MATH 2050B 2017-18

Mathematical Analysis I

Test Solution

(Q1) Using MI (mathematical induction) show the Bernoulli Inequality:

If $x > -1$, then $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

(A1) Use Induction on n, it is obvious when $n = 1$. $\Big((1+x)^1 = 1 + (1)x\Big)$

Suppose the inequality holds for some $n = k \in \mathbb{N}$, i.e. $(1+x)^k \geq 1 + kx$. Then

$$\begin{aligned}(1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) && \text{By Induction Hypothesis} \\ &= 1 + kx + x + kx^2 \\ &\geq 1 + (k+1)x && \text{since } x^2 \geq 0,\end{aligned}$$

the statement is true when $n = k+1$,

by principal of M.I., $(1+x)^n \geq 1 + nx \ \forall\, n \in \mathbb{N}$.

(Q2) Let $\alpha, \beta \in \mathbb{R}$ and $A \subset \mathbb{R}$ be such that

$$\alpha < \text{Sup} A < \beta$$

Prove/Disprove for each of the following assertions:

(i) $\alpha \leq a$ for all $a \in A$.

(ii) $\alpha \leq a$ for some $a \in A$.

(iii) $\alpha < a$ for some $a \in A$.

(iv) $a \leq \beta$ for some $a \in A$.

(v) $a < \beta$ for all $a \in A$.

(A2) Note $\alpha < \text{Sup} A < \beta$,

(i) False. Think about the counter example $A = \{0, 1\}$, $\alpha = 0.5$.
Note Sup $A = 1 > \alpha$ and $0 \in A$, but $0 < \alpha$.

(ii) True follow by (iii).

(iii) True.
If α were an upper bound of A, it will contradict the definition of Sup A (least upper bound).
Hence, α is NOT an upper bound of A, that is, $\exists\, a \in A$, s.t. $a > \alpha$.

(iv) True follow by (v).

(v) True.
Note Sup A is an upper bound of A, so $a \leq \text{Sup } A < \beta \ \forall a \in A$.

(Q3) Let $A \in \mathbb{R}$ be bounded below but not bounded above with convex ordering:

$$\text{If } x, y \in A \text{ with } x < z < y, \text{ then } z \in A.$$

Show that A is an interval.

(A3) Since A bounded below, by Completeness Axiom of $\mathbb{R}$, $\alpha := \text{Inf } A$ exists in $\mathbb{R}$.

That is, $\alpha \leq a \ \forall a \in A$. Hence, $A \subset [\alpha, +\infty)$.

Pick any $z \in (\alpha, +\infty)$, by definition of Inf, z is NOT a lower bound of A, that is, $\exists\, x \in A$, s.t. $x < z$.

Since A is NOT bounded above, $\exists\ y \in A$, s.t. $y > z$.

By convex ordering of A, we have $z \in A$. Hence, $(\alpha, +\infty) \subset A$.

Note we have proved $(\alpha, +\infty) \subset A \subset [\alpha, +\infty)$.

There are only two possible cases: $A = (\alpha, +\infty)$ or $A = [\alpha, +\infty)$, both are interval.

(Q4) Give the definition for $\lim_n x_n = x$ $\big($sequence (x_n) converges to $x \in \mathbb{R}\big)$.

Show, by definition, that

$$\text{If } \alpha < \lim_n x_n < \beta, \text{ then } \exists\ N \in \mathbb{N}, \text{ s.t. } \alpha < x_n < \beta\ \forall\ n \geq N.$$

$\Big[$Hint: Can you find $\varepsilon > 0$ s.t. $V_\varepsilon(x) \subset (\alpha, \beta)$, where $V_\varepsilon(x) := \big\{z \in \mathbb{R} : |z - x| < \varepsilon\big\}\Big]$

(A4) A sequence (x_n) converges to $x \in \mathbb{R}$, denoted by $\lim_n x_n = x$, if

$\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$, s.t. $\forall n \geq N$, we have $\left|x_n - x\right| < \varepsilon$ $\big($Or, $x_n \in V_\varepsilon(x)\big)$.

