Solutions to Homework V

1. (1) First we find separated solutions. Suppose that $u(t,x) = T(t)X(x)$, then

$$T'(t)X(x) = T(t)X''(x),$$

giving that

$$-\frac{T'(t)}{T(t)} = -\frac{X''(x)}{X(x)} = \lambda.$$

We first solve $X(x)$. $X(x)$ satisfies

$$\begin{cases} -X'' = \lambda X; \\ X(0) = X(\ell) = 0. \end{cases}$$

If $\lambda = 0$, $X(x) = Ax + B$. By the boundary conditions, $A = B = 0$. If $\lambda \neq 0$, suppose that $-\lambda = \gamma^2$, then $X(x) = A\mathrm{e}^{\gamma x} + B\mathrm{e}^{-\gamma x}$. By the boundary conditions,

$$\begin{vmatrix} 1 & 1 \\ \mathrm{e}^{\gamma\ell} & \mathrm{e}^{-\gamma\ell} \end{vmatrix} = 0.$$

So $\gamma = n\pi\mathrm{i}/\ell$ and $\lambda = (n\pi/\ell)^2$ for $n \in \mathbb{Z} \setminus \{0\}$. So

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right)$$

for $n \in \mathbb{N}^+$. Then

$$T_n(t) = \mathrm{e}^{-n^2\pi^2 t/\ell^2}.$$

Therefore, a solution is

$$\sum_{n=1}^{\infty} A_n \mathrm{e}^{-n^2\pi^2 t/\ell^2} \sin\left(\frac{n\pi x}{\ell}\right).$$

(2) (a) Suppose that $u(t,x) = T(t)X(x)$, then

$$T''(t)X(x) = T(t)X''(x),$$

giving that

$$-\frac{T''(t)}{T(t)} = -\frac{X''(x)}{X(x)} = \lambda.$$

$X(x)$ satisfies

$$\begin{cases} -X'' = \lambda X; \\ X'(0) = X(\ell) = 0. \end{cases}$$

If $\lambda = 0$, $X(x) = Ax + B$. By the boundary conditions, $A = B = 0$. If $\lambda \neq 0$, suppose that $-\lambda = \gamma^2$, then $X(x) = A\mathrm{e}^{\gamma x} + B\mathrm{e}^{-\gamma x}$. By the boundary conditions,

$$\begin{vmatrix} 1 & -1 \\ \mathrm{e}^{\gamma\ell} & \mathrm{e}^{-\gamma\ell} \end{vmatrix} = 0.$$

So $\gamma = (n + \frac{1}{2})\pi\mathrm{i}/\ell$ and $\lambda = (n + \frac{1}{2})^2\pi^2/\ell^2$ for $n \in \mathbb{Z}$. So

$$X_n(x) = \cos\left[\frac{\left(n + \frac{1}{2}\right)\pi x}{\ell}\right]$$

for $n \in \mathbb{N}$.

(b)

$$T_n(t) = A\cos\left[\frac{\left(n + \frac{1}{2}\right)\pi t}{\ell}\right] + B\sin\left[\frac{\left(n + \frac{1}{2}\right)\pi t}{\ell}\right].$$

Therefore, a solution is

$$\sum_{n=0}^{\infty}\left\{A_n\cos\left[\frac{\left(n + \frac{1}{2}\right)\pi t}{\ell}\right] + B_n\sin\left[\frac{\left(n + \frac{1}{2}\right)\pi t}{\ell}\right]\right\}\cos\left[\frac{\left(n + \frac{1}{2}\right)\pi x}{\ell}\right].$$

(3) (a) Suppose that $u(t, x) = T(t)X(x)$, then

$$T'(t)X(x) = T(t)X''(x),$$

giving that

$$-\frac{T'(t)}{T(t)} = -\frac{X''(x)}{X(x)} = \lambda.$$

$X(x)$ satisfies

$$\begin{cases} -X'' = \lambda X; \\ X(-\ell) = X(\ell),\ X'(-\ell) = X'(\ell). \end{cases}$$

If $\lambda = 0$, $X(x) = Ax+B$. By the boundary conditions, $A = 0$. If $\lambda \neq 0$, suppose that $-\lambda = \gamma^2$, then $X(x) = A\mathrm{e}^{\gamma x} + B\mathrm{e}^{-\gamma x}$. By the boundary conditions,

$$(\mathrm{e}^{-\gamma\ell} - \mathrm{e}^{\gamma\ell})\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 0.$$

