THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 9 (April 8)

Definition. Let $f : [a, b] \to \mathbb{R}$, and let $\pi = \{a = x_0 < x_1 < \cdots < x_n = b\} \in \operatorname{Par}[a, b]$. Define

$$t(f; \pi) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

and the **total variation** of f over $[a, b]$ by

$$T_a^b(f) := \sup\{t(f; \pi) : \pi \in \operatorname{Par}[a, b]\}.$$

We also set $T_a^a(f) := 0$. Similarly, we define the **positive** and **negative variation** $P_a^b(f), N_a^b(f)$ by replacing $|\cdot|$ with $[\cdot]^+$ and $[\cdot]^-$, respectively.

Proposition 1. Let $f, g \colon [a, b] \to \mathbb{R}$. Then

(a) $T_a^b(f) = T_a^c(f) + T_c^b(f)$ for any $c \in [a, b]$;

(b) $x \mapsto T_a^x(f)$ is non-negative and increasing on $[a, b]$;

(c) $T_a^b(f + g) \leq T_a^b(f) + T_a^b(g)$.

Corresponding results hold for $P_a^x(f)$ and $N_a^x(f)$.

Proposition 2. If $f \colon [a, b] \to \mathbb{R}$, then $T_a^b(f) = P_a^b(f) + N_a^b(f)$.

Definition. We say that a function $f \colon [a, b] \to \mathbb{R}$ is of **bounded variation** over $[a, b]$ if $T_a^b(f) < +\infty$. In symbol, we write $f \in \operatorname{BV}[a, b]$.

Proposition 3. If $f \in \operatorname{BV}[a, b]$, then $f(b) - f(a) = P_a^b(f) - N_a^b(f)$. In particular, any function of bounded variation can be written as a difference of two increasing functions.

Theorem 1. *Let $f \colon [a, b] \to \mathbb{R}$ be monotone increasing. Then*

(a) $f'(x)$ exists a.e.;

(b) f' is measurable;

(c) $f' \in \mathcal{L}[a, b]$ with $\displaystyle\int_a^b f' \leq f(b) - f(a)$.

Remark. The last inequality can be strict. For example, consider the Cantor function.

Corollary. *If $f \in \operatorname{BV}[a, b]$, then $f'(x)$ exists a.e. and $f' \in \mathcal{L}[a, b]$.*

Example 1. Let $f \in \mathrm{BV}[a,b]$. Show that $\displaystyle\int_a^b |f'| \le T_a^b(f)$.

Solution. It suffices to show that $\int_a^b (f')^+ \le P_a^b$ and $\int_a^b (f')^- \le N_a^b(f)$. By Proposition 3, $f(x) - f(a) = P_a^x(f) - N_a^x(f) \triangleq P_a^x - N_a^x$ for all $x \in [a,b]$. Since both P_a^x and N_a^x are increasing, it follows from Theorem 1 that $(P_a^x)'$, $(N_a^x)'$ exist a.e. and hence $f' = (P_a^x)' - (N_a^x)'$ a.e. Since $(a-b)^+ \le a^+ + b^-$ and $(a-b)^- \le a^- + b^+$, we have

$$\begin{cases} (f')^+ \le [(P_a^x)']^+ + [(N_a^x)']^- = (P_a^x)' \\ (f')^- \le [(P_a^x)']^- + [(N_a^x)']^+ = (N_a^x)' \end{cases} \quad \text{a.e.}$$

Theorem 1 now yields $\int_a^b (f')^+ \le \int_a^b (P_a^x)' \le P_a^b$ and $\int_a^b (f')^- \le \int_a^b (N_a^x)' \le N_a^b$. ◀

Definition. A function $f\colon [a,b] \to \mathbb{R}$ is said to be **absolutely continuous** on $[a,b]$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum\limits_{i=1}^{n} |f(x_i) - f(x_i')| < \varepsilon$ whenever $\{(x_i, x_i')\}_{i=1}^n$ is a finite collection of non-overlapping intervals such that $\sum\limits_{i=1}^{n} |x_i - x_i'| < \delta$. In this case, we write $f \in \mathrm{ABC}[a,b]$.

Remark. We can replace "finite" by "countable".

Example 2. Let $f \in \mathrm{ABC}[a,b]$. Then

(a) f is continuous on $[a,b]$.

(b) f is of bounded variation over $[a,b]$;

(c) f has the Luzin N property: f maps sets of measure zero to sets of measure zero.

Remark. In fact, a function satisfying all three conditions is absolutely continuous. This is Banach-Zarecki Theorem.

Solution. (a) It is clear from that that definition that if $f \in \mathrm{ABC}[a,b]$, then f is uniformly continuous on $[a,b]$.

(b) Let $\varepsilon = 1$. Choose $\delta > 0$ such that the condition in $\mathrm{ABC}[a,b]$ is satisfied. Let $N \in \mathbb{N}$ such that $(b-a)/N < \delta$. Divide $[a,b]$ into N-many subintervals with equal length with partition points $a = x_0 < x_1 < \cdots < x_N = b$. Then $t(f\big|_{[x_{i-1},x_i]}; \pi) \le \varepsilon = 1$ for any $\pi \in \mathrm{Par}[x_{i-1}, x_i]$. So $T_{x_{i-1}}^{x_i}(f) \le 1$ and hence $T_a^b(f) = \sum_{i=1}^N T_{x_{i-1}}^{x_i}(f) \le N$.

(c) Let $E \subseteq (a,b)$ such that $m(E) = 0$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that the condition in $\mathrm{ABC}[a,b]$ is satisfied. By outer regularity, there is an open set $G \supseteq E$ such that $m(G) < \delta$. Write $G = \bigcup_{i=1}^\infty (a_i, b_i)$, a countable union of disjoint open intervals. Then $\sum_{i=1}^\infty |a_i - b_i| = m(G) < \delta$. Since f is continuous on each $[a_i, b_i]$, there are $x_i^*, y_i^* \in [a_i, b_i]$ such that $f([a_i, b_i]) \subseteq [f(x_i^*), f(y_i^*)]$. Now $\sum_{i=1}^\infty |x_i^* - y_i^*| \le \sum_{i=1}^\infty |a_i - b_i| < \delta$ implies that $\sum_{i=1}^\infty |f(x_i^*) - f(y_i^*)| \le \varepsilon$. Hence

$$m^*(f(E)) \le m^*(f(G)) \le \sum_{i=1}^\infty m^*(f((a_i, b_i))) \le \sum_{i=1}^\infty |f(x_i^*) - f(y_i^*)| \le \varepsilon.$$

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