THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 2 (February 14)

Definition. The *outer measure* is a function $m^*\colon \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ defined by

$$m^*(A) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : \{I_n\}_{n=1}^{\infty} \text{ is a countable open interval cover of } A\right\}.$$

Here $\ell(I)$ denotes the length of an interval I.

Proposition 1. *The outer measure m^* satisfies the following properties:*

(a) $m^*(\emptyset) = 0$ and $m^*(\{x\}) = 0$;

(b) m^* is monotonic;

(c) m^* is translational invariance;

(d) $m^*(A) = \inf\{m^*(O) : A \subseteq O,$ an open set in $\mathbb{R}\}$;

(e) m^* is countably subadditive.

Definition (Caratheodory's criterion)**.** A subset $E \subseteq \mathbb{R}$ is said to be *measurable* if

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \widetilde{E}) \qquad \forall A \subseteq \mathbb{R}.$$

Let $\mathcal{M}$ denote the family of all measurable sets.

Proposition 2. *$\mathcal{M}$ and m satisfy the following properties:*

(a) $\mathcal{M}$ is a σ-algebra that contains all sets of m^*-measure zero;

(b) $m^*\big|_{\mathcal{M}}$ is a measure;

(c) $\mathcal{B} \subseteq \mathcal{M}$.

Definition (Lebesgue measure)**.** The restriction of m^* to $\mathcal{M}$ is called the Lebesgue measure on $\mathbb{R}$ and is denoted by m.

Remark. (1) By outer regularity, for any $A \subseteq \mathbb{R}$, there exists a G_δ-set $G \supseteq A$ such that $m^*(A) = m^*(G)$:

(2) $\mathcal{M} \subsetneq \mathcal{P}(\mathbb{R})$: By the Axiom of Choice, one can show that there is a **non-measurable set** $P \subseteq [0, 1]$ such that

$$(P + r) \cap (P + s) = \emptyset \qquad \forall r, s \in \mathbb{Q} \text{ s.t. } r \neq s.$$

(3) $\mathcal{B} \subsetneq \mathcal{M}$.

Montone Convergence Lemma for Measures. *Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of measurable sets.*

(a) If $E_n \uparrow E$ (i.e. $E_n \subseteq E_{n+1}$ $\forall n$ and $E = \bigcup_n E_n$), then $m(E_n) \uparrow m(E)$.

(b) Suppose $m(E_{n_0}) < +\infty$ for some n_0. If $E_n \downarrow E$ (i.e. $E_n \supseteq E_{n+1}$ $\forall n$ and $E = \bigcap_n E_n$), then $m(E_n) \downarrow m(E)$.

Remark. The result in (b) does not hold if the condition $m(E_{n_0}) < +\infty$ $\exists n_0$ is dropped: Take $E_n = [n, +\infty)$. Then $m(E_n) = +\infty$ for all n. However, $m(\bigcap_n E_n) = 0$ since $\bigcap_n E_n = \emptyset$.

Example 1. Give an example of a sequence of set $\{E_n\}_{n=1}^{\infty}$ such that $m^*(E_1) < +\infty$, $E_n \downarrow E$ but $m^*(E) < \lim_n m^*(E_n)$.

Solution. Let $P \subseteq [0,1]$ be the non-measurable set above. Let $\{r_n\}$ be an enumeration of $\mathbb{Q} \cap [-1,1]$. Define $E_n = \bigcup_{k\geq n}(P + r_k)$. Then $E_n \supseteq E_{n+1}$ and

$$m^*(E_n) \leq m^*(\bigcup_{k\geq n}(P + r_k)) \leq m^*([-1,2]) = 3.$$

Moreover $E \coloneqq \bigcap_n E_n = \emptyset$ since $(P + r_k)$'s are pairwise disjoint. Hence $m^*(E) = 0$.

However, by translation invariance of m^* and non-measurability of P, we have

$$m^*(E_n) \geq m^*(P + r_n) = m^*(P) > 0.$$

Thus

$$m^*(E) = 0 < m^*(P) \leq \lim_n m^*(E_n).$$

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Example 2. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of set such that $E_n \uparrow E$. Show that $\lim_n m^*(E_n) = m^*(E)$.