So $\gamma = n\pi\mathrm{i}/\ell$ and $\lambda = (n\pi/\ell)^2$ for $n \in \mathbb{Z}$. Hence, the eigenvalues are $(n\pi/\ell)^2$ for $n \in \mathbb{N}$. Moreover,

$$X_0(x) = 1,$$

and

$$X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right) \text{ or } \sin\left(\frac{n\pi x}{\ell}\right)$$

for $n \in \mathbb{N}^+$.

(b)

$$T_n(t) = \mathrm{e}^{-n^2\pi^2 t/\ell^2}.$$

Therefore, a solution is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} \mathrm{e}^{-n^2\pi^2 t/\ell^2}\left[A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right)\right].$$

2. (a) When $\lambda = 0$, $v(x) = Ax + B$. It is easy to see that $Ax + B$ satisfies the problem, so 0 is a double eigenvalue.

   (b) For $\lambda > 0$, $v(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. The boundary condition could be regarded as $v_x(0) = v_x(\ell)$ and $v_x(0) = [v(\ell) - v(0)]/\ell$. So

$$\begin{vmatrix} \sin\sqrt{\lambda}\ell & 1 - \cos\sqrt{\lambda}\ell \\ \frac{\cos\sqrt{\lambda}\ell - 1}{\ell} & \frac{\sin\sqrt{\lambda}\ell}{\ell} - \sqrt{\lambda} \end{vmatrix} = 0.$$

   It follows that

$$2(1 - \cos\sqrt{\lambda}\ell) = \sin\sqrt{\lambda}\ell\sqrt{\lambda}\ell,$$

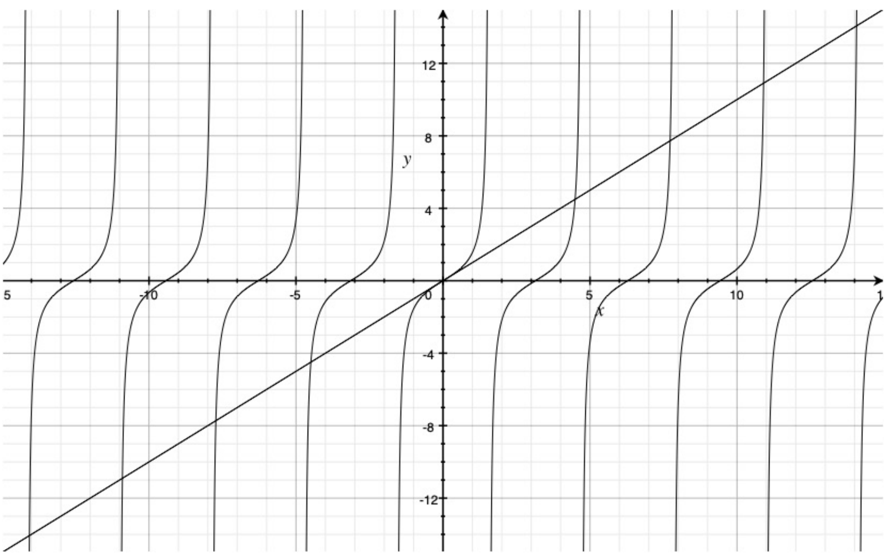
   that is,

$$4\sin^2\left(\frac{\sqrt{\lambda}\ell}{2}\right) = \sin\sqrt{\lambda}\ell\sqrt{\lambda}\ell.$$

   (c) Let $\gamma = \sqrt{\lambda}\ell/2$, then

$$\sin^2\gamma = \sin\gamma\cos\gamma\,\gamma.$$

   (d) If $\sin\gamma = 0$, $\gamma = n\pi$ and $\lambda = (2n\pi/\ell)^2$ for $n \in \mathbb{N}^+$. If $\sin\gamma \neq 0$, $\tan\gamma = \gamma$. The graph of γ is as follows:



