THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 6 (March 11)

Construction of Lebesgue Integral in Four Steps.

Step 1. Integral of simple functions.

Step 2. Integral of bounded measurable functions on E with $m(E) < \infty$.

Step 3. Integral of nonnegative measurable functions.

Step 4. Integral of integrable measurable functions.

Fatou's Lemma. *Let $\{f_n\}$ be a sequence of* ***nonnegative*** *measurable functions on E such that $\lim_n f_n(x) = f(x)$ for a.e. $x \in E$. Then*

$$\int_E f \leq \liminf_n \int_E f_n.$$

Remark. (1) The inequality can be strict. For example, if $E = [0,1]$, $f_n = n\chi_{[0,1/n]}$, then $f_n \to 0$ a.e. but $\int_E f_n = 1$ for all n.

(2) Nonnegativity of $\{f_n\}$ is crucial. For example, consider $f_n = -n\chi_{[0,1/n]}$.

(3) A simple generalization: If $\{f_n\}$ be a sequence of **nonnegative** measurable functions on E, then $\displaystyle\int_E \liminf_n f_n \leq \liminf_n \int_E f_n$. To see this, apply Fatou's Lemma on $g_n := \inf_{k \geq n} f_n$.

Monotone Convergence Theorem. *Let $\{f_n\}$ be a sequence of measurable functions on E such that $0 \leq f_n \uparrow f$ a.e. on E. Then*

$$\int_E f = \lim_n \int_E f_n.$$

Remark. (1) MCT may not hold for $\downarrow$ sequence. For example, take $E = \mathbb{R}$ and $f_n = \chi_{[n,\infty)}$. Then $f_n \downarrow 0$ but $\int_E f_n = +\infty$ for all n.

(2) MCT holds for $\downarrow$ sequence if we further assume that $\int_E f_1 < +\infty$.

(3) Nonnegativity of $\{f_n\}$ is necessary. For example, consider $f_n = -\chi_{[n,\infty)}$.

Example 1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be additive, that is

$$f(x+y) = f(x) + f(y) \qquad \text{for any } x, y \in \mathbb{R}. \qquad (*)$$

Show that if f is measurable, then f is continuous.

From $(*)$, it is easy to see that $f(0) = 0$ and

$$f(ax+by) = af(x) + bf(y) \qquad \text{for any } x, y \in \mathbb{R} \text{ and } a, b \in \mathbb{Q}.$$

Solution. It suffices to show that f is continuous at 0. By applying Lusin's Theorem (in Tutorial 5) on $[-1,1]$, there is a closed set $F \subseteq [-1,1]$ such that $m([-1,1] \setminus F) < 1$ and $f\big|_F$ is continuous. In particular, $m(F) > 2 - 1 = 1 > 0$. Moreover $f\big|_F$ is uniformly continuous since F is compact.

Let $\varepsilon > 0$. By uniform continuity, there is $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \qquad \text{whenever } x, y \in F \text{ and } |x - y| < \delta. \tag{$**$}$$

By Steinhaus Theorem (see Tutorial 2), $m(F) > 0$ implies that $F - F \supseteq (-\delta', \delta')$ for some $\delta' > 0$. Take $\delta'' = \min\{\delta, \delta', 1\}$. Now, if $|v| < \delta''$, we have

$$v \in F - F \implies v = x - y \quad \text{for some } x, y \in F.$$

As $|x - y| = |v| < \delta'' \leq \delta$, it follows from $(**)$ that

$$|f(v) - f(0)| = |f(x - y) - 0| = |f(x) - f(y)| < \varepsilon.$$

Therefore f is continuous at 0. ◀

Remark. Using the Axiom of Choice, one can show the existence of additive functions that are not continuous, and hence the existence of non-measurable functions and non-measurable sets by the above result.