Solutions of Homework I

1. (i) Since $4^2 - 4 \times 4 = 0$, it is of the parabolic type.

   (ii) Since $6^2 - 4 \times 9 = 0$, it is of the parabolic type.

   (iii) Since $4^2 - 4 > 0$, it is of the hyperbolic type.

2. (i) Fix (t, x). Let $z(s) = u(t+s, x+\frac{3}{2}s)$, then $\dot{z} = 0$. Since $z(-t) = u(0, x-\frac{3}{2}t) = \sin(x-\frac{3}{2}t)$, $u(t, x) = z(0) = \sin(x-\frac{3}{2}t)$. And it is easy to verify that $\sin(x-\frac{3}{2}t)$ is a solution.

   (ii) Fix (t, x). Let $z(s) = u(t+s, x+s)$, then $\dot{z}+z = 0$. Since $z(-t) = u(0, x-t) = g(x - t)$, $u(t, x) = z(0) = \mathrm{e}^{-(0-(-t))}g(x - t) = \mathrm{e}^{-t}g(x - t)$. And it is easy to verify that $\mathrm{e}^{-t}g(x - t)$ is a solution.

   (iii) Fix (t, x). Let $z(s) = u(t + s, x + s)$, then $\dot{z} + z = \mathrm{e}^{t+2x+3s}$. Since $z(-t) = u(0, x - t) = 0$, $u(t, x) = z(0) = \int_{-t}^{0} \mathrm{e}^{-(0-s)}\mathrm{e}^{t+2x+3s}\,\mathrm{d}s = \frac{1}{4}(\mathrm{e}^{t+2x} - \mathrm{e}^{-3t+2x})$. And it is easy to verify that $\frac{1}{4}(\mathrm{e}^{t+2x} - \mathrm{e}^{-3t+2x})$ is a solution.

3. (i) For a solution u, fixing (t, x), let $z(s) = u(t + s, x + 2s)$, then $\dot{z} = 0$. Since $z(-t) = u(0, x - 2t) = g(x - 2t)$, $u(t, x) = z(0) = g(x - 2t)$. Now it is clear that for each fixed x, $u(t, x) = g(x - 2t)$ tends to 0 as $t \to \infty$ since $g(x) \to 0$ as $x \to -\infty$.

   (ii) For a solution u, fixing (t, x), let $z(s) = u(t + s, x + 2s)$, then $\dot{z} + z = 0$. Since $z(-t) = u(0, x - 2t) = g(x - 2t)$, $u(t, x) = z(0) = \mathrm{e}^{-(0-(-t))}g(x - 2t) = \mathrm{e}^{-t}g(x - 2t)$. So it is clear that for each fixed x, $u(t, x) \to 0$ as $t \to \infty$ since g is bounded.

4. For (i) and (ii), they could be verified by direct calculation. Here we verify them from another point of view.

   (i) Firstly, the domains are (a) $\mathbb{R}^2$, (b) $\mathbb{R}^2$, (c) $\mathbb{R}^2 \setminus \{(0, 0)\}$, (d) $\mathbb{R}^2 \setminus \{(0, 0)\}$. We could observe that these functions are real parts of holomorphic functions (a) e^z, (b) $1 + z^2$, (d) $1/z$. (c) is somewhat subtle. In fact, we can't find a holomorphic function whose real part is $\log(x^2 + y^2)$ on $\mathbb{R}^2 \setminus \{(0, 0)\}$. But since harmonicity is just a local property, we could show that $\log(x^2+y^2)$ is harmonic by regarding it as the real parts of $2\log z$ defined on $\{\arg z\in(-\pi,\pi)\}$ and $\{\arg z\in(0,2\pi)\}$. We could also prove it by regrading it as the real part of $2\log z$ defined on $\{\arg z\in(-\pi,\pi)\}$ and then by continuity of Δu to cover the line $\arg z=\pi$.

(ii) An important observation is that traveling waves $f(x+2t)$ and $g(x-2t)$ are solutions of the wave equations. (In fact, we will learn that solutions of one-dimensional wave equations are linear superpositions of these two kinds of functions.) Now, (a) $4t^2+x^2=\frac{1}{2}((2t+x)^2+(2t-x)^2)$; (b) is clear; (c) $\sin 2t\cos x=\frac{1}{2}(\sin(2t+x)+\sin(2t-x))$; (d) is clear.

(iii) By the polar coordinate representation of Laplacian, we have

$$f''(r)+\frac{1}{r}f'(r)=0.$$

We have $f'(r)=C/r$ and $f(r)=a\log r+b$. And it is easy to check that $a\log r+b$ are harmonic on $\mathbb{R}^2\setminus\{(0,0)\}$.

5. Firstly, u^2 also satisfies the equation. So we could assume that $u\geq 0$ and we only need to prove that $u\leq 0$. u has a maximum M on D and suppose that u attains it at z_0. If z_0 is in the interior of D, then $u_x(z_0)=u_y(z_0)=0$, and by the equation we have $M=0$. Suppose that z_0 is on the boundary of D. $a(x,y)x+b(x,y)y>0$ tells us that $v=(a,b)$ is in the same direction of normal vector. In fact, we have $z_0-tv(z_0)\in D$ if $t>0$ is small enough.

$$|z_0|^2-|z_0-tv|^2=t(2(z_0,v)-t|v|^2),$$

where (z_0,v) denotes the inner product. Let $z_0=(x_0,y_0)$. Since $(z_0,v)=a(x_0,y_0)x_0+b(x_0,y_0)y_0>0$, if $t>0$ is small enough, we have $z_0-tv(z_0)\in D$. Since u attains the maximum at z_0,

$$\frac{u(z_0)-u(z_0-tv(z_0))}{t}\geq 0$$

for small positive t. Let $t\to 0$, then we obtain that

$$(v(z_0),\nabla u)=a(z_0)u_x(z_0)+b(z_0)u_y(z_0)\geq 0.$$

By the equation, we have $M\geq 0$. So in each case, $M\leq 0$, and we have $u\leq 0$.

6. (i) Fix (t,x). Let $z(s)=u(t+s,x+s)$, then $\dot z+z^2=0$. Since

$$-\frac{\dot z}{z^2}=1,$$

$$\frac{1}{z(0)} - \frac{1}{z(-t)} = t.$$

Since $z(-t) = u(0, x-t) = g(x-t)$,

$$u(t,x) = z(0) = \frac{1}{1/g(x-t)+t} = \frac{g(x-t)}{tg(x-t)+1}.$$

We could check that it is a general formula for the equation.

(ii) If the initial data is positive, it is clear that the solution exists for all time. Since

$$|u(t,x)| \le \frac{1}{t},$$

we have $u(t,x) \to 0$ as $t \to \infty$ for each fixed x.

(iii) For (iii) and (iv), we just consider the cases g has a minimum because otherwise there may be no solutions to the equation. Suppose that g attain its minimum $m < 0$ at x_0. Then before $T = -1/m$, we could imply that

$$u(t,x) = \frac{g(x-t)}{tg(x-t)+1}.$$

For $y = x_0 - 1/m$, we have

$$\lim_{t \to T^-} u(t,y) = -\infty.$$

(iv) If $m = \min g < 0$, it is easy to check that $g(x-t)/(tg(x-t)+1)$ is a solution before $T = -1/m$. So by (iii), $T_* = -(\min g)^{-1}$. If $m = \min g \ge 0$, by (ii), $T_* = \infty$. Therefore,

$$T_* = \begin{cases} -\dfrac{1}{\min g} & \min g < 0; \\ \infty & \min g \ge 0. \end{cases}$$