MATH2050B 1920 HW4

TA's solutions to selected problems

Q1. Let $x(1) = 8$ and $x(n+1) := 2 + x(n)/2$. Show the sequence is decreasing and positive and hence converges. Find the limit.

Solution. We are going to show by induction that $x(n)$ is decreasing and $x(n) > 4$ for every n.

One computes $x(2) = 6$. So $x(1) \geq x(2) \geq 4$. Suppose now $x(1) \geq x(2) \geq \cdots \geq x(N) > 4$. One computes $x(N+1) = 2 + x(N)/2 > 2 + 4/2 = 4$, and $x(N) - x(N+1) = x(N)/2 - 2 > 0$. By MI we conclude $x(n)$ is decreasing and $x(n) > 4$ for every n.

The sequence $x(n)$ is decreasing and bounded below, hence (by MCT) is convergent. Let $x = \lim_{n\to\infty} x(n)$. Using the relation $x(n+1) = 2 + x(n)/2$ we have $x = 2 + x/2$, so $x = 4$.

Q2. Let $f(x) = 2 - 1/x$ for each positive $x > 1$. Show that $f(x) < x$ and hence that the sequence $x(n)$ defined in Q2 of Section 3.3 is decreasing to limit 1: $x(1) > 1$ and $x(n+1) = 2 - 1/x(n)$.

Solution. Since $x + \frac{1}{x} - 2 = (\sqrt{x} - \frac{1}{\sqrt{x}})^2 > 0$ for $x > 1$, so $f(x) < x$. Using this, we see that $x(n+1) \leq x(n)$ for all n, so that $x(n)$ is decreasing.

We claim that $x(n)$ is bounded below by 1: $x(1) > 1$ is by definition. Suppose that $x(1), \ldots, x(N) > 1$, then $1/x(N) < 1$, so $x(N+1) = 2 - 1/x(N) > 2 - 1 = 1$. The claim is concluded by MI.

By MCT $x(n)$ is convergent to a real number x. Using $x(n+1) = 2 - 1/x(n)$, we have $x = 2 - 1/x$. This gives $x = 1$.

Q3. Let $g(x)$ be defined by

$$g(x) = 1 + \sqrt{x-1}, \qquad \forall x \in [2, \infty).$$

Show that $g(x)$ is dominated by x and that the sequence $x(n)$ defined by $x(1) \in [2, \infty)$ and $x(n+1) = g(x(n))$ is decreasing. Find its limit.

Solution. The sentence "$g(x)$ is dominated by x" means "$g(x) \leq x$" (where $x \in [2, \infty)$). So we need to show

$$1 + \sqrt{x-1} \leq x, \qquad \forall x \geq 2.$$

This inequality holds iff $\sqrt{x-1} \leq x - 1$ for all $x \geq 2$ iff $x - 1 \leq (x-1)^2$ for all $x \geq 2$ iff $y \leq y^2$ for all $y \geq 1$. Because the inequality "$y \leq y^2$ for all $y \geq 1$" holds, therefore $g(x) \leq x$, for all $x \geq 2$.

Let $x(1)$ be any number ≥ 2. Then $x(2) = g(x(1)) \leq x(1)$ because $g(x) \leq x$. Inductively $x(n+1) \leq x(n)$ for all n, so that $x(n)$ is decreasing. Notice the range of g is $[2, \infty)$, so $x(n)$ is bounded below by 2. By MCT $x(n) \to x$ for some real x. Using $x(n+1) = 1 + \sqrt{x(n) - 1}$ we have

$$x = 1 + \sqrt{x-1}.$$

Solving this equation gives $x = 2$.

Q4. Let $x_1 = 1$ and $x_{n+1} = \sqrt{2 + x_n}$ for all n. Show that x_n is increasing and bounded above by $1 + \sqrt{2}$. Find the limit if exists.

Solution. It is clear that $\sqrt{3} = x_2 \geq x_1 = 1$. Suppose that $x_N \geq x_{N-1}$. Then $x_N + 2 \geq x_{N-1} + 2$, and so $x_{N+1} \geq x_N$. By MI, x_n is increasing.

To show x_n is bounded by $1+\sqrt{2}$, note x_1 is clearly bounded by $1+\sqrt{2}$. Suppose that $x_N \le 1+\sqrt{2}$, then

$$x_N + 2 \le 1+\sqrt{2}+2 \le (1+\sqrt{2})^2,$$

so that $x_{N+1} \le 1+\sqrt{2}$. By MI $x_n \le 1+\sqrt{2}$ for all n.

Q5. Let $p > 0$ and $y_1 = \sqrt{p}, y_{n+1} = \sqrt{p+y_n}$. Show that y_n is increasing and bounded by $1+\sqrt{p}$. Find the limit if exists.

Solution. The method is the same as **Q4**.

Q6. Let $a, z_1 > 0$, and $z_{n+1} = \sqrt{a+z_n}$ for all n. Show that

$$z_n \le \max\{1, z_1\} + \sqrt{a}, \qquad \forall n \in \mathbb{N}.$$

Show that z_n is monotone, so that the limit exists.

Solution. Let $M = \max\{1, z_1\}$. Then $z_1 \le M + \sqrt{a}$. Suppose that $z_N \le M + \sqrt{a}$. Then

$$z_N \le M + \sqrt{a},$$

so that

$$z_N + a \le M + a + \sqrt{a} \le (M + \sqrt{a})^2.$$

Therefore $z_{N+1} \le M + \sqrt{a}$. By MI, $z_n \le M + \sqrt{a}$ for all n.

