THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 12 (April 29)

Let E be a measurable subset of $\mathbb{R}$.

Definition. For $1 \leq p < \infty$, define

$$L^p(E) = \left\{ f \text{ measurable function on } E : \|f\|_p := \left(\int_E |f|^p \right)^{1/p} < \infty \right\},$$

where we identify functions that are equal almost everywhere on E. Then $(L^p(E), \|\cdot\|_p)$ is a normed vector space.

Example 1. Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Prove Young's inequality: $a \cdot b \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$ for $a, b \geq 0$. Show that equality holds if and only if $a^p = b^q$.

(b) Prove Hölder's inequality using Young's inequality.

(c) Show that Hölder's inequality becomes an equality if and only if there exist $\alpha, \beta \geq 0$, not both zero, such that $\alpha |f|^p = \beta |g|^q$ a.e.

Example 2. Deduce the Minkowski inequality from Hölder's inequality.

Definition. Let

$$L^\infty(E) = \{\text{a.e. bounded measurable functions on } E\},$$

where we again identify functions that are equal almost everywhere on E. Define

$$\|f\|_\infty := \inf\{M : |f(x)| \leq M \text{ a.e. on } E\}.$$

Then it is easy to see that

$$\|f\|_\infty \leq \lambda \quad \iff \quad |f(x)| \leq \lambda \quad \text{for a.e. } x \in E.$$

Now $(L^\infty(E), \|\cdot\|_\infty)$ is also a normed vector space and Hölder's inequality is true for $(p, q) = (1, \infty)$ or $(\infty, 1)$.

Theorem (Riesz-Fisher)**.** *For $1 \leq p \leq \infty$, $(L^p(E), \|\cdot\|_p)$ is complete.*

Example 3. Suppose $1 \leq p < q \leq \infty$. Show that $L^p(E) \not\subseteq L^q(E)$ and $L^q(E) \not\subseteq L^p(E)$ in general.

Example 4. Suppose $m(E) < \infty$. If $1 \leq p < q \leq \infty$, show that $L^q(E) \subseteq L^p(E)$.

Example 5. Suppose $\|f\|_r < \infty$ for some $1 \leq r < \infty$. Show that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$