Solution. It suffices to show that $\lim_n m^*(E_n) \geq m^*(E)$. WLOG, we assume that $m^*(E_n) < +\infty$. By outer regularity of m^*, for each $n \in \mathbb{N}$, there exists G_δ-set $G_n \supseteq E_n$ such that

$$m^*(E_n) = m^*(G_n) = m(G_n). \tag{1}$$

Let

$$F_k \coloneqq \bigcap_{n\geq k} G_n \quad \text{and} \quad F \coloneqq \bigcup_{k\geq 1} F_k.$$

Then $\{F_k\}$ is an increasing sequence of measurable sets such that $F_k \uparrow F$. Furthermore

$$E_k \subseteq E_n \subseteq G_n \quad \forall n \geq k \implies E_k \subseteq F_k \implies E \subseteq F,$$

and

$$F_k \subseteq G_n \quad \forall n \geq k. \tag{2}$$

Now

$$\begin{aligned}
m^*(E) &\le m^*(F) = m(F) \\
&= \lim_k m(F_k) && \text{(by MCL for measure)} \\
&\le \lim_k \inf_{n\ge k} m(G_n) && \text{(by (2))} \\
&= \liminf_n m(G_n) \\
&= \liminf_n m^*(E_n) && \text{(by (1))} \\
&= \lim_n m^*(E_n).
\end{aligned}$$

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Example 3. Let E be a subset of $\mathbb{R}$ with $m^*(E) > 0$. Show that for each $0 < \alpha < 1$, there exists an open interval I so that

$$m^*(E \cap I) > \alpha m^*(I).$$

Solution. WLOG we may assume $m^*(E) < +\infty$ since there is some $n \in \mathbb{Z}$ such that

$$0 < m^*(E \cap [n, n+1]) \le 1.$$

Let $0 < \alpha < 1$. By the outer regularity of m^*, there exists open $O \supseteq E$ such that

$$m^*(E) > \alpha m^*(O) = \alpha m(O).$$

By the structure theorem for open sets, we can write

$$O = \bigcup_{n=1}^{\infty} I_n,$$

where $\{I_n\}$ is a sequence of pairwise disjoint open intervals. Now

$$\sum_{n=1}^{\infty} m^*(E \cap I_n) \ge m^*(E \cap \bigcup_{n=1}^{\infty} I_n) = m^*(E) > \alpha m(\bigcup_{n=1}^{\infty} I_n) = \sum_{n=1}^{\infty} \alpha m(I_n).$$

This implies that

$$m^*(E \cap I_n) > \alpha m(I_n) \qquad \text{for some } n \in \mathbb{N}.$$

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Example 4 (Steinhaus Theorem)**.** Suppose $E \subseteq \mathbb{R}$ is measurable with $m(E) > 0$. Prove that the difference set of E,

$$E - E := \{x - y \in \mathbb{R} : x, y \in E\},$$

contains an open interval centered at the origin.

Solution. By the previous Exercise, there exists an open interval I such that

$$m(E \cap I) > \frac{7}{8} m(I) > 0. \tag{3}$$

Let $E_0 \coloneqq E \cap I$. Since $E_0 - E_0 \subseteq E - E$, it suffices to show that $E_0 - E_0$ contains an open interval centered at the origin.

Suppose it is not true. Then for any $\delta > 0$, there exists a with $0 < |a| < \delta$ such that $a \notin E_0 - E_0$. Hence E_0 and $E_0 + a$ are disjoint measurable sets such that

$$E_0 \cup (E_0 + a) \subseteq I \cup (I + a).$$

Now, if we take $\delta = \ell(I)/2$, then

$$2m(E_0) = m(E_0 \cup (E_0 + a)) \leq m(I \cup (I + a)) \leq \frac{3}{2} m(I).$$

So $m(E_0) \leq \dfrac{3}{4} m(I)$, contradicting (3). ◀