To show z_n is monotone,

Case 1. $z_1 \ge z_2$. In this case, suppose that $z_{N-1} \ge z_N$. Then

$$z_{N-1} + a \ge z_N + a$$

so that $z_N \ge z_{N+1}$. By MI z_n is decreasing.

Case 2. $z_1 \le z_2$. Similar to **Case 1**, z_n is increasing in this case.

Q7. Let $x_1 = a > 0$, and $x_{n+1} = x_n + \frac{1}{x_n}$. Then x_n is increasing, with $x = \lim x_n \le \infty$. Show that $x = \infty$.

Solution. Note that $x_2 = x_1 + \frac{1}{x_1} \ge x_1$, and in general $x_{n+1} = x_n + \frac{1}{x_n} \ge x_n$. So x_n is increasing by MI.

The set $\{x_n : n = 1, 2, \dots\}$ is either bounded or unbounded. If it is bounded, then $x = \lim x_n < \infty$. If it is unbounded, then $x = \infty$. So $x \le \infty$.

If x is finite. Since $0 < x_1 \le x$, so $x \ne 0$. Take limit on both sides in $x_{n+1} = x_n + \frac{1}{x_n}$,

$$x = x + \frac{1}{x}.$$

This will give a contradiction. So $x = \infty$.

Q8. Let $\varnothing \ne A \subset \mathbb{R}$, bounded with $x := \sup A \in \mathbb{R}$. Show that there is a sequence (x_n) in A such that $\lim_n x_n = x$. Moreover, if $x \notin A$ show that you can have your (x_n) satisfying additionally that $x_n < x_{n+1}$ for all n.

Solution. If $x \in A$, we simply take $x_n = x$ for all n.

Assume $x \notin A$.

Let $\delta_0 > 0$. There is $x_1 \in A$ so that

$$x > x_1 > x - \delta_0.$$

Because $x \notin A$, so the number $\delta_1 := \min(x - x_1, \frac{1}{1})$ is positive.

There is $x_2 \in A$ so that

$$x > x_2 > x - \delta_1.$$

Note that $x_2 > x_1$. The number $\delta_2 := \min(x - x_2, \frac{1}{2})$ is positive, so there is $x_3 \in A$ so that

$$x > x_3 > x - \delta_2.$$

Note $x_3 > x_2$. To repeat the same step, we can inductively define x_n so that x_n is strictly increasing and $x - x_n \leq \frac{1}{n}$. This shows $x_n \to x$.

Q9. Let (a_n) be a bounded sequence, and

$$t_n = \inf\{a_m : m \geq n\} = \inf\{a_n, a_{n+1}, a_{n+2}, \dots\},$$

$$s_n = \sup\{a_m : m \geq n\} = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}.$$

Show that $(t_n), (s_n)$ are monotone and

$$\lim_n t_n = \sup\{t_n : n \in \mathbb{N}\} \leq \inf\{s_k : k \in \mathbb{N}\} = \lim_k s_k.$$

Solution. For each n, $t_n \leq a_m$ for all $m \geq n$. In particular, $t_n \leq a_m$ for all $m \geq n+1$, and so

$$t_n \leq \inf\{a_m : m \geq n+1\} = t_{n+1}.$$

This shows that (t_n) is increasing.

For each n, $s_n \geq a_m$ for all $m \geq n$. In particular, $s_n \geq a_m$ for all $m \geq n+1$, and so

$$s_n \geq \sup\{a_m : m \geq n+1\} = s_{n+1}.$$

This shows that (s_n) is decreasing.

Since (a_n) is bounded, so (t_n) and (s_n) are also bounded. By MCT, $\lim_n t_n$ exists, $\lim_n t_n = \sup\{t_n : n \in \mathbb{N}\}$, and $\lim_n s_n$ exists, $\lim_n s_n = \inf\{s_n : n \in \mathbb{N}\}$. Because $t_n \leq s_n$ for all n, it follows also $\lim_n t_n \leq \lim_n s_n$.

Q10. Let $(a_n), (t_n), (s_n)$ be as in **Q9**. Show that (a_n) converges iff $\lim_n t_n = \lim_n s_n$.

($\lim_n t_n$ is usually denoted by $\liminf_n a_n$. $\lim_n s_n$ is usually denoted by $\limsup_n a_n$.)

Solution. ($\Rightarrow$)Suppose that (a_n) is convergent to $a \in \mathbb{R}$. We claim that $\lim_n t_n = \lim_n s_n = a$.

Let $\epsilon > 0$, then there is N so that $|a_n - a| < \epsilon$ for all $n \geq N$. This is to say for all $n \geq N$,

$$a - \epsilon < a_n < a + \epsilon.$$

It follows that for all $n \geq N$, $a - \epsilon \leq t_n \leq a + \epsilon$ and $a - \epsilon \leq s_n \leq a + \epsilon$. This implies that for all $n \geq N$, $|t_n - a| \leq \epsilon$ and $|s_n - a| \leq \epsilon$, which shows the claim.

($\Leftarrow$) Suppose that $\lim_n t_n = \lim_n s_n = a \in \mathbb{R}$. We claim that a_n converges to a. Let $\epsilon > 0$. Then there is N so that for all $n \geq N$,

$$a - \epsilon < t_n < a + \epsilon$$

and

$$a - \epsilon < s_n < a + \epsilon.$$

This two conditions imply that for all $n \geq N$, $a - \epsilon < a_n < a + \epsilon$. Hence $\lim_n a_n$ exists and equals a.