MATH4050 Real Analysis
Assignment 7

There are 5 questions in this assignment. The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

1. (3rd: P.89, Q3)

   Let f be a nonnegative measurable function. Show (by the monotonicity) that $\int f = 0$ implies $f = 0$ a.e. , and that $\int f \in \mathbb{R}$ implies that $f(x) \in \mathbb{R}$ a.e.

2. (3rd: P.89, Q4; 4th: P.85, Q24)

   Let f be a nonnegative measurable function.

   a. Show that there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions each of which vanishes outside a set of finite measure such that $f = \lim \varphi_n$.

   b. Show that $\int f = \sup \int \varphi$ over all nonnegative simple functions $\varphi \leq f$ with φ vanishing outside a set of finite measure.

3. (3rd: P.89, Q5)

   Let f be a nonnegative integrable function. Show that the function F defined by

   $$F(x) = \int_{-\infty}^{x} f$$

   is continuous by using Theorem 10 (3rd. ed).
   (Note: Theorem 10 is the monotone convergence theorem)

4. (3rd: P.89, Q6; 4th: P.85, Q25)

   Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converge to f, and suppose $f_n \leq f$ for each n. Show that

   $$\int f = \lim \int f_n.$$

5. (3rd: P.89, Q7; 4th: P.85, Q25 for part b.)

   a. Show that we may have strict inequality in Fatou's lemma. (Consider the sequence $\{f_n\}$ defined by $f_n(x) = 1$ if $n \leq x < n+1$, with $f_n(x) = 0$ otherwise)

   b. Show that the Monotone Convergence Theorem need not hold for decreasing sequences of functions. (Let $f_n(x) = 0$ if $x < n$, $f_n(x) = 1$ for $x \geq n$)