THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 7 (March 18)

Dominated Convergence Theorem. *Let $\{f_n\}$ be a sequence of measurable functions on E such that $f_n \to f$ a.e. on E. Suppose there is an integrable function g on E such that*

$$|f_n| \leq g \qquad \text{a.e. on } E, \text{ for all } n.$$

Then

$$\lim_n \int_E |f_n - f| = 0.$$

Remark. The dominance condition cannot be dropped. For example $f_n := \chi_{[n,n+1]} \to 0$ but $\int_{\mathbb{R}} f = 1$ for all n.

Example 1. Evaluate $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{n \sin x}{1 + n^2\sqrt{x}}\, dx$.

Solution. Note that for $x \in (0, 1]$, $n \in \mathbb{N}$,

$$\left| \frac{n \sin x}{1 + n^2\sqrt{x}} \right| \leq \frac{n}{1 + n^2\sqrt{x}} \leq \frac{1}{n\sqrt{x}} \leq \frac{1}{\sqrt{x}}.$$

Hence $f_n(x) := \frac{n \sin x}{1+n^2\sqrt{x}} \to 0$ a.e. on $[0, 1]$, and $|f_n(x)| \leq 1/\sqrt{x}$ a.e. on $[0, 1]$. Here $1/\sqrt{x} \in \mathcal{L}([0, 1])$ since

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{k\to\infty} \int_{1/k}^1 \frac{1}{\sqrt{x}}\, dx = \lim_{k\to\infty} 2\sqrt{x}\Big|_{1/k}^1 = 2 < \infty.$$

By Dominated Convergence Theorem, $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{n \sin x}{1 + n^2\sqrt{x}}\, dx = 0$. ◀

Example 2 (Improper Riemann Integral)**.** Let $f : [0, \infty) \to \mathbb{R}$ be Riemann integrable on any closed bounded subintervals of $[0, \infty)$. Then the improper Riemann integral is defined by

$$(\mathcal{R}) \int_0^\infty f(x)\, dx := \lim_{a\to\infty} \int_0^a f(x)\, dx.$$

Does $\displaystyle(\mathcal{R}) \int_0^\infty f(x)\, dx = (\mathcal{L}) \int_0^\infty f(x)\, dx$?

Solution. No. For example, $\displaystyle(\mathcal{R}) \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$. However

$$\begin{aligned}\int_0^\infty \left(\frac{\sin x}{x}\right)^+ dx &\geq \sum_{n=1}^\infty \int_{2n\pi+\pi/4}^{2n\pi+3\pi/4} \frac{1/\sqrt{2}}{2n\pi + 3\pi/4}\, dx \\ &\geq \sum_{n=1}^\infty \frac{1}{\sqrt{2}} \cdot \frac{1}{4n\pi} \cdot \frac{\pi}{2} = \frac{1}{8\sqrt{2}} \sum_{n=1}^\infty \frac{1}{n} = \infty.\end{aligned}$$

Similarly $\int_0^\infty \left(\frac{\sin x}{x}\right)^- dx = \infty$. So $(\mathcal{L}) \int_0^\infty \frac{\sin x}{x} dx$ does not even exist. ◀

Remark. If $f \geq 0$ or $f \in \mathcal{L}([0,1])$, then $(\mathcal{R}) \int_0^\infty f(x)\, dx = (\mathcal{L}) \int_0^\infty f(x)\, dx$.

Example 3. Evaluate $\lim_{n\to\infty} \int_0^\infty n^2 e^{-nx} \tan^{-1} x\, dx$.

Solution. Note that the integral makes sense in view of Riemann and Lebesgue since $n^2e^{-nx} \tan^{-1} x$ is nonnegative and continuous on $[0,\infty)$. Substituting $y = nx$, we have

$$\int_0^\infty n^2 e^{-nx} \tan^{-1} x\, dx = \int_0^\infty ne^{-y} \tan^{-1}(y/n)\, dy.$$

Note that $g_n(y) := ne^{-y} \tan^{-1}(y/n)$ is continuous on $[0,\infty)$ and satisfies

$$g_n(y) \to ye^{-y} \quad \text{and} \quad g_n(y) \leq ye^{-y} \quad \text{for any } y \in [0,\infty).$$

Here ye^{-y} is integrable on $[0,\infty)$ since $\int_0^\infty ye^{-y}\, dy = 1$. Hence, by Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^\infty n^2 e^{-nx} \tan^{-1} x\, dx = \lim_{n\to\infty} \int_0^\infty g_n(y)\, dy = \int_0^\infty ye^{-y}\, dy = 1.$$

◀

Let E be a measurable subset of $\mathbb{R}$, let f, f_n be measurable functions on E. Consider the following modes of convergence:

(I) We say that $f_n \to f$ **pointwise almost everywhere** if there exists $A \subseteq E$ such that $m(E \setminus A) = 0$ and $f_n(x) \to f(x)$ for all $x \in A$.

(II) We say that $f_n \to f$ in L^1 if $\|f_n - f\|_1 \to 0$, where $\|g\|_1 := \int_E |g|$.

Example 4. (i) (I) does not imply (II): For example $f_n := \chi_{[n,n+1]}$.

(ii) (II) does not imply (I): For example

$$f_n := \chi_{\left[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}\right]}, \qquad \text{where } 2^k \leq n < 2^{k+1},\ k = 0,1,2,\ldots$$

Borel-Cantelli Lemma. *Let $\{A_n\}$ be a sequence of measurable sets in E such that $\sum_{n=1}^\infty m(A_n) < \infty$. Then*

$$m(\{x : x \in A_n \textit{ for infinitely many } n\}) = 0.$$

Proof. Let $g(x) = \sum_{n=1}^\infty \chi_{A_n}$. Then $g(x) = \infty$ if and only if $x \in A_n$ for infinitely many n. Since $\int_E g = \sum_{n=1}^\infty m(A_n) < \infty$, g is finite a.e. The result follows. □

Example 5. If $f_n \to f$ in L^1, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ pointwise a.e.

Solution. For any $\varepsilon > 0$, $\displaystyle\int_E |f_n - f| \geq \varepsilon m(\{x : |f_n(x) - f(x)| \geq \varepsilon\})$, so that

$$m(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon}\|f_n - f\|_1.$$

Take $\varepsilon = 2^{-k}$. Since $\|f_n - f\|_1 \to 0$, there is $N_k \in \mathbb{N}$ such that

$$\|f_n - f\|_1 < 2^{-2k} \qquad \text{for all } n \geq N_k.$$

Hence we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that for each k,

$$m(E_k) := m(\{x : |f_{n_k}(x) - f(x)| \geq 2^{-k}\}) < 2^k \cdot 2^{-2k} = 2^{-k}.$$

Note $\sum_k m(E_k) \leq \sum_k 2^{-k} < \infty$. By Borel-Cantelli Lemma, we have

$$m(\{x : x \in E_k \text{ for infinitely many } k\}) = 0.$$

Hence, for a.e. $x \in E$, there is $N(x) \in \mathbb{N}$ such that

$$|f_{n_k}(x) - f(x)| < 2^{-k} \qquad \text{for all } k \geq N(x),$$

which implies that $f_{n_k}(x) \to f(x)$. ◀