HW IV A

1. Let (x_n) be a bounded seq and $\beta \in \mathbb{R}$. Show that $\beta = \limsup_n x_n$ iff it has the following properties:

(i) $\forall \beta' > \beta$, $\exists N \in \mathbb{N}$ s.t. $x_n \leq \beta'$ $\forall n \geq N$;

(ii) $\forall \beta'' < \beta$, $\exists$ infinitely many n s.t. $\beta'' < x_n$.

State and prove the corresponding result for $\liminf_n x_n$.

2.* (Ratio-test). Let (x_n) be a positive seq and $r, R \in \mathbb{R}$. Show that

(i) $\sum_{n=1}^{\infty} x_n$ converges if $\limsup_n \frac{x_{n+1}}{x_n} < r < 1$;

(ii) $\sum_{n=1}^{\infty} x_n$ diverges if $1 < R < \liminf_n \frac{x_{n+1}}{x_n}$.

3 (Root-test) Use $\limsup_n x_n^{1/n}$, $\liminf_n x_n^{1/n}$ in place of $\limsup_n \frac{x_{n+1}}{x_n}$, $\liminf_n \frac{x_{n+1}}{x_n}$ in Q3

4.* Show that (for positive seq (x_n))

$$\liminf_n \frac{x_{n+1}}{x_n} \leq \liminf_n x_n^{1/n} \leq \limsup_n x_n^{1/n} \leq \limsup_n \frac{x_{n+1}}{x_n}$$

(with the usual conventions, e.g. the last ineq holds if $\limsup_n \frac{x_{n+1}}{x_n}$ not exists in $\mathbb{R}$).

Thus the Root-test is "better" than the Ratio-test.

5. When $\lim_n \frac{x_{n+1}}{x_n} = 1$ or $\lim_n x_n^{1/n} = 1$, no conclusion can be drew from Q2, Q3: $\sum_{n=1}^{\infty} x_n$ can be converge/divergent (Can you provide examples? (Hint: harmonic series))

6. Let $x_{2n-1} = \frac{1}{3^{2n-1}}$ and $x_{2n} = \frac{1}{2^{2n}}$ $\forall n$. Can the tests be applied?