Solutions to Final Exam

1. (a)
$$\int_{\mathbb{R}} S(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-x^2/4t}\,\mathrm{d}x.$$
By change of variables and the Gaussian integral,
$$\int_{\mathbb{R}} S(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} \frac{1}{\sqrt{\pi}}\mathrm{e}^{-x^2}\,\mathrm{d}x = 1.$$

(b)
$$\max_{\delta<|x|<\infty} S(t,x) \le \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\delta^2/4t}.$$
So
$$\max_{\delta<|x|<\infty} S(t,x) \to 0$$
as $t \to 0$.

2. Suppose that $f_n \to f$ uniformly, that is,
$$\sup_I |f_n - f| \to 0.$$
Since
$$\int_I |f_n - f|^2 \le |I| \left(\sup_I |f_n - f|\right)^2,$$
$f_n \to f$ in the L^2 sense. Since
$$|f_n(x) - f(x)| \le \sup_I |f_n - f|$$
for $x \in I$, $f_n \to f$ in the pointwise sense.

3. (a) For $(t,x) \in (0,\infty)\times(0,1)$, since $0 \le u \le 1$ on the parabolic boundary $\overline{(0,1)\times(0,t]} \setminus (0,1)\times(0,t]$, by the maximum principle, $0 \le u(t,x) \le 1$.

(b) Let $u_1(t,x) = u(t,x)$ and $u_2(t,x) = u(t,1-x)$, then u_1 and u_2 satisfy the heat equation, $u_1(t,0) = u_2(t,0)$, $u_1(t,1) = u_2(t,1)$, and $u_1(0,x) = u_2(0,x)$. So by the uniqueness of initial boundary problems of heat equations, we have $u_1(t,x) = u_2(t,x)$ for all $(t,x) \in [0,\infty)\times[0,1]$.

(c) Since
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u^2(t,x)\,\mathrm{d}x = \int_0^1 2uu_t\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^1 2uu_{xx}\,\mathrm{d}x \\
&= -\int_0^1 2u_x^2\,\mathrm{d}x \\
&\le 0,
\end{aligned}
$$

where the boundary terms of the integration by parts vanish since u vanishes on the boundary, $\int_0^1 u^2(t,x)\,\mathrm{d}x$ is a decreasing function of t.

4. (a) For u_1 and u_2 satisfying the problem, consider $v = u_1 - u_2$, then v satisfies that

$$
\begin{cases}
v_t - v_{xx} = 0 & (t,x) \in (0,\infty) \times (0,\ell); \\
v_x(t,0) = v_x(t,\ell) = 0 & t \in (0,\infty); \\
v(0,x) = 0 & x \in [0,1].
\end{cases}
$$

Consider

$$
E(t) = \int_0^\ell v^2(t,x)\,\mathrm{d}x.
$$

Then

$$
\begin{aligned}
\frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \int_0^\ell 2vv_t\,\mathrm{d}x \\
&= \int_0^\ell 2vv_{xx}\,\mathrm{d}x \\
&= -\int_0^\ell 2v_x^2\,\mathrm{d}x \\
&\le 0,
\end{aligned}
$$

where the boundary terms of the integration by parts vanish since v_x vanishes on the boundary. Since $E(0) = 0$, $E(t) = 0$ and $v = 0$.

(b) Let $u = \mathrm{e}^{-t}v$, then

$$
\mathrm{e}^{-t}v_t - \mathrm{e}^{-t}v - \mathrm{e}^{-t}v_{xx} + \mathrm{e}^{-t}v = 0,
$$

giving that

$$
v_t - v_{xx} = 0.
$$

Moreover,

$$
v(0,x) = u(0,x) = \phi(x).
$$

So

$$
v(t,x) = \int_{\mathbb{R}} S(t,x-y)\phi(y)\,\mathrm{d}y.
$$

Therefore

$$u(t,x) = \mathrm{e}^{-t} v(t,x) = \mathrm{e}^{-t} \int_{\mathbb{R}} S(t, x-y)\phi(y)\,\mathrm{d}y,$$

and it is easy to verify that u is a solution.

