MATH 4050 Real Analysis

Suggested Solution of Homework 5

Only the solutions to $*$ questions are provided.

4.* (3rd: P.71, Q24; 4th: P.59, Q7)
Let f be measurable and B a Borel set. Show that $f^{-1}[B]$ is a measurable set. [Hint: The class of sets for which $f^{-1}[E]$ is measurable is a σ-algebra.]

Solution. Let $\mathcal{A} = \{E \in \mathcal{P}(\mathbb{R}) : f^{-1}[E] \in \mathcal{M}\}$, where $\mathcal{M}$ is the σ-algebra of all measurable sets. Then it is straightforward to check that $\mathcal{A}$ is a σ-algebra. By the definition of measurable functions, $(\alpha, \infty) \in \mathcal{A}$ for any $\alpha \in \mathbb{R}$. Hence $\mathcal{B} \subseteq \mathcal{A}$, and the result follows. ◀

5.* (3rd: P.71, Q25; 4th: P.59, Q10)
Show that if f is a measurable real-valued function and g a continuous function defined on $(-\infty, \infty)$, then $g \circ f$ is measurable.

Solution. For any $\alpha \in \mathbb{R}$, $g^{-1}[(\alpha, \infty)]$ is open since g is continuous, so that $(g \circ f)^{-1} = f^{-1}\left[g^{-1}[(\alpha, \infty)]\right]$ is measurable by Q4. Thus $g \circ f$ is measurable. ◀

6.* (3rd: P.73, Q29)
Give an example to show that we must require $m(E) < \infty$ in Proposition 23 (3rd ed.).

Proposition 23 is the claim $(*)$ in the proof of Egoroff's theorem in the lecture notes (Ch3, P.25), except the pointwise convergence a.e. on E is replaced by pointwise convergence on E.

Solution. Take $E = \mathbb{R}$ and $f_n = \chi_{[n,\infty)}$ on $\mathbb{R}$. Then $\lim\limits_n f_n(x) = 0$ for any $x \in E$. However, for any $A \subseteq E$ with $m(A) < 1$, there is a sequence $(x_n) \in E \setminus A$ such that for all n, $x_n \geq n$, and thus $|f_n(x_n) - 0| = 1$. ◀

8.* (3rd: P.74, Q31)
Prove Lusin's Theorem: Let f be a measurable real-valued function on an interval $[a, b]$. Then given $\delta > 0$, there is a continuous function φ on $[a, b]$ such that $m(x : f(x) \neq \varphi(x)) < \delta$. Can you do the same on the interval $(-\infty, \infty)$?

Solution. By a corollary of Littlewood's 2nd Principle, there is a sequence of continuous function g_n on $[a, b]$ that converges to f almost everywhere. Let $\delta > 0$. By Egoroff's Theorem, there is a subset $S \subseteq [a, b]$ with $m(S) < \delta$ such that g_n converges to f uniformly on $[a, b] \setminus S$. Using outer regularity of measure, we can further assume that S is open. As the uniform limit of a continuous sequence, $f\big|_{[a,b]\setminus S}$ is continuous. Now S is a countable union of disjoint open intervals. Define $\varphi(x) = f(x)$ for $x \in [a, b] \setminus S$; and define φ to be linear on each of the above intervals so that its values at the end points coincide with those of f. Then φ is continuous on $[a, b]$ and satisfies $m(\{x : f(x) \neq \varphi(x)\}) \leq m(S) < \delta$.

Suppose f is defined on $(-\infty, \infty)$. By the above argument, we can find a continuous function φ_n on $[n, n+1]$ that approximate f (say with an error of ε_n). Then we modify φ_n near the end points (say in ε_n-neighbourhoods), so that φ_n can be glued together to form a continuous function φ on $\mathbb{R}$. If we take $\varepsilon_n = \delta \cdot 10^{-|n|-1}$, then

$$m(\{x : f(x) \neq \varphi(x)\}) \leq \sum_{n=-\infty}^{\infty} (\varepsilon_n + 2\varepsilon_n) < \delta.$$

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