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(e) For $\lambda = 0$, $v_0 = 1$ or x. For $\lambda = (2n\pi/\ell)^2$, $v_n = \cos(2n\pi x/\ell)$. For $\lambda = (2\gamma_n/\ell)^2$, where $\gamma_1 < \gamma_2 < \ldots$ are the positive solutions to $\tan\gamma = \gamma$,

$$\begin{aligned} v_n &= -\gamma_n \cos\left(\frac{2\gamma_n x}{\ell}\right) + \sin\left(\frac{2\gamma_n x}{\ell}\right) \\ &= -\frac{-\sin\gamma_n \cos\left(\frac{2\gamma_n x}{\ell}\right) + \cos\gamma_n \sin\left(\frac{2\gamma_n x}{\ell}\right)}{\cos\gamma_n} \\ &= \frac{\sin\left(\frac{2\gamma_n x}{\ell} - \gamma_n\right)}{\cos\gamma_n}. \end{aligned}$$

(f) Suppose that $u(t,x) = T(t)v(x)$, then

$$T'(t)v(x) = T(t)v''(x),$$

giving that

$$-\frac{T'(t)}{T(t)} = -\frac{v''(x)}{v(x)} = \lambda.$$

v satisfies the boundary conditions. For λ, $T_\lambda = \mathrm{e}^{-\lambda t}$. So

$$A_0 + B_0 x + \sum_{n=1}^{\infty} C_n \mathrm{e}^{-(2n\pi)^2 t/\ell^2} \cos\left(\frac{2n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} D_n \mathrm{e}^{-(2n\pi)^2 t/\ell^2} \frac{\sin\left(\frac{2\gamma_n x}{\ell} - \gamma_n\right)}{\cos\gamma_n},$$

provided that

$$\phi(x) = A_0 + B_0 x + \sum_{n=1}^{\infty} C_n \cos\left(\frac{2n\pi x}{\ell}\right) + \sum_{n=1}^{\infty} D_n \frac{\sin\left(\frac{2\gamma_n x}{\ell} - \gamma_n\right)}{\cos\gamma_n}.$$

(g) When $t \to \infty$, $u(t,x) \to A_0 + B_0 x$.

3. (a) Suppose that $\lambda = \beta^4$, then $X(x) = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x$. By the boundary conditions,

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ \cosh\beta\ell & \sinh\beta\ell & \cos\beta\ell & \sin\beta\ell \\ \cosh\beta\ell & \sinh\beta\ell & -\cos\beta\ell & -\sin\beta\ell \end{vmatrix} = 0.$$

So

$$\begin{vmatrix} \sinh\beta\ell & \sin\beta\ell \\ \sinh\beta\ell & -\sin\beta\ell \end{vmatrix} = 0,$$

giving that

$$\sin\beta\ell = 0.$$

So $\beta = n\pi/\ell$ and $\lambda = (n\pi/\ell)^4$ for $n \in \mathbb{N}^+$. And the corresponding eigenfunctions are $\sin(n\pi x/\ell)$.

(b) Suppose that $\lambda = \beta^4$, then $X(x) = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x$.

By the boundary conditions,

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \cosh\beta\ell & \sinh\beta\ell & \cos\beta\ell & \sin\beta\ell \\ \sinh\beta\ell & \cosh\beta\ell & -\sin\beta\ell & \cos\beta\ell \end{vmatrix} = 0.$$

Letting column 3 $-$ column 1 and column 4 $-$ column 2, we have

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cosh\beta\ell & \sinh\beta\ell & \cos\beta\ell - \cosh\beta\ell & \sin\beta\ell - \sinh\beta\ell \\ \sinh\beta\ell & \cosh\beta\ell & -\sin\beta\ell - \sinh\beta\ell & \cos\beta\ell - \cosh\beta\ell \end{vmatrix} = 0.$$

So

$$\begin{vmatrix} \cos\beta\ell - \cosh\beta\ell & \sin\beta\ell - \sinh\beta\ell \\ -\sin\beta\ell - \sinh\beta\ell & \cos\beta\ell - \cosh\beta\ell \end{vmatrix} = 0,$$

giving that

$$\cosh\beta\ell\cos\beta\ell = 1.$$

Hence, the eigenvalues are positive solutions to the above equation and the eigenfunctions are

$$(\sinh\beta\ell - \sin\beta\ell)(\cosh\beta x - \cos\beta x) - (\cosh\beta\ell - \cos\beta\ell)(\sinh\beta x - \sin\beta x).$$