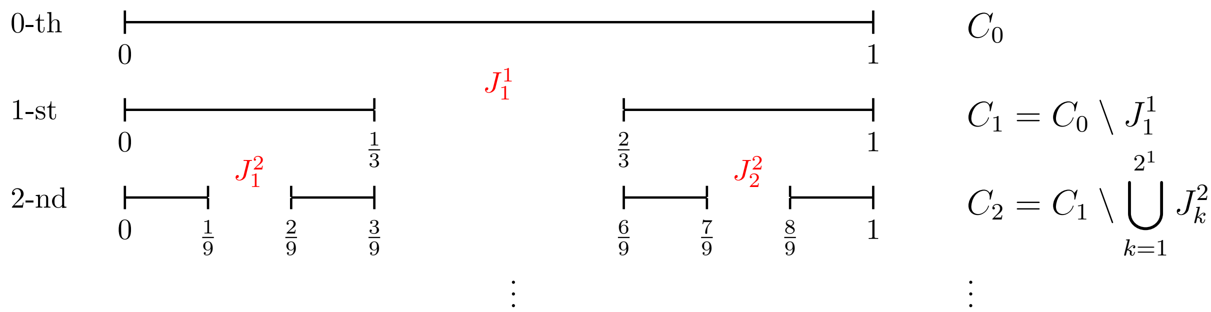
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 4 (February 26)

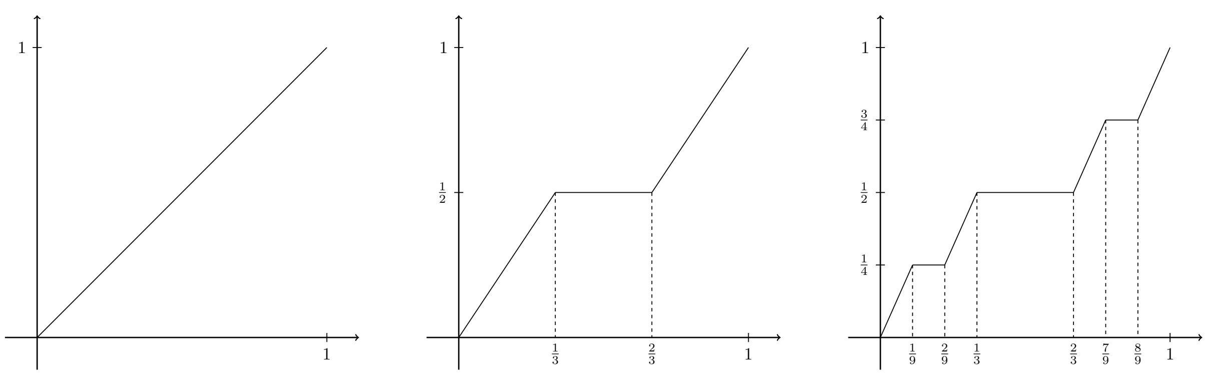
Last time we constructed the Cantor set $\mathcal{C} \coloneqq \bigcap_n C_n$, where $\{C_n\}$ is a sequence of sets given by:



We showed that $\mathcal{C}$ is a nonempty compact set that is perfect and $m(\mathcal{C}) = 0$. In particular, $\mathcal{C}$ is an uncountable set of measure zero.

Example 1 (The Cantor function)**.** Define a sequence of functions $\{g_n\}$ on $[0,1]$ by

$$g_0(x) = x, \qquad g_{n+1}(x) = \begin{cases} \frac{1}{2} g_n(3x) & \text{if } 0 \le x \le \frac{1}{3}, \\ \frac{1}{2} & \text{if } \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2} g_n(3x-2) & \text{if } \frac{2}{3} \le x \le 1. \end{cases}$$



g_0 g_1 g_2

Then each g_n is continuous, increasing on $[0,1]$ satisfying

$$\begin{aligned} \sup_{0 \le x \le 1} |g_{n+1}(x) - g_n(x)| &\le \frac{1}{2} \sup_{0 \le x \le 1} |g_n(x) - g_{n-1}(x)| \\ &\le \cdots \le \frac{1}{2^n} \sup_{0 \le x \le 1} |g_1(x) - g_0(x)| \\ &\le \frac{1}{2^n}. \end{aligned}$$

By Weierstrass M-test, the series $g_0 + \sum_{n=0}^{\infty}(g_{n+1} - g_n) = \lim_n g_n$ converges uniformly on $[0, 1]$ to a function g. As the uniform limit of $\{g_n\}$, g is (a) continuous, (b) increasing, and (c) constant on each J_k^m, $k = 1, 2, \ldots, 2^m$, $m \in \mathbb{N}$. We call g the Cantor function.

Example 2. Let $\varphi\colon [0, 1] \to \mathbb{R}$ be given by $\varphi(x) = x + g(x)$, where $g(x)$ is the Cantor function defined above. Then it is clear that φ is continuous, strictly increasing and maps $[0, 1]$ onto $[0, 2]$. Show that

(a) $\varphi(\mathcal{C})$ is Borel.

(b) $m(\varphi(\mathcal{C})) = 1$.

Solution. (a) Extend φ appropriately to a continuous injective function on $\mathbb{R}$. Then, by Example 3 in Tutorial 1, φ maps the Borel set $\mathcal{C}$ to a Borel set.

(b) Since φ is injective,

$$\begin{aligned}\varphi(\mathcal{C}) &= \varphi\left([0,1] \setminus \bigcup_{m=0}^{\infty}\bigcup_{k=1}^{2^m} J_k^{m+1}\right) \\ &= \varphi([0,1]) \setminus \varphi\left(\bigcup_{m=0}^{\infty}\bigcup_{k=1}^{2^m} J_k^{m+1}\right) \\ &= [0,2] \setminus \bigcup_{m=0}^{\infty}\bigcup_{k=1}^{2^m} \varphi\left(J_k^{m+1}\right).\end{aligned}$$

If $J_k^{m+1} = (a, b)$, then

$$\begin{aligned}m(\varphi(a,b)) &= m(\varphi(a), \varphi(b)) \\ &= \varphi(b) - \varphi(a) \\ &= b + g(b) - a - g(a) \\ &= b - a.\end{aligned}$$

As J_k^{m+1}'s are all disjoint, we have

$$m(\varphi(\mathcal{C})) = 2 - \sum_{m=0}^{\infty}\sum_{k=1}^{2^m} m(J_k^{m+1}) = 2 - 1 = 1.$$

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We assume the following result at the moment.

Proposition. Every measurable set in $\mathbb{R}$ with positive measure contains a non-measurable subset.

Example 3. Show that there is a measurable set that is not Borel.

Solution. By the Proposition, there is a non-measurable set $E \subseteq \varphi(\mathcal{C})$. Now, $\varphi^{-1}(E)$ is measurable since

$$m^*(\varphi^{-1}(E)) \leq m^*(\mathcal{C}) = 0.$$

However $\varphi^{-1}(E)$ is not Borel. For otherwise, $E = \varphi(\varphi^{-1}(E))$ is also Borel, hence measurable, which is a contradiction. ◀