\frac{1}{|a_n - 2|} \leq \frac{2}{1} = 2.$$

Because (a_n) is convergent, it is also bounded, say $M > 0$ is a bound for (a_n), i.e. $-M < a_n < M$ for all n.

We show that $\frac{a_n^2+1}{a_n-2}$ converges to 10:

Let $\epsilon > 0$. Consider the positive number $\epsilon\frac{1}{2(M+7)}$. Then there is N_1 such that

$$|a_n - 3| < \epsilon\frac{1}{2(M+7)}.$$

Now, for all $n > \max(N_1, N)$:

$$\begin{aligned} |\frac{a_n^2+1}{a_n-2} - 10| &= |\frac{(a_n-7)(a_n-3)}{a_n-2}| \\ &\leq \frac{(|a_n|+7)|a_n-3|}{|a_n-2|} \\ &< 2(M+7)\frac{\epsilon}{2(M+7)} = \epsilon. \end{aligned}$$

Q4.

(i) Let $a_n = n^{1/n}$ (so $a_n^n = n$). Find the limit of (a_n). (Hint. Set $b_n = n^{1/n} - 1$ and make use of the Binomial Theorem)

(ii) Let $x_{n+1} = \frac{1}{3}x_n + 5\,\forall n \in \mathbb{N}$ and let $x_1 = 3$. Show that $\lim x_n$ exists and find its value.

Solution. (i): Let $b_n = a_n - 1$. Then $b_n \geq 0$ for all n and for all $n \geq 2$:

$$n = (b_n + 1)^n \geq \binom{n}{2}b_n^2 = \frac{n(n-1)}{2}b_n^2,$$

so that

$$\sqrt{\frac{2}{n-1}} \geq b_n \geq 0.$$

Apply the Squeeze Theorem (see **Q2**) to $x_n = 0$, $z_n = b_n$, $y_n = \sqrt{2/(n-1)}$, we get $\lim b_n$ exists and equals 0. Hence $a_n = b_n + 1$ converges to 1.

(ii): We are going to show by induction that $x_n < 15/2$ and x_n is increasing.

One finds that $x_2 = 6$. So $x_2 < 15/2$ and $x_2 - x_1 > 0$. Suppose that $x_N < 15/2$ and $x_N > x_{N-1}$, then

$$x_{N+1} = \frac{1}{3}x_N + 5 \leq \frac{1}{3}\frac{15}{2} + 5 = \frac{15}{2},$$

and

$$x_{N+1} - x_N = 5 - \frac{2}{3}x_N > 5 - \frac{2}{3}\frac{15}{2} = 0.$$

Hence the claim is proved by induction. Since (x_n) is bounded and monotone, by Monotone Convergence Theorem, (x_n) is convergent.

Let $x = \lim x_n$. Since $x_{n+1} = \frac{1}{3}x_n + 5$, so

$$x = \frac{1}{3}x + 5$$

and hence $x = 15/2$.