# MATH4050 Real Analysis

(Revised) **Assignment 3**

There are 8 questions in this assignment. The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

*

1. (3rd: P.52, Q51)
   (Upper and lower envelopes of a function) Let $f$ be a real-valued function defined on $[a, b]$. We define the *lower envelope* $g$ of $f$ to be the function $g$ defined by
   $$g(y) = \sup_{\delta>0} \inf_{|x-y|<\delta} f(x),$$
   and the *upper envelope* $h$ by
   $$h(y) = \inf_{\delta>0} \sup_{|x-y|<\delta} f(x).$$
   Prove the following:
   a. For each $x \in [a, b]$, $g(x) \leq f(x) \leq h(x)$, and $g(x) = f(x)$ if and only if $f$ is lower semicontinuous at $x$, while $g(x) = h(x)$ if and only if $f$ is continuous at $x$.
   b. If $f$ is bounded, the function $g$ is lower semicontinuous, while $h$ is upper semicontinuous.
   c. If $\varphi$ is any lower semicontinuous function such that $\varphi(x) \leq f(x)$ for all $x \in [a, b]$, then $\varphi(x) \leq g(x)$ for all $x \in [a, b]$.

*

2. (3rd: P.53, Q52)
   Let $f$ be a lower semicontinuous function defined for all real numbers. What can you say about the sets $\{x : f(x) > a\}$, $\{x : f(x) \geq a\}$, $\{x : f(x) < a\}$, $\{x : f(x) \leq a\}$, and $\{x : f(x) = a\}$?

*

3. (3rd: P.53, Q53; 4th: P.28, Q56)
   Let $f$ be a real-valued function defined for all real numbers. Prove that the set of points at which $f$ is continuous is a $G_\delta$. (Hint: $\forall \varepsilon > 0$, let $G(\varepsilon) := \{z : \exists \delta > 0$ s.t. $|f(x_1) - f(x_2)| < \varepsilon \ \forall x_1, x_2 \in V_\delta(z)\}$. Then $G(\varepsilon)$ is open and $\bigcap_{\varepsilon>0} G(\varepsilon)$ consists of all continuity pts)

*

4. (3rd: P.53, Q54; 4th: P.28, Q57)
   Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Show that the set $C$ of points where this sequence converges is a $F_{\sigma\delta}$. Hint: $C = \bigcap_{\varepsilon>0} \bigcup_{N\in\mathbb{N}} \{x : |f_m(x) - f_n(x)| < \varepsilon \ \forall m, n \geq N\}$

For Question 5-7, let $m$ be a countably additive measure defined for all sets in a $\sigma$-algebra $\mathfrak{M}$. Prove that:

5. (3rd: P.55, Q1; 4th: P.31, Q1)
   If $A$ and $B$ are two sets in $\mathfrak{M}$ with $A \subset B$, then $m(A) \leq m(B)$. This property is called monotonicity.

6. (3rd: P.55, Q2; 4th: P.31, Q2)
   Let $\{E_n\}$ be any sequence of sets in $\mathfrak{M}$. Then $m(\bigcup E_n) \leq \sum mE_n$.

7. (3rd: P.55, Q3; 4th: P.31, Q3)
   If there is a set $A$ in $\mathfrak{M}$ such that $mA < \infty$, then $m\phi = 0$.

8. (3rd: P.55, Q4; 4th: P.31, Q4)
   Let $nE$ be $\infty$ for an infinite set $E$ and be equal to the number of elements in $E$ for a finite set. Show that $n$ is a countably additive set function that is translation invariant and defined for all sets of real numbers. This measure is called the **counting measure**.

$G(\varepsilon) = \{z : \exists \delta > 0$ s.t. $|f(x_1) - f(x_2)| < \varepsilon \ \forall x_1, x_2 \in V_\delta(z)\}$.
Let $z \in G(\varepsilon)$ with the corresponding $\delta$. Let $z' \in V_{\delta/2}(z)$. Then $z' \in G(\varepsilon)$ with the corr. n'd determined by $\delta/2$ (since $V_{\delta/2}(z') \subseteq V_\delta(z)$)