Suppose $x = \lim_n x_n$ and $\alpha < x < \beta$.

Take $\varepsilon' = \text{Min}\ \{\beta - x, x - \alpha\} > 0$, in particular, $\varepsilon' < \beta - x$ and $\varepsilon' < x - \alpha$.

By convergence of (x_n), $\exists\ N \in \mathbb{N}$, s.t. $\forall n \geq N$, we have $\left|x_n - x\right| < \varepsilon'$.

Note that $\forall\ n \geq N$, we have

$$\alpha = x - (x - \alpha) < x - \varepsilon' < x_n < x + \varepsilon' < x + (\beta - x) = \beta$$

(Q5) Show by $\varepsilon - N$ terminology that if $(x_n), (y_n)$ converge to real x and y respectively,

then $\lim_n \left(x_n + y_n\right) = x + y$.

(A5) Fixed any $\varepsilon > 0$, note $\frac{\varepsilon}{2} > 0$,

By (x_n) converge to x, $\exists\ N_1 \in \mathbb{N}$, s.t. $\left|x_n - x\right| < \frac{\varepsilon}{2}\ \forall\ n \geq N_1$.

By (y_n) converge to y, $\exists\ N_2 \in \mathbb{N}$, s.t. $\left|y_n - y\right| < \frac{\varepsilon}{2}\ \forall\ n \geq N_2$.

Note $\forall n \geq N := \text{Max}\ \left\{N_1, N_2\right\}$, we have

$$\left|(x_n + y_n) - (x + y)\right| \leq \left|x_n - x\right| + \left|y_n - y\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\lim_n \left(x_n + y_n\right) = x + y$.

(Q5*) Show by $\varepsilon - N$ terminology that if $(x_n), (y_n)$ converge to real x and y respectively,

then $\lim_n \left(x_n y_n\right) = xy$.

(A5*) We first show that (x_n) is a bounded sequence.

Take $\varepsilon_0 = 1$, by (x_n) converge to x, $\exists\ N' \in \mathbb{N}$, s.t. $\left|x_n - x\right| < \varepsilon_0 = 1\ \forall\ n \geq N'$. That is,

$$\begin{aligned} x - 1 < x_n &< x + 1 && \forall\ n \geq N' \\ \left|x_n\right| &< \text{Max}\ \left\{\left|x + 1\right|, \left|x - 1\right|\right\} && \forall\ n \geq N' \\ \left|x_n\right| &< \text{Max}\ \left\{\left|x_1\right|, \left|x_2\right|, ..., \left|x_{N'-1}\right|, \left|x + 1\right|, \left|x - 1\right|\right\} =: M && \forall\ n \in \mathbb{N} \end{aligned}$$

Note $M \in \mathbb{R}$ since the set inside the Max is a finite set.

Hence (x_n) is bounded by M (assume $M > 0$ WLOG).

Now, Fixed any $\varepsilon > 0$, note $\frac{\varepsilon}{2M} > 0$ and $\frac{\varepsilon}{2(|y|+1)} > 0$.

By (x_n) converge to x, $\exists\ N_1 \in \mathbb{N}$, s.t. $\left|x_n - x\right| < \frac{\varepsilon}{2(|y|+1)}\ \forall\ n \geq N_1$.

By (y_n) converge to y, $\exists\ N_2 \in \mathbb{N}$, s.t. $\left|y_n - y\right| < \frac{\varepsilon}{2M}\ \forall\ n \geq N_2$.

Note $\forall n \geq N\ := \text{Max}\left\{N_1, N_2\right\}$, we have

$$\begin{aligned}\left|x_n y_n - xy\right| &= \left|(x_n y_n - x_n y) + (x_n y - xy\right| \leq \left|x_n\right|\left|y_n - y\right| + |y|\left|x_n - x\right| \\ &< M\frac{\varepsilon}{2M} + |y|\frac{\varepsilon}{2\left(|y|+1\right)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence, $\lim_{n} \left(x_n y_n\right) = xy$.