(c) Let $u = \mathrm{e}^{-t^3/3} v$, then

$$\mathrm{e}^{-t^3/3} v_t - t^2 \mathrm{e}^{-t^3/3} v - \mathrm{e}^{-t^3/3} v_{xx} + t^2 \mathrm{e}^{-t^3/3} v = 0,$$

giving that

$$v_t - v_{xx} = 0.$$

Moreover,

$$v(0,x) = u(0,x) = \phi(x).$$

So

$$v(t,x) = \int_{\mathbb{R}} S(t, x-y)\phi(y)\,\mathrm{d}y.$$

Therefore

$$u(t,x) = \mathrm{e}^{-t^3/3} v(t,x) = \mathrm{e}^{-t^3/3} \int_{\mathbb{R}} S(t, x-y)\phi(y)\,\mathrm{d}y,$$

and it is easy to verify that u is a solution.

5. (a) By d'Alembert's formula,

$$\begin{aligned} u(t,x) &= \frac{\mathrm{e}^{x+t} + \mathrm{e}^{x-t}}{2} + \frac{1}{2}\int_{x-t}^{x+t} \sin s\,\mathrm{d}s \\ &= \frac{\mathrm{e}^{x+t} + \mathrm{e}^{x-t}}{2} + \frac{\cos(x-t) - \cos(x+t)}{2}. \end{aligned}$$

(b) By d'Alembert's formula,

$$\begin{aligned} u(t,x) &= \frac{\log[1+(x+t)^2] + \log[1+(x-t)^2]}{2} + \frac{1}{2}\int_{x-t}^{x+t} (4+s)\,\mathrm{d}s \\ &= \frac{\log[1+(x+t)^2] + \log[1+(x-t)^2]}{2} + \frac{(4+x+t)^2 - (4+x-t)^2}{4} \\ &= \frac{\log[1+(x+t)^2] + \log[1+(x-t)^2]}{2} + t(4+x). \end{aligned}$$

6. (a)

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,\mathrm{d}x = 0; \\ B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x = \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

So

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx.$$

(b) First we find separated solutions $u(t,x) = T(t)X(x)$. Then

$$T''(t)X(x) = T(t)X''(x),$$

giving that

$$-\frac{T''(t)}{T(t)} = -\frac{X''(x)}{X(x)} = \lambda.$$

$X(x)$ satisfies that $X(0) = X(\pi) = 0$. If $\lambda = 0$, then $X(x) = Ax + B$. From the boundary conditions, $A = B = 0$. If $\lambda \neq 0$, suppose that $-\lambda = \beta^2$, then $X(x) = A\mathrm{e}^{\beta x} + B\mathrm{e}^{-\beta x}$. From the boundary conditions,

$$\begin{vmatrix} 1 & 1 \\ \mathrm{e}^{\beta\pi} & \mathrm{e}^{-\beta\pi} \end{vmatrix} = 0.$$

So $\beta = n\mathrm{i}$ and $\lambda = n^2$ for $n \in \mathbb{Z} \setminus \{0\}$. Then $X_n(x) = \sin nx$ for $n \in \mathbb{N}^+$. Then $T_n(t) = A_n \cos nt + B_n \sin nt$. By (a),

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx.$$

Moreover,

$$0 = \sum_{n=1}^{\infty} 0 \sin nx.$$

Therefore,

$$u(t,x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \cos nt \sin nx.$$

7. (a) We first prove that

$$v(x) \leq \frac{1}{4\pi r^2} \int_{\partial B_r(x)} v(y)\,\mathrm{d}y.$$

$$\frac{1}{4\pi r^2} \int_{\partial B_r(x)} v(y)\,\mathrm{d}y = \frac{1}{4\pi} \int_{\partial B_1(0)} v(x+ry)\,\mathrm{d}y.$$

