THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH4050 Real Analysis

Tutorial 10 (April 15)

Theorem 1. *Let $f \in \mathcal{L}[a,b]$. Then*

$$\frac{d}{dx}\int_a^x f = f(x) \quad \text{for a.e. } x \in [a,b].$$

Theorem 2. *If $F \in \mathrm{ABC}[a,b]$, then $F'(x)$ exists a.e. in $[a,b]$, $F' \in \mathcal{L}[a,b]$ and*

$$F(x) = \int_a^x F' + F(a) \quad \text{for all } x \in [a,b].$$

Conversely, if $f \in \mathcal{L}[a,b]$, then the "indefinite integral" defined by

$$x \mapsto \int_a^x f + \text{constant}$$

is absolutely continuous on $[a,b]$.

Example 1. Let $f \in \mathrm{BV}[a,b]$. Show that

(a) $\displaystyle\int_a^b |f'| \le T_a^b(f)$;

(b) $f \in \mathrm{ABC}[a,b]$ if and only if $\displaystyle\int_a^b |f'| = T_a^b(f)$.

Decomposition for Absolutely Continuous Functions. *Let $f\colon [a,b] \to \mathbb{R}$. Then $f \in \mathrm{ABC}[a,b]$ if and only if there is a pair (g,h) of increasing absolutely continuous functions on $[a,b]$ such that $f = g - h$.*

Example 2. Let $f : [a,b] \to \mathbb{R}$ be a measurable function. Suppose $E \subseteq [a,b]$ is a measurable set and $f'(x)$ exists for all $x \in E$. Show that

$$m^*(f(E)) \le \int_E |f'|.$$

Solution. Note that f' is a measurable function on E. For each $n \in \mathbb{N}$, let

$$E_n = \{x \in E : |f'(x)| < n\},$$
$$E_{n,k} = \{x \in E : (k-1)2^{-n} \le |f'(x)| < k2^{-n}\} \quad \text{for } k = 1, 2, \ldots, n2^n.$$

Then $\{E_{n,k}\}_{k=1}^{n2^n}$ are disjoint, measurable and $E_n = \mathring{\bigcup}_{k=1}^{n2^n} E_{n,k}$. Moreover $E_n \uparrow E$. Hence, by Example 2 in Tutorial 8,

$$\begin{aligned} m^*(f(E_n)) &\le \sum_k m^*(f(E_{n,k})) \le \sum_k \frac{k}{2^n} m^*(E_{n,k}) \\ &\le \sum_k \frac{k-1}{2^n} m(E_{n,k}) + \frac{1}{2^n} \sum_k m(E_{n,k}) \\ &\le \sum_k \int_{E_{n,k}} |f'| + \frac{1}{2^n} m(E_n) \\ &\le \int_{E_n} |f'| + \frac{1}{2^n} m(E) \\ &\le \int_E |f'| + \frac{b-a}{2^n}. \end{aligned}$$

Since $f(E_n) \uparrow f(E)$, it follows from the Monotone Convergence Lemma for m^* (see Tutorial 2) that $\lim_n m^*(f(E_n)) = m^*(f(E))$. Taking $n \to \infty$ gives $m^*(f(E)) \le \int_E |f'|$. ◀

Theorem 3 (Banach-Zarecki)**.** *Let $f : [a,b] \to \mathbb{R}$. Then $f \in \mathrm{ABC}[a,b]$ if and only if the following conditions are all satisfied.*

(a) f is continuous on $[a,b]$.

(b) $f \in \mathrm{BV}[a,b]$.

(c) f has Lusin N property, i.e. f maps a set of measure zero to a set of measure zero.

Proof. Suppose f satisfies (a), (b) and (c). Since $f \in \mathrm{BV}[a,b]$, f' exists a.e. on $[a,b]$ and $f' \in \mathcal{L}[a,b]$. Assume that $f'(x)$ exists on E with $m([a,b] \setminus E) = 0$. Note that E must be measurable. Let $\varepsilon > 0$. Then the absolute continuity of integral implies that there is $\delta > 0$ for which

$$F \in \mathcal{M},\ F \subseteq [a,b] \text{ and } m(F) < \delta \implies \int_F |f'| < \varepsilon. \tag{1}$$

Let $\{(x_k, y_k)\}_{k=1}^n$ be non-overlapping intervals in $[a,b]$ such that $\sum_{k=1}^n |x_k - y_k| < \delta$. By continuity of f, there is $I_k := [c_k, d_k] \subseteq [x_k, y_k]$ such that $f([x_k, y_k]) \subseteq f(I_k)$, and hence $|f(x_k) - f(y_k)| \le m^*(f(I_k))$. Now, by Example 2 and (c), we have

$$m^*(f(I_k)) \le m^*(f(I_k \cap E)) + m^*(f(I_k \setminus E)) \le \int_{I_k \cap E} |f'| + 0 = \int_{I_k \cap E} |f'|.$$

Since

$$m(\bigcup_{k=1}^n (I_k \cap E) \le \sum_{k=1}^n m(I_k) \le \sum_{k=1}^n |x_k - y_k| < \delta,$$

it follows from (1) that

$$\sum_{k=1}^n |f(x_k) - f(y_k)| \le \sum_{k=1}^n \int_{I_k \cap E} |f'| = \int_{\bigcup_{k=1}^n (I_k \cap E)} |f'| < \delta.$$

Therefore $f \in \mathrm{ABC}[a,b]$. □