THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 11 (April 22)

Example 1. If $f \in \text{ABC}[a, b]$, show that f maps any measurable set to a measurable set.

Solution. Let $E \subseteq [a, b]$ be measurable. By inner regularity, there is an F_σ-set $F \subseteq E$ such that $m(E \setminus F) = 0$. Then $F = \bigcup_{n=1}^{\infty} F_n$, where $F_n \subseteq [a, b]$, hence compact. Since f is continuous $f(F_n)$ is compact, and thus $f(F) = \bigcup_{n=1}^{\infty} f(F_n) \in \mathcal{B}$. By Lusin N property, $m(f(E \setminus F)) = 0$, so that $f(E \setminus F) \in \mathcal{M}$. Hence $f(E) = f(F) \cup f(E \setminus F) \in \mathcal{M}$. ◀

Example 2. Let $f \in \text{ABC}[a, b]$ and $g \in \text{ABC}[c, d]$ such that $g([c, d]) \subseteq [a, b]$.

(a) Is it true that $f \circ g \in \text{ABC}[c, d]$?

(b) Suppose further that f satisfies a Lipschitz condition on $[a, b]$. Show that $f \circ g \in \text{ABC}[c, d]$.

(c) Suppose further that g is monotone. Show that $f \circ g \in \text{ABC}[c, d]$.

Solution. (a) No. For example, let $f, g\colon [0, 1] \to [0, 1]$ be defined by $f(x) = \sqrt{x}$ and $g(x) = (x \sin \frac{1}{x})^2 \chi_{(0,1]}$. Then $f \in \text{ABC}[0, 1]$ since $\frac{1}{2\sqrt{x}} \in \mathcal{L}([0, 1])$ and $f(x) = \sqrt{x} = \int_0^x \frac{1}{2\sqrt{u}}\, du$. For g, it is continuous and satisfies that if $0 < y \le x \le 1$, then

$$\begin{aligned}|g(x) - g(y)| &\le |(x^2 - y^2) \sin^2 \frac{1}{x}| + |y^2(\sin \frac{1}{x} - \sin \frac{1}{y})| \\ &\le 2|x - y| + 2y^2|\frac{1}{x} - \frac{1}{y}| \\ &\le 4|x - y|.\end{aligned}$$

So g is Lipschitz continuous and hence absolutely continuous on $[0, 1]$.

However $f \circ g(x) = |x \sin \frac{1}{x}|\chi_{(0,1]}$ is not even of bounded variation. Indeed, let $x_k := (\pi + k\pi/2)^{-1}$ for $k = 0, 1, \cdots$. Then $f \circ g(x_k) = x_k$ if k is odd; and $f \circ g(x_k) = 0$ if k is even. So, for $k \ge 1$,

$$|f \circ g(x_k) - f \circ g(x_{k-1})| = x_k \text{ or } x_{k-1} \ge x_k = \frac{1}{\pi + k\pi/2} \ge \frac{1}{3k\pi}.$$

Since $0 < x_n < x_{n-1} < \cdots < x_1 < x_0 < 1$, we have

$$T_0^1(f \circ g) \ge T_{x_n}^{x_0}(f \circ g) \ge \sum_{k=1}^{n} |f \circ g(x_k) - f \circ g(x_{k-1})| \ge \sum_{k=1}^{n} \frac{1}{3k\pi} \to \infty \quad \text{as } n \to \infty.$$

Thus $T_0^1(f \circ g) = \infty$, and $f \circ g$ is not of bounded variation.

(b) Since f is Lipschitz on $[c, d]$, there is $L > 0$ such that

$$|f(u) - f(v)| \le L|u - v| \qquad \text{for all } u, v \in [c, d].$$

Hence

$$|f(g(x)) - f(g(y))| \le L|g(x) - g(y)| \qquad \text{for all } x, y \in [a, b].$$

The result then follows readily from the definition of absolute continuity.

(c) WLOG, assume that g is increasing. So if $\{(x_i, x_i')\}_{i=1}^n$ is a finite collection of non-overlapping intervals in $[a, b]$, then $\{(g(x_i), g(x_i'))\}_{i=1}^n$ is a finite collection of non-overlapping intervals in $[c, d]$. The result then follows readily from the definition of absolute continuity.

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Example 3 (Change of Variables formula)**.** Let g be strictly increasing and absolutely continuous on $[a, b]$ such that $g(a) = c$ and $g(b) = d$. Show that for any integrable function f over $[c, d]$,

$$\int_c^d f(y)\, dy = \int_a^b f(g(x))g'(x)\, dx.$$