MATH 4050 Real Analysis

Suggested Solution of Homework 3

Only the solutions to $*$ questions are provided.

1.* (3rd: P.52, Q51) (Upper and lower envelopes of a function) Let f be a real-valued function defined on $[a, b]$. We define the *lower envelope* g of f to be the function g defined by

$$g(y) = \sup_{\delta>0} \inf_{|x-y|<\delta} f(x),$$

and the *upper envelope* h by

$$h(y) = \inf_{\delta>0} \sup_{|x-y|<\delta} f(x).$$

Prove the following:

(a) For each $x \in [a, b]$, $g(x) \leq f(x) \leq h(x)$, and $g(x) = f(x)$ if and only if f is lower semicontinuous at x, while $g(x) = h(x)$ if and only if f is continuous at x.

(b) If f is bounded, the function g is lower semicontinuous, while h is upper semicontinuous.

(c) If φ is any lower semicontinuous function such that $\varphi(x) \leq f(x)$ for all $x \in [a, b]$, then $\varphi(x) \leq g(x)$ for all $x \in [a, b]$.

Solution. (a) Let $x \in [a, b]$. Then $\inf\limits_{|y-x|<\delta} f(y) \leq f(x) \leq \sup\limits_{|y-x|<\delta} f(y)$ for any $\delta > 0$.

Hence

$$g(x) = \sup_{\delta>0} \inf_{|y-x|<\delta} f(y) \leq f(x) \leq \inf_{\delta>0} \sup_{|y-x|<\delta} f(y) = h(x).$$

Suppose $g(x) = f(x)$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) - \varepsilon = g(x) - \varepsilon < f(y) \qquad \text{whenever } y \in [a, b] \text{ and } |y - x| < \delta.$$

So f is lower semicontinuous at x.

Conversely, suppose f is lower semicontinuous at x. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $y \in [a, b]$ and $|y - x| < \delta$, we have $f(x) - \varepsilon < f(y)$. Thus

$$f(x) - \varepsilon \leq \inf_{|y-x|<\delta} f(y) \leq g(x).$$

As ε is arbitrary and $g(x) \leq f(x)$, we have $g(x) = f(x)$.

Similarly, one can show $h(x) = f(x)$ if and only if f is upper semicontinuous at x. Finally f is continuous at x if and only if f is both upper semicontinuous and lower semicontinuous at x if and only if $h(x) = f(x) = g(x)$.

(b) Let $x \in [a, b]$. Since f is bounded, we have $g(x) \in \mathbb{R}$. Thus given $\varepsilon > 0$, there exists $\delta > 0$ such that $g(x) < \inf\limits_{|y-x|<\delta} f(y) + \varepsilon$. Note that $(z - \delta/2, z + \delta/2) \subseteq (x - \delta, x + \delta)$ if $|x - z| < \delta/2$. So if $|z - x| < \delta/2$, we have

$$g(x) < \inf_{|y-x|<\delta} f(y) + \varepsilon \leq \inf_{|y-z|<\delta/2} f(y) + \varepsilon \leq g(z) + \varepsilon.$$

Hence g is lower semicontinuous at x.

Similarly we can show that h is upper semicontinuous.

(c) Let $\underline{\varphi}$ be the lower envelope of ϕ. Then for any $x \in [a, b]$,

$$\underline{\varphi}(x) = \sup_{\delta>0} \inf_{|y-x|<\delta} \varphi(y) \leq \sup_{\delta>0} \inf_{|y-x|<\delta} f(y) = g(x),$$

and hence $\varphi(x) = \underline{\varphi}(x) \leq g(x)$, since φ is lower semicontinuous.

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2.* (3rd: P.52, Q52)

Let f be a lower semicontinuous function defined for all real numbers. What can you say about the sets $\{x : f(x) > \alpha\}$, $\{x : f(x) \geq \alpha\}$, $\{x : f(x) < \alpha\}$, $\{x : f(x) \leq \alpha\}$, and $\{x : f(x) = \alpha\}$?

Solution. Since f is lower semicontinuous, $\{x : f(x) > \alpha\}$ is open.
$\{x : f(x) \geq \alpha\} = \bigcap_n \{x : f(x) > \alpha - 1/n\}$ is G_δ.
$\{x : f(x) \leq \alpha\} = \{x : f(x) > \alpha\}^c$ is closed.
$\{x : f(x) < \alpha\} = \{x : f(x) \geq \alpha\}^c$ is F_σ.
$\{x : f(x) = \alpha\} = \{x : f(x) \geq \alpha\} \cap \{x : f(x) \leq \alpha\}$ is the intersection of a G_δ set with a closed set, so it is also G_δ. ◀

3.* (3rd: P.52, Q53; 4th: P.28, Q56)

Let f be a real-valued function defined for all real numbers. Prove that the set of points at which f is continuous is a G_δ.

Solution. Let C be the set of points at which f is continuous. Let g and h be the lower and upper envelope of f, respectively.

By Q1(a), we have

$$\begin{aligned} C &= \{x \in [a, b] : f \text{ continuous at } x\} \\ &= \{x \in [a, b] : g(x) = h(x)\} \\ &= \bigcap_{n=1}^{\infty} \{x \in [a, b] : h(x) - g(x) < 1/n\}. \end{aligned}$$

(Note that $g(x)$ and $h(x)$ cannot be both $+\infty$ or $-\infty$.)

To see that C is a G_δ-set (in $[a, b]$), it suffices to show that, given any $\lambda > 0$, $A := \{x \in [a, b] : h(x) - g(x) < \lambda\}$ is an open set in $[a, b]$.

Let $x \in A$. Then $h(x), g(x) \in \mathbb{R}$. Set $\varepsilon_0 := (\lambda - (h(x) - g(x))/4 > 0$. By the same argument in 1(b), we can find $\delta > 0$ such that

$$g(x) \leq g(z) + \varepsilon_0 \quad \text{and} \quad h(x) \geq h(z) - \varepsilon_0 \quad \text{for any } z \in V_\delta(x) \cap [a, b].$$

Hence for any $z \in V_\delta(x)$, we have

$$h(z) - g(z) \leq h(x) - g(x) + 2\varepsilon_0 = \frac{h(x) - g(x) + \lambda}{2} < \lambda.$$

Thus $V_\delta(x) \cap [a, b] \subseteq A$, and A is open. ◀

4.* (3rd: P.52, Q54; 4th: P.28, Q57)
Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Show that the set C of points where this sequence converges is a $F_{\sigma\delta}$.

Solution. Note that $\{f_n\}$ converges at x if and only if for each $k \in N$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have $|f_n(x) - f_m(x)| \leq 1/k$. So

$$C = \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n,m\geq N} A_{k,m,n},$$

where $A_{k,m,n} := \{x : |f_n(x) - f_m(x)| \leq 1/k\}$.

Since $|f_n - f_m|$ is continuous, $A_{k,m,n}$ is closed and so is $\displaystyle\bigcap_{n,m\geq N} A_{k,m,n}$.

Thus $\{x : (f_n) \text{ converges at } x\}$ is $F_{\sigma\delta}$. ◀