So

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{4\pi r^2} \int_{\partial B_r(x)} v(y)\,\mathrm{d}y\right) &= \frac{1}{4\pi} \int_{\partial B_1(0)} \nabla v(x+ry) \cdot y\,\mathrm{d}y \\
&= \frac{1}{4\pi r^2} \int_{\partial B_r(x)} \nabla v(y) \cdot \mathbf{n}\,\mathrm{d}y \\
&= \frac{1}{4\pi r^2} \int_{B_r(x)} \Delta v(y)\,\mathrm{d}y \\
&\geq 0.
\end{aligned}$$

Moreover,

$$\lim_{r\to 0} \frac{1}{4\pi r^2} \int_{\partial B_r(x)} v(y)\,\mathrm{d}y = v(x).$$

Therefore,

$$v(x) \le \frac{1}{4\pi r^2} \int_{\partial B_r(x)} v(y)\,\mathrm{d}y.$$

So

$$4\pi r^2 v(x) \le \int_{\partial B_r(x)} v(y)\,\mathrm{d}y.$$

Integrating r, we have

$$\frac{4\pi r^3}{3} v(x) \le \int_{B_r(x)} v(y)\,\mathrm{d}y,$$

giving that

$$v(x) \le \frac{3}{4\pi r^3} \int_{B_r(x)} v(y)\,\mathrm{d}y.$$

(b) Suppose that v attains its maximum M at x_0. If $x_0 \in \partial U$, the proof ends. If $x_0 \in U$, we then show that $v = M$ on $\overline{U}$. Let

$$E = \{x \in U \mid v(x) = M\}.$$

Then E is closed. Next we show that E is open. Suppose that $x \in E$ and $\overline{B_r(x)} \subset U$. By (a),

$$M = v(x) \le \frac{3}{4\pi r^3} \int_{B_r(x)} v(y)\,\mathrm{d}y.$$

So $v = M$ on $B_r(x)$ and $B_r(x) \subset E$. Therefore E is open. Since U is connected and E is nonempty, $E = U$. Hence $v = M$ on $\overline{U}$ and $\max_{\partial U} v = M$.

8. Consider $v = \pm u + \frac{|x|^2}{6}\lambda$ where $\lambda = \max_{\overline{U}}|f|$. Then

$$\Delta v = \mp f + \lambda \ge 0.$$

By 7,

$$\max_{\overline{U}} v = \max_{\partial U} v \le \max_{\partial U}|g| + C(U)\lambda.$$

So

$$\pm u \le \max_{\overline{U}} v \le C(U)\left(\max_{\partial U}|g| + \max_{\overline{U}}|f|\right)$$

on $\overline{U}$, giving that

$$\max_{\overline{U}}|u| \le C(U)\left(\max_{\partial U}|g| + \max_{\overline{U}}|f|\right).$$

9. (a)

$$\begin{aligned}\widehat{f'}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot \mathrm{i}n\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x \\ &= \mathrm{i}n\hat{f}(n).\end{aligned}$$

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = 0.$$

By Parseval's equality,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,\mathrm{d}x &= \sum_{n\neq 0} |\hat{f}(n)|^2 \\
&\leq \sum_{n\neq 0} |\mathrm{i}n\hat{f}(n)|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2\,\mathrm{d}x.
\end{aligned}$$

(b) Consider the odd extension $\tilde{f}$ of f across 0. Then $\tilde{f}$ is C^1 on $[-\pi, \pi]$, $\tilde{f}(-\pi) = \tilde{f}(\pi)$, and $\int_{-\pi}^{\pi} \tilde{f}(x)\,\mathrm{d}x = 0$. By (a),

$$\int_{-\pi}^{\pi} |\tilde{f}(x)|^2\,\mathrm{d}x \leq \int_{-\pi}^{\pi} |\tilde{f}'(x)|^2\,\mathrm{d}x,$$

giving that

$$\int_{0}^{\pi} |f(x)|^2\,\mathrm{d}x \leq \int_{0}^{\pi} |f'(x)|^2\,\mathrm{